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ON THE SSQM AND NONLINEAR EQUATIONS

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1. INTRODUCTION

The study of particle-like behaviour of nonlinear fields originally initiated by Einstein to systematically derive the motion equations of particle in an external field, made a new turn with the discovery of the soliton solutions /1/.

The soliton-type properties have been found by now in a great variety of nonlinear physical systems such as Korteveg de Vries (KdV), sine-Gordon, nonlinear Schrödinger (NLS), etc.

In the seventies theoretical physics has developed a new fruitful conception of sypersymmetry with main idea to treat bosons and fermions equally /2/.

The interesting advantage of sypersymmetry is the natural way of incorporating fermions into the soliton system; it was firstly done for nonlinear equations via direct sypersymmetrization in refs. /3,4/.

From this supersoliton theory, which is given by the supersoliton Lagrangian in (1+1) space-time dimensions:

$$L = \frac{1}{2}[(\partial_\mu \phi)^2 - V^2(\phi) + \bar{\psi}(i\partial + V'(\phi))\psi], \qquad (1.1)$$

where ϕ is a Bose field and ψ is a Fermi field, we can obtain SSQM as a restriction to (0+1) space time dimension (the prime denotes differentiation with respect to the argument).

Really if we substitute in (1.1) the following restriction:

$$\phi \to x(t), \quad \partial_\mu \to \partial_t,$$
$$\bar{\psi} \to \psi^T \sigma_2, \quad i\partial\!\!\!/ \to i\partial_t \sigma_2,$$

where $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ with components being interpreted as anticommuting c-numbers, σ_k denote the Pauli matrices, then $L \to L_{SSQM}$ and

$$L_{SSQM} = \frac{1}{2}[(\partial_t x)^2 - V^2(x) + \psi^T(i\partial_t + \sigma_2 V'(x))\psi]. \qquad (1.2)$$

The corresponding Hamiltonian has the known form

$$H_{SSQM} = \frac{1}{2}p^2 + \frac{1}{2}V^2(x) + \frac{1}{2}i[\psi_1, \psi_2]V'(x), \qquad (1.3)$$

which was proposed by Witten [5] and also by Salamonson and Van Holten [6].

In the present work we want to show the role of SSQM for the nonlinear equations such as NLS and KdV.

The application of SSQM to the Zakharov equations [7] and the generalization, first given in [8], are discussed.

We also demonstrate the correspondence between a new class of the soliton solutions for the U(N) NLS [9] and the corresponding results in SSQM.

2. SUPERSYMMETRIC QUANTUM MECHANICS

We shall start with the Schrödinger factorization in QM: Assume the one-dimensional Schrödinger eq.

$$\left(-\frac{d^2}{dx^2}+U(x)\right)\psi(x) = E\psi(x), \qquad (2.1)$$

and the factorization in the form

$$\left(\frac{d}{dx}+v\right)\left(-\frac{d}{dx}+v\right)\psi = E\psi. \qquad (2.2)$$

If we denote

$$A^{\pm} = \pm\frac{d}{dx}+v, \qquad (2.3)$$

we can write $A^+A^-\psi = E\psi = H_+\psi$, but it gives

$$A^+A^- = H_+ = -\frac{d^2}{dx^2}+v^2+v_x = -\frac{d^2}{dx^2}+V_+.$$

Let us choose the ground state ψ_o^+ to be $H_+\psi_o^+ = A^+A^-\psi_o^+ = 0$ implying from

$$A^-\psi_o^+ = 0. \qquad (2.4)$$

This is a first-order differential eq.

$$\left(-\frac{d}{dx}+v(x)\right)\psi_o^+ = 0,$$

leading to

$$v = \frac{\psi^+_{ox}}{\psi^+_o}. \qquad (2.5)$$

If we consider the factorization in the form

$$\left(-\frac{d}{dx}+v\right)\left(\frac{d}{dx}+v\right)\psi = E\psi,$$

we get

$$A^-A^+ = -\frac{d^2}{dx^2}+v^2-v_x = -\frac{d^2}{dx^2}+V_- = H_-. \qquad (2.6)$$

Now suppose ψ^+ to be any eigenfunction of H_+

$$H_+\psi^+ = E_+\psi^+, \qquad (2.7)$$

then

$$A^-H_+\psi^+ = E_+(A^-\psi^+) = A^-A^+A^-\psi^+.$$

Either $A^-\psi^+ = 0$ (so that $E_+ = 0$ and ψ_+ is the ground state) or $H_-(A^-\psi^+) = E_+(A^-\psi^+)$.

Thus, every eigenstate of H_+ except for the ground state gives rise (via A^-) to an eigenstate of H_- with the same eigenvalue.

The ground state in H_+ with the zero energy does not correspond to any eigenstate of H_-.

It means that the Hamiltonian H_+ has the same spectrum as H_- plus one ground state more.

If we denote the solution by ψ_o^- of the zero-energy Schrödinger eq. with H_-

$$\left(-\frac{d^2}{dx^2}+V_-\right)\psi_o^- = -\psi^-_{oxx}+(v^2-v_x)\psi_o^- = 0, \qquad (2.8)$$

we get

$$v = -\frac{\psi^-_{ox}}{\psi^-_o} \qquad (2.9)$$

and by comparing with (2.5), we have

$$\psi_o^+ \sim \frac{1}{\psi_o^-}. \qquad (2.10)$$

The factorization presented here can be written in a supersymmetric way.

In the matrix formulation H_{SSQM} (1.3) becomes a 2x2 matrix as well

$$H_{SSQM} = \frac{1}{2}\begin{pmatrix} -\frac{d^2}{dx^2}+v^2(x)+v_x(x) & 0 \\ 0 & -\frac{d^2}{dx^2}+v^2(x)-v_x(x) \end{pmatrix} \qquad (2.11)$$

Then,

$$H_S = 2H_{SSQM} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \begin{pmatrix} A^+A^- & 0 \\ 0 & A^-A^+ \end{pmatrix} = \{Q^-, Q^+\}, \quad (2.12)$$

where the "supercharges" are defined as

$$Q^- = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}.$$

The other relations are

$$(Q^-)^2 = (Q^+)^2 = 0, \quad [H_S, Q^-] = [H_S, Q^+] = 0. \quad (2.13)$$

The eigenfunctions of H_S are

$$\psi_S = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}$$

and they have the properties

$$Q^-\psi_S = \begin{pmatrix} 0 \\ \psi^- \end{pmatrix} \quad \text{unless } A^-\psi^+ = 0,$$

$$Q^+\psi_S = \begin{pmatrix} \psi^+ \\ 0 \end{pmatrix}.$$

We can call the levels $\begin{pmatrix} \psi^+ \\ 0 \end{pmatrix}$ "bosonic" and the levels $\begin{pmatrix} 0 \\ \psi^- \end{pmatrix}$ "fermionic" in the view of the "fermionic" nature of the "superalgebra" in the relations (2.12) and (2.13).

In the theory of the spectral transforms and solitons [10] there is shown that the Schrödinger factorization (2.2) is equivalent to the Miura transformation between V_+ and v:

$$V_+ = v^2 + v_x,$$

coupling KdV and modified KdV (MKdV). The same is valid for V_-, because MKdV is invariant under the transformation $v \to -v$.

In this sense the Miura transformation represents the supersymmetric "square root".

There a deep connection between the N-soliton solution of the KdV, reflectionless potentials

$$U_N(x) = -N(N+1)\,b^2 \operatorname{sech}^2 bx, \quad N = 1,2\ldots, \quad (2.14)$$

and SSQM exists:

i) Let us take N=1, $b = \frac{1}{L\sqrt{2}}$ in the symmetric reflectionless potential (2.14)

$$u(x) = -\frac{1}{L^2} \operatorname{sech}^2 \frac{x}{L\sqrt{2}}.$$

Then, $u(x)$ can be regarded as a one-soliton solution of the KdV eq. for $t = 0$, i.e. of the eq.

$$u_t - 6uu_x + u_{xxx} = 0. \quad (2.15)$$

The KdV one-soliton solution for all t is

$$u(x,t) = -\frac{1}{L^2} \operatorname{sech}^2 \left(\frac{x - (2/L^2)\,t}{L\sqrt{2}}\right).$$

The same is valid for higher N.

ii) Let us consider now a function $v(x,t)$ satisfying MKdV

$$v_t + 6\left(\frac{1}{2L^2} - v^2\right) v_x + v_{xxx} = 0. \quad (2.16)$$

Then, if we define

$$V_- = v^2 - v_x - \frac{1}{2L^2} \quad (2.17)$$

as is usual in SSQM, it can easily be shown that V_- satisfies KdV. The same is valid for

$$V_+ = v^2 + v_x - \frac{1}{2L^2}. \quad (2.18)$$

In this way we can see that there exists the general connection between N-soliton solutions of KdV, SSQM, inverse scattering method and the construction of the reflectionless potentials.

We shall now be concerned in the application of these results of SSQM to the Schrödinger eq. with selfconsistent potentials.

3. THE APPLICATION OF SSQM TO THE NONINTEGRABLE SYSTEMS

Here we discuss the nonintegrable system [7]

$$i\psi_t + \psi_{xx} - n\psi = 0, \quad (3.1a)$$

$$n_{tt} - n_{xx} = |\psi|^2_{xx} \quad (3.1b)$$

and the system [8]

$$i\psi_t + \psi_{xx} - n\psi = 0, \quad (3.2a)$$

$$n_{tt} - n_{xx} - \alpha(n^2)_{xx} - \beta n_{xxxx} = 0. \quad (3.2b)$$

Here $\psi(x,t)$ and $n(x,t)$ are respectively complex and real functions; α, β are the real parameters.

These nonintegrable (in general case) systems have applications in the interesting areas in Physics /11/.

We shall now demonstrate using (3.1) that the basic role for finding soliton-like solutions plays eq.(3.1a) with $n(x,t)$ being the symmetric reflectionless potential.

The same will be valid for system (3.2)

Now we shall discuss the so-called quasi-static limit of the Zakharov (Z) eqs. (3.1a,b) neglecting the term n_{tt}. Then, eq. (3.1b) has the form

$$(n + |\psi|^2)_{xx} = 0, \tag{3.3}$$

which implies $n = -|\psi|^2$ if n and $|\psi|^2$ are square-integrable.

Substitution of this expression for n into (3.1a) yields the NLS eq.

$$i\psi_t + \psi_{xx} + |\psi|^2\psi = 0.$$

It is well known that the Z eqs.(3.1a,b) have a one-soliton solution

$$\psi = \frac{1}{L}\,\mathrm{sech}\left[\frac{x - x_0 - vt}{L\sqrt{2(1-v^2)}}\right]\exp\left[\frac{1}{2}ivx - i\left(\frac{1}{4}v^2 - \frac{1}{2L^2(1-v^2)}\right)t + i\theta_0\right] \tag{3.4a}$$

and

$$n = -\frac{|\psi|^2}{1-v^2}, \tag{3.4b}$$

where $L > 0$, v, x_0 and θ_0 are constants.

It is clear that solution (3.4a) tends to the particular one-soliton solution

$$\psi(x,t) = \exp\left(\frac{it}{2L^2}\right)\frac{1}{L}\,\mathrm{sech}\,\frac{x}{L\sqrt{2}} \tag{3.5}$$

and

$$n(x) = -|\psi|^2, \tag{3.6}$$

for $v = 0$, $x_0 = 0$, $\theta_0 = 0$.

If we put solutions (3.5) and (3.6) into eq.(3.1a), we obtain

$$\left(-\frac{d^2}{dx^2} - \frac{1}{L^2\,\mathrm{ch}^2(x/L\sqrt{2})}\right)\psi_1(E_1) = E_1\psi_1(E_1), \tag{3.7}$$

that is the eigenvalue eq. $H_1\psi_1 = E_1\psi_1$ corresponding to eq.(3.1a).

In eq.(3.7) we denote the eigenvalue by $E_1 = -\gamma_1^2 = -\frac{1}{2L^2}$, and it corresponds to the eigenfunction

$$\psi_1 = \frac{1}{L}\,\mathrm{sech}\,\frac{x}{L\sqrt{2}}.$$

Now we shall think of the $n(x)$:

$$n(x) = -|\psi_1|^2 = -\frac{1}{L^2}\,\mathrm{sech}^2\frac{x}{L\sqrt{2}}$$

as the symmetric reflectionless potential in the eigenvalue problem (3.7), and using the results of SSQM we shall construct other symmetric reflectionless potentials as "superpartners".

We can see that $H_1 = -\frac{d^2}{dx^2} + n_1$ is the superpartner to the $H_0 = -\frac{d^2}{dx^2} + n_0$, where the potential n_1 supports a simple bound state at the energy $E = -\frac{1}{2L^2}$ meanwhile n_0 supports no bound states.

Choosing $n_0 = 0$, H_0 is then the free particle Hamiltonian and reflection coefficient of n_0 is $R_0(k) = 0$ for the positive energies $E = k^2$.

The reflection coefficient of the H_1 is given by

$$R_1(k) = \frac{\gamma_1 - ik}{\gamma_1 + ik}R_0(k),$$

which is zero for $R_0(k) = 0$. But it is the case of the reflectionless potential in (2.14) for $N = 1$, $b = (1/L\sqrt{2})$.

From SSQM let us suppose

$$V_- = v^2 - v_x = \frac{1}{2L^2}. \tag{3.8}$$

Eq.(3.8) is a very simple Riccati eq. whose solution is given by substituting

$$v = -\frac{\psi_{0x}}{\psi_0} \tag{3.9}$$

and we have

$$\frac{\psi_{0xx}}{\psi_0} = \frac{1}{2L^2}. \tag{3.10}$$

Here, ψ_o is the solution of the zero-energy Schrödinger eq. with H_-

$$(-\frac{d^2}{dx^2} + v^2 - v_x)\psi_o = 0,$$

The solution ψ_o form eq.(3.10) is

$$\psi_o = \text{const. ch}\frac{x}{\sqrt{2}L} \qquad (3.11)$$

and from (3.9) it follows

$$v = -\frac{1}{\sqrt{2}L}\tanh\frac{x}{\sqrt{2}L}. \qquad (3.12)$$

The superpartner to the V_- has the form

$$V_+ = v^2 + v_x = \frac{1}{2L^2} - \frac{1}{L^2}\,\text{sech}^2\frac{x}{L\sqrt{2}}. \qquad (3.13)$$

Now, if we denote

$$n_o(x) = v^2 - v_x - \frac{1}{2L^2} = 0,$$

$$n_1(x) = v^2 + v_x - \frac{1}{2L^2} = -\frac{1}{L^2}\,\text{sech}^2\frac{x}{L\sqrt{2}},$$

we can see that H_1 is the superpartner to H_o.

Using the receipt from SSQM we shall now demonstrate how to construct a symmetric reflectionless $n_j(x)$, $j = 1,2...N$.

For arbitrary j we may now assume $n_{j-1}(x)$ to be known and define v_j by

$$n_{j-1} = v_j^2 - v_{jx} - E_j.$$

Then, the superpartner has the form

$$n_j = v_j^2 + v_{jx} - E_j.$$

The crucial point for the construction is that the super-symmetric reflectionless partner can be expressed via the eigenfunctions of the corresponding Hamiltonian.

This can be seen from the following ($j = 1$):

$$H_+ = A^+A^- = H_- + [A^+, A^-] = H_- + 2\frac{d}{dx}v = H_- - 2\frac{d^2}{dx^2}\ln\psi_o. \qquad (3.14)$$

From this

$$H_+ = -\frac{d^2}{dx^2} + E_1 + n_o - 2\frac{d^2}{dx^2}\ln\psi_o(E_1)$$

and

$$n_1 = -2\frac{d^2}{dx^2}\ln\psi_o(E_1). \qquad (3.15)$$

for $n_o = 0$.

We can apply this procedure to the Z system (3.1).

It was shown by Sukumar [18] that the symmetric reflection-less $n_N(x)$ may be expressed in terms of the normalised bound state eigenfunctions in the form

$$n_N(x) = -4\sum_{i=1}^{N}[\gamma_i\psi_N^2(E_i)]. \qquad (3.16)$$

Using the results of SSQM a vector version of the NLS (VNLS) was presented [12] in the form:

$$i\partial_t\psi_N + \psi_{Nxx} - n_N\psi_N = 0, \qquad (3.17a)$$

$$\{n_N + 4\sum_{i=1}^{N}[\gamma_i|\psi_N(E_i)|^2]\}_{xx} = 0, \qquad (3.17b)$$

where $\psi_N(E_i)$ are bound states given by eq.

$$(-\frac{d^2}{dx^2} + n_N(x))\psi_N(E_i) = E_i\psi_N(E_i) \qquad (3.18)$$

with n_N being "symmetric" reflectionless potentials.

For the physical application the interesting case is when

$$n_N(x) = -N(N+1)\frac{1}{2L^2}\,\text{sech}^2\frac{x}{L\sqrt{2}} = -\frac{N(N+1)}{2}n_1(x). \qquad (3.19)$$

Mathematically, formula (3.19) corresponds to the so-called Lame-Ains N-zones elliptical potential [13].

In this case the VNLS system (3.17) has the form:

$$i\partial_t\psi_N + \psi_{Nxx} + \frac{N(N+1)}{2}|\psi_1|^2\psi_N = 0, \qquad (3.20a)$$

$$[n_N(x) + \frac{N(N+1)}{2}|\psi_1|^2]_{xx} = 0. \qquad (3.20b)$$

In these eqs. there exist the envelope solitary wave solutions

$$\psi_N = e^{\frac{it}{2L^2}}\frac{1}{L}\,\text{sech}\frac{\sqrt{N(N+1)}}{2L}x,$$

for arbitrary $N = 1,2...$.

We now show that this admits another presentation first done in[9].

4. THE U(N) VECTOR NONLINEAR SCHRÖDINGER EQUATION, FACTORIZATION AND THE RELATION WITH THE SSQM RESULTS

The U(N) vector NLS has the form

$$i\phi_{Nt} + \phi_{Nxx} + (\bar{\phi}_N \phi_N)\phi_N = 0, \tag{4.1}$$

where

$$\phi_N(x,t) = (\phi_{N,1},\dots,\phi_{N,m})^T, \quad \bar{\phi}_N\phi_N = \sum_{j=1}^{m} |\phi_{N,j}|^2, \tag{4.2}$$

$m \geq N$

A new particular class of the soliton solutions of eq.(4.1) has recently been obtained[9] via the so-called factorization method and the technique in a sense similar to that developed by Krichever[14].

We show that these solutions are equivalent to the reflectionless symmetric potentials of the one-dimensional Schrödinger eq. and in the case when the potentials $n_N(x)$ have the form (2.14) it exactly corresponds to the results given in Sections 2. and 3. via SSQM.

We can show this by the following way:

Write the solutions of eq.(4.1) in the form

$$\phi_N(x,t) = Ce^{iW}\,\Phi_N(y), \tag{4.3}$$

where $\Phi_N(y) = (\Phi_{N,1},\dots,\Phi_{N,m})^T$, $C = \mathrm{diag}(C_1,\dots,C_m)$,

$W = \mathrm{diag}(\theta_1,\dots,\theta_m),\ \theta_j = \frac{v}{2}(x - \frac{v}{2}t) - \Lambda_j t,\quad y = x - vt,$

and put (4.3) into (4.1) to get:

$$\Phi''_{N,j} - n_N\Phi_{N,j} = -\Lambda_j\Phi_{N,j}, \tag{4.4}$$

$$n_N = -\sum_{j=1}^{m} |C_j|^2\,\Phi^2_{N,j}.$$

Suppose the potential n_N to be in the form (2.14), then (4.4) becomes

$$\Phi''_{N,j} + N(N+1)\,b^2\,\mathrm{sech}^2\,by\,\Phi_{N,j} = -\Lambda_j\Phi_{N,j}. \tag{4.5}$$

It is known (see, e.g. ref.[15]) that eq.(4.5) for arbitrary N has N eigenvalues $\Lambda_j = -j^2b^2$, $j = 1,2\dots N$.

The corresponding eigenfunctions may be found by using the factorization which is equivalent to the SSQM "square root", as it has been mentioned:

We can define $A^{\pm}_{\ell}$ in the same way as in (2.3), namely

$$A^{\pm}_{\ell} = \pm\frac{d}{dy} + \ell\, b\,\mathrm{th}\, by = \pm\frac{d}{dy} + v_{\ell}(y),$$

where v_{ℓ} has the form (3.12) for $\ell = 1$, $b = \frac{1}{L\sqrt{2}}$.

Then in the same way as in SSQM we define

$$A^{+}_{\ell+1}\Phi_{\ell,j} = \Phi_{\ell+1,j}, \tag{4.6}$$

$$A^{-}_{\ell+1}\Phi_{\ell,j} = \Phi_{\ell-1,j}. \tag{4.7}$$

From (4.6) and (4.7), using

$$\Phi_{\ell,j} \equiv 0 \tag{4.8}$$

for $\ell > N$, we obtain all the solutions to eq. (4.5).

Some of them follow directly:

for $N = j = \ell$ we get

$$A^{+}_{N}\Phi_{N,N} = 0,$$

and from this

$$\Phi_{N,N} \sim \mathrm{sech}^N\, by. \tag{4.9}$$

Generally, we have the recurrent formula

$$\Phi_{\ell,j} = A^{+}_{\ell}A^{+}_{\ell-1}\dots A^{+}_{j+1}\Phi_{j,j}. \tag{4.10}$$

Thus we obtain for $N = 1 = m$, i.e. $\ell = j = 1$ from (4.9)

$$\Phi_{1,1} \sim \mathrm{sech}\, by, \tag{4.11}$$

but it corresponds to the SSQM relation (2.10) with ψ_0^- from (3.11).

Really, the potential $V_+ = v_1^2 + v_{1x}$ has a zero-energy bound state whose eigenfunction is

$$\psi_0^+ \sim \mathrm{sech}\,\frac{x}{L\sqrt{2}}.$$

For $N = 2$ we have the two solutions corresponding to $\Lambda_1=-b^2$ $\Lambda_2=-4b^2$. Then, from (4.9) it follows

$$\Phi_{2,2} \sim \mathrm{sech}^2\, by, \qquad (4.12a)$$

and from (4.10) and (4.11) we get

$$\Phi_{2,1} = A_2^{+}\Phi_{1,1} \sim \mathrm{th}\, by\, \mathrm{sech}\, by. \qquad (4.12b)$$

So, we obtain from the relations (4.3) and (4.11) the known one-soliton solution of the U(1) NLS eq.

$$\phi_1(x,t) = C e^{i\theta_1}\,\mathrm{sech}\, by, \qquad (4.13)$$

where $|C|^2 = 2b^2$.

For $N = m = 2$ it follows for the soliton solution to the U(2) VNLS the expression:

$$\phi_2(x,t) = \begin{pmatrix} C_1 e^{i\theta_1}\,\mathrm{sh}\, by \\ C_2 e^{i\theta_2} \end{pmatrix} \mathrm{sech}^2\, by, \qquad (4.14)$$

where $|C_1|^2 = |C_2|^2 = 6b^2$.

Analogously for $N = m = 3$ and so on.

The general expression for the symmetric reflectionless potentials $n_N(x,t)$ in (4.4) can be given following Sukumar[16]

$$n_N = -2\frac{d^2}{dx^2}\ln\det D_N, \qquad (4.15)$$

where the elements of the matrix D_N are given by

$$[D_N]_{JK} = \frac{1}{2}(\gamma_K)^{J-1}[e^{\gamma_K x} + (-1)^{J+K}e^{-\gamma_K x}] \qquad (4.16)$$

and the normalised eigenfunctions for the eigenenergy $E_j = -\gamma_j^2 \equiv \equiv \Lambda_j$ may be written in the form

$$\tilde\phi_N(E_j) = \Big[\frac{\gamma_j}{2}\sum_{K\neq j}^{N}|\gamma_K^2 - \gamma_j^2|\Big]^{1/2}[D_N^{-1}]_{jN}, \qquad (4.17)$$

where $j = 1,2\ldots N$.

For $N = 2$ from the relations (4.15-17) it follows

$$D_2 = \begin{pmatrix} \mathrm{ch}\,\gamma_1 x & \mathrm{sh}\,\gamma_2 x \\ \gamma_1\,\mathrm{sh}\,\gamma_1 x & \gamma_2\,\mathrm{ch}\,\gamma_2 x \end{pmatrix}, \qquad (4.18)$$

$$n_2(x) = -2(\gamma_2^2 - \gamma_1^2)\frac{\gamma_2^2\,\mathrm{ch}^2\gamma_1 x + \gamma_1^2\,\mathrm{sh}^2\gamma_2 x}{(\gamma_2\,\mathrm{ch}\,\gamma_2 x\,\mathrm{ch}\,\gamma_1 x - \gamma_1\,\mathrm{sh}\,\gamma_2 x\,\mathrm{sh}\,\gamma_1 x)^2} \qquad (4.19)$$

$$\tilde\phi_2(E_1) = \Big[\frac{\gamma_1}{2}(\gamma_2^2 - \gamma_1^2)\Big]^{1/2}\frac{\mathrm{ch}\,\gamma_2 x}{\det D_2}, \qquad (4.20a)$$

$$\tilde\phi_2(E_2) = \Big[\frac{\gamma_2}{2}(\gamma_2^2 - \gamma_1^2)\Big]^{1/2}\frac{\mathrm{ch}\,\gamma_1 x}{\det D_2}. \qquad (4.20b)$$

Formulae (4.19) and (4.20) coincide with (22) and (25), (26) of ref.[9], when $\kappa = -(\gamma_1 - \gamma_2)$, $\nu = \gamma_1 + \gamma_2$.

We can also see that

$$n_2(x) = -4[\gamma_2\tilde\phi_2^2(E_2) + \gamma_1\tilde\phi_2^2(E_1)], \qquad (4.21)$$

and in particular, if $\gamma_2^2 = 4\gamma_1^2$ the resulting potential is

$$n_2(x) = -6\gamma_1^2\,\mathrm{sech}^2\gamma_1 x, \qquad (4.22)$$

i.e. $n_2(x)$ exactly corresponds to the potential (2.14) for $N = 2$. There exist two bound states at $\Lambda_1 = -\gamma_1^2 = -b^2$, $\Lambda_2 = -4\gamma_1^2 = -4b^2$ and (4.20a,b) correspond to the (4.14).

5. CONCLUSIONS

In this paper after short introduction to the SSQM we applied the methods of SSQM to obtain soliton-like solutions to some nonlinear evolution equations.

The application of SSQM to the vector version of the NLS gives a possibility to investigate soliton sector of certain nonintegrable systems such as Z system and system (3.2).

The symmetric reflectionless potentials are obtained here as linear combinations of the eigenvalue solutions.

This gives the possibility of looking for the physical realization of these new soliton solutions in the whole area, where eqs. (3.1a,b) and (3.2a,b) can be applied.

It should be noted that the symmetric reflectionless SSQM potentials and those obtained via the familiar factorization method naturally coincide up to reparametrization.

Ultimately since above vector solitons are direct descendants of soliton-like solutions with the selfconsistent potentials, they may be utilized in the study of localized excitations in many-layer crystals as well as of power Langmuir turbulence in plasma[11,12].

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Грубы Я., Маханьков В.Г. E2-87-890

О суперсимметричной квантовой механике и нелинейных уравнениях

Обсуждается метод получения суперсимметричных потенциалов в рамках суперсимметричной квантовой механики /ССКМ/ в связи с изучением нелинейных эволюционных уравнений с безотражательными потенциалами. Обсуждается соответствие между новым классом солитонных решений U(n) нелинейного уравнения Шредингера и решений, полученных с помощью методов ССКМ.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1987

Hruby J., Makhankov V.G. E2-87-890

On the SSQM and Nonlinear Equations

The method for obtaining the superpartner potential in the supersymmetric quantum mechanics (SSQM) is discussed in connection with the nonlinear equations and the reflectionless potentials. The correspondence between a new class of the soliton solutions to the U(N) nonlinear Schrödinger equation is obtained via application of SSQM and those known earlier are also discussed.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1987