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THE PROBLEM OF MOMENTS AND THE MEANING OF PERTURBATION EXPANSIONS IN QUANTUM PHYSICS

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1. Introduction

Let us start with the remark that is widely and often successfully used in physical applications methods of perturbation calculus line Rayleigh-Schödinger expansion in quantum mechanics or Feynman graphs technique in quantum field theory have purely formal character if their mathematical structure is considered. These methods allow one to compute the coefficients of the formal Taylor-Maclaurin expansions in the coupling constant g without any investigation (and, in the consequence, any knowledge) of the analytic structure of the expanded quantities as functions of g in the neighbourhood of the point $g=0$. The analytic properties of the expanded functions play the crucial role in the interpretation of the obtained perturbation series. The analyticity of the original function for $g=0$ is necessary to identify it with the sum of its expansion even for convergent series. The case of divergent series, treated as asymptotic ones, is far more complicated. For unique resummation of them the knowledge of the analytic properties of the expanded function is necessary but not enough and the behaviour of the difference between the function and the truncated expansion should be examined. The realisation of such a program in a mathematically rigorous way is very difficult and the methods used are strongly model dependent. The general solution is known only for regular perturbations (exemplified by a two-electron atom) in quantum mechanics described in the framework of the Kato-Rellich [1] theory which cannot be applied to the divergent asymptotic perturbation expansions. In this case, the problem is well understood for some simple models like anharmonic oscillator[2], superrenormalizable euclidean scalar field models $P\varphi_2$, φ^4_3 [3] and partly for renormalizable euclidean scalar field model φ^4_4 [4]. The analytic properties of the energy levels (for anharmonic oscillator) and the Schwinger functions (for field models) are known as well as the asymptotic character of the perturbation expansions. They are the alternating series with factorially growing coefficients, i.e., having the Lipatov form /5/ and may be summed using the Borel or Pade methods with the choice of the resummation method dependent on the model.

For our purposes we propose to understand the meaning of the perturbation expansions in a different way. The basic fact is that the coefficients of such expansions may (at least in principle and without the analysis of technical problems) be computed without detailed investigation of the analytic structure of the expanded function. If they have the Lipatov form we are able to construct explicitly the functions with known analytic properties for which the original series are the asymptotic ones. Due to the ambiguity of the reconstruction procedure (whose origin is fundamental and connected with the existence of the non trivial functions with zero asymptotic expansions in the neighbourhood of $g=0$), the result cannot be identified with the original object without further information about its structure. The last, in any case, cannot be deduced only from the knowledge of the set of values of derivatives (even if we know them all) computed at the point which is not the point of analyticity. This means that the asymptotic perturbation calculus from the beginning does not contain the full information and has to be completed to give a unique result. In the proposed scheme such complementary information is given by the method of reconstruction itself and the uniqueness conditions may be expressed only in terms of the properties of the expansion coefficients.

2. General Formalism

The subject of our analysis is the perturbation expansions whose coefficients have the Lipatov form, i.e., they are elements of the class of equivalence

$$f_n = a^n \Gamma(\mu n+\beta+1)\, F(n) \tag{2.1}$$

with any function $F(n)$ obeying the asymptotic expansion

$$F(n) \underset{n\to\infty}{\sim} \sum_{k=0}^{\infty} \alpha_k n^{-k} \tag{2.2}$$

for $n\to\infty$. We assume that the parameters $a,\mu,\beta,\ \{1,\alpha_k\}_{k=1}^{\infty}$ are real, i.e., we restrict ourselves to the expansions of physical quantities. It is obvious that the Taylor-Maclaurin series constructed with f_n's as its coefficients

$$f(z) = \sum_{n=0}^{\infty} f_n (-z)^n \tag{2.3}$$

is purely formal because it is divergent for all nonzero values of the variable z. The only way to give such a series mathematical meaning (and physical interpretation) is to consider it as asymptotic expansion of $f(z)$ for $z\to 0$ in some subset of the complex z plane and to find $f(z)$ fulfilling this condition. Such a problem is solvable /6/, in a suitable sector $\{z : |\arg z|<\beta,\ 0<|z|<R\}$ for any series of the type (2.3). The choice of the method of reconstruction depends on the properties of the expansion coefficients considered. In the case of (2.1) the methods of the theory of moments seem to be the most useful tool to solve the problem.

As the first step let us define the quantities

$$\tilde f_n = \frac{f_n}{F(n)} = a^n \Gamma(\mu n+\beta+1) \tag{2.4}$$

which for all natural n and real a,μ,β are the moments of the positive measure on $\mathbb{R}$. The same holds for $\tilde f_{k+j}$ with fixed k and $j=0,1,2,\ldots$, if $a>0$ and fixed even k and $j=0,1,2,\ldots$, if $a<0$. For positive a we have

$$\tilde f_{k+j} = \frac{1}{\mu}\int_0^{\infty} u^{j+k-1}\left(\frac{u}{a}\right)^{\frac{\beta+1}{\mu}} \exp\left(-\left(\frac{u}{a}\right)^{\frac{1}{\mu}}\right) du \tag{2.5}$$

i.e., $\tilde f_{k+j} = \int_0^{\infty} u^j\, d\tau_k(u)$, $d\tau_k \geq 0$ for $u\in[0,\infty)$ while for negative a

$$\tilde f_{k+j} = -\frac{1}{\mu}\int_{\infty}^{0} u^{j+k-1}\left(\frac{u}{a}\right)^{\frac{\beta+1}{\mu}} \exp\left(-\left(\frac{u}{a}\right)^{\frac{1}{\mu}}\right) du \tag{2.6}$$

which gives $\tilde f_{2k+j} = \int_{-\infty}^{\infty} u^j\, \theta(-u)\, d\tau_{2k}(u)$, $\theta(-u)\, d\tau_{2k} \geq 0$ for $u\in(-\infty,\infty)$. According to Hamburger's theorem, in the moment theory /7/ these conditions are sufficient and necessary for the positivity of all quadratic forms $\sum_{p,s=0}^{m} \tilde f_{k+p+s}\, x_p x_s$ with the only restriction for k to be even for negative a. This means that the determinants of the $\tilde{\mathcal{D}}$ for m matrices

$$\tilde{D}(k,m) = \det \begin{vmatrix} \tilde{f}_k & \tilde{f}_{k+1} & \dots & \tilde{f}_{k+m} \\ \tilde{f}_{k+1} & \tilde{f}_{k+2} & \dots & \tilde{f}_{k+m+1} \\ & & \dots & \\ \tilde{f}_{k+m} & \tilde{f}_{k+m+1} & \dots & \tilde{f}_{k+2m} \end{vmatrix} \tag{2.7}$$

fulfill the inequalities

$$\tilde{D}(k+j,m) > 0 \qquad j,m = 0,1,2,\dots, \quad a > 0 \tag{2.8'}$$

$$\tilde{D}(k+j,m) > 0 \qquad j,m = 0,1,2,\dots; \quad k \text{ even}, \quad a < 0. \tag{2.8''}$$

With the leading asymptotical term considered, let us take the full form of (2.1). Instead of (2.7) we have the determinants

$$D(k,m) = \det \begin{vmatrix} f_k & f_{k+1} & \dots & f_{k+m} \\ f_{k+1} & f_{k+2} & \dots & f_{k+m+1} \\ & & \dots & \\ f_{k+m} & f_{k+m+1} & & f_{k+2m} \end{vmatrix} \tag{2.9}$$

corresponding to the quadratic forms $\sum_{p,s=0}^{m} f_{k+p+s} x_p x_s$. Let us choose k big enough to replace the exact formula (2.1) by its asymptotics (2.2). Since the asymptotic expansions of $F(k)$ and $F(k+m)$ for $k \to \infty$ are the same for any function and any m, we have with the accuracy to the zero asymptotic expansion

$$D(k,m) \underset{k\to\infty}{\sim} \tilde{D}(k,m) \Big[\sum_{j=0}^{\infty} \alpha_j k^{-j} \Big]^{m+1} \tag{2.10}$$

i.e., the determinants $D(k,m)$ for k sufficiently large fulfill the same positivity conditions as $\tilde{D}(k,m)$. Moreover, for $a > 0$, the determinants $D(k+1,m)$ are also positive for large k . Indeed, we have

$$D(k+1,m) \underset{k\to\infty}{\sim} a\, \tilde{D}'(k,m) \Big[\sum_{j=0}^{\infty} \alpha_j k^{-j} \Big]^{m+1}, \tag{2.11}$$

where $\tilde{D}'(k,m)$ are the determinants of the type (2.7) computed for $\tilde{f}'_j = \tilde{f}_{j+1}$ which does not change the properties (2.8'). Such a positivity condition is not true for $a < 0$, which may immediately be seen from (2.11), and it is the source of differences between two cases considered. The positivity of the determinants $D(k,m)$ and $D(k+1,m)$ for $a > 0$ is a sufficient and necessary condition for the sequences $\{f_{k+j}\}_{j=0}^{\infty}$ to be Stieltjes moments (i.e. moments on the positive semiaxis) while the positivity of the determinants $D(2k,m)$ for $a < 0$ enables one to interprete the sequences $\{f_{2k+j}\}_{j=0}^{\infty}$ as Hamburger's moments (i.e. moments on the whole real axis). These statements are equivalent to the existence of the positive measures on $\mathbb{R}$ such that

$$f_{k+j} = \int_0^\infty u^j \theta(u)\, d\sigma_k(u) \qquad a > 0 \tag{2.12'}$$

$$f_{2k+j} = \int_{-\infty}^{\infty} u^j\, d\sigma_{2k}(u) \qquad a < 0 \tag{2.12''}$$

and the series

$$f^{[k]}(z) = \sum_{n=k}^{\infty} f_n (-z)^n = (-z)^k \sum_{j=0}^{\infty} f_{k+j} (-z)^j \tag{2.13}$$

are the series of Stieltjes or Hamburger, respectively. The properties of these classes of divergent series are well known /7,8/ because they play the crucial role in the theory of moments and in the theory of Pade approximations. The fundamental result, namely the theorem of Nevanlinna and Hamburger (N.-H.), has different formulations for the series of Stieltjes and Hamburger; so we have to analyse them separately. For the series of Stieltjes (2.12'), i.e. for $a > 0$ and the expansion (2.3) alternating for $n \geq k$ we obtain

$$f^{[k]}(z) = (-z)^k \int_0^\infty \frac{\theta(u)\, d\sigma_k(u)}{1 + uz} \tag{2.14}$$

and the function $(-z)^{-k} f^{[k]}(z)$ real symmetric and analytic in $S = \{ z : |\arg z| < \pi - \delta,\ \delta > 0 \}$ admitting the series in (2.13) as its asymptotic expansion uniformly in S . For the series of Hamburger (2.12''), i.e. for $a < 0$ and (2.3) nonalternating for

$m > k$, the analogous formula is

$$f^{[2k]}(z) = (-z)^{2k}\int_0^\infty \frac{d\tau_{2k}(u)}{1+uz} \qquad (2.15)$$

and $(-z)^{2k} f^{[2k]}(z)$ real symmetric and analytic in $S=\{z:\ \varepsilon < |\arg z| < \pi-\varepsilon,\ \varepsilon>0\}$ with (2.13) uniformly asymptotic in S. Using these results we may come to the conclusion that for formal perturbation expansion (2.3) with the coefficients (2.1) obeying the asymptotics (2.2) we have found the function with known analytic properties and admitting the representation

$$f(z) = \sum_{n=0}^{k-1} f_n(-z)^n + (-z)^k\int_0^\infty \frac{d\tau_k(u)}{1+uz}, \qquad (2.16)$$

where $d\tau_k(u)$ is the solution of the moments problem

$$f_{k+j} = \int_0^\infty u^j\, d\tau_k(u) \qquad j=0,1,2,\dots, \qquad (2.17)$$

with the given series as its asymptotic expansion

$$f(z) \underset{S\ni z\to 0}{\sim} \sum_{n=0}^\infty f_n(-z)^n \qquad (2.18)$$

uniformly in the domain of definiteness.

3. Applications

With the basic formula (2.16) obtained, we should analyse its applicability in practical calculations. The problem has two different aspects. The first is the direct computation of the measure $d\tau_k(u)$ in (2.17) from the sequence $\{f_{k+j}\}_{j=0}^\infty$ given and the second is the application of the standard resummation methods, like Padé or Borel methods, to obtain unknown function $f(z)$. Owing to the differences between the alternating and nonalternating series, we shall consider them separately.

3.1. The alternating series

Let us begin with the simplest, exactly solvable, example of the one-dimensional integral

$$I(g) = \int_{-\infty}^{\infty} dx\, e^{-x^2-gx^4} \qquad (3.1)$$

often called the zero dimensional analog of the φ^4 model. $I(g)$ defines the parabolic cylinder function $D_{-\frac12}$ equivalent to the McDonald function $K_{\frac14}$

$$I(g) = \frac{1}{2\sqrt{g}}\exp\left(\frac{1}{8g}\right)K_{\frac14}\left(\frac{1}{8g}\right). \qquad (3.2)$$

The perturbation expansion of (3.1) is given by

$$I(g) = \sum_{k=0}^\infty \frac{\Gamma(2k+\frac12)}{\Gamma(k+1)}(-g)^k \qquad (3.3)$$

and is formal, everywhere divergent alternating series whose coefficients have the Lipatov form $\frac{\Gamma(2k+1)}{\Gamma(k+1)} \sim \frac{1}{\sqrt{\pi}}4^k\Gamma(k)$ for $k\to\infty$. The coefficients in (3.3) are the Stieltjes moments for all k because

$$\frac{\Gamma(2k+\frac12)}{\Gamma(k+1)} = \frac{1}{\sqrt{2\pi}}4^k\frac{\Gamma(k+\frac14)\Gamma(k+\frac34)}{\Gamma(k+1)} = \int_0^\infty x^k\left[\frac{1}{2\pi\sqrt{2}}x^{-\frac12}\exp\left(-\frac{x}{8}\right)K_{\frac14}\left(\frac{x}{8}\right)\right]dx \qquad (3.4)$$

which, according to (2.16), means that

$$I(g) = \frac{1}{2\pi\sqrt2}\int_0^\infty \frac{x^{-\frac12}\exp(-\frac{x}{8})K_{\frac14}(\frac{x}{8})\,dx}{1+xg} = \frac{1}{2\sqrt g}\exp\left(\frac{1}{8g}\right)K_{\frac14}\left(\frac{1}{8g}\right) \qquad (3.5)$$

with (3.3) as an asymptotic expansion /9/. The above construction is unique, which is guaranteed by the criterion of Carleman for the uniqueness of the solution of the Stieltjes moment problem /7/. The sufficient condition for the uniqueness is the divergency of the series $\sum_{n=0}^\infty (f_n)^{-\frac{1}{2n}}$ which is, roughly speaking, equivalent to the bound $|f_n| \le C\,\Gamma(2n+\frac12)$ which is fulfilled for the example considered.

An analogous procedure would be applied (using methods of Mellin's transform theory) for physically realistic models if the sequence $\{f_{k+j}\}_{j=0}^\infty$ were exactly known /10/. Unfortunately, in practice, only the leading asymptotic terms (2.4) are known which give the measure $d\tau_k(u)$ with the accuracy to the leading asymptotical term for $u\to\infty$

$$(-z)^k\int_0^\infty \frac{d\tau_k(u)}{1+uz} = \frac{(-z)^k}{\mu}\int_0^\infty \frac{u^{k-1}\left(\frac{u}{a}\right)^{\frac{\beta+1}{\mu}}\exp\left(-\left(\frac{u}{a}\right)^{\frac1\mu}\right)du}{1+uz} \qquad (3.6)$$

Such an approximation, formally identical with the generalized Borel sum of (2.13) with the coefficients (2.4), in general does not give satisfactory results /11/ but for some simple models, like anharmonic oscillator gx^{2N}, $N=1,3$ the 5% accuracy, in comparison with the exact results may be achieved /12/. Better occuracy, about 1%, can be obtained (within the same approximation for the coefficients used) if the Borel or Pade methods are applied to sum the sequence (2.13). The Borel summability of (2.13) to the integral in (2.16) may be proved /10/ because the N.-H. theorem guarantees the assumptions of the Watson theorem to be fulfilled if the coefficients in (2.13) satisfy the bound $|f_n| \le c^n (2n)!$. The uniqueness criterion of Carleman gives a little bit stronger bound: $|f_n| \le c^n \Gamma(2n+\frac{1}{2})$. Among the models satisfying these conditions the best accuracy was obtained in the framework of the Padé-Borel method applied to an anharmonic oscillator, funnel-like potential /13/ and Gell-Mann-Low function and critical exponents in the scalar field models /11/. The Padé summability of (2.13) to the integral in (2.16) follows directly the general theorems of the Padé approximant theory /8/ similar to the N.-H. theorem. Moreover, the approximants $[N-1, N]$ and $[N,N]$ give for $z > 0$ the lower and upper bounds for the integral in (2.16) with the accuracy growing with N. The uniqueness condition demands the divergency of the series $\sum_{n=0}^{\infty} (f_n)^{-\frac{1}{2n+1}}$ and gives $|f_n| \le c^n \Gamma(2n+1)$. The method, applied to such quantum mechanical models like anharmonic oscillator and funnel-like potential gives results more accurate then obtained by the Borel method /14/. It is also understood why the method does not work for an anharmonic oscillator gx^8 /15/ which breaks the uniqueness condition. The models of this type need further development of the formalism based on the solutions of the indefinite problem of moments /7/.

3.2. The nonalternating series

The simplest example, where the nonalternating perturbation expansion appears, is the integral

$$Y(g^2) = \int_{-\infty}^{\infty} dx \, e^{-x^2(1-gx)^2} \tag{3.7}$$

whose exact value is

$$Y(g^2) = \frac{\pi}{\sqrt{16g^2}} \exp\left(-\frac{1}{32g^2}\right)\left(I_{\frac{1}{4}}\left(\frac{1}{32g^2}\right) + I_{-\frac{1}{4}}\left(\frac{1}{32g^2}\right)\right) \tag{3.8}$$

and formal perturbation expansion

$$Y(g^2) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(2k+\frac{1}{2})}{\Gamma(k+1)} (-4g^2)^k \tag{3.9}$$

everywhere divergent and nonalternating, obeying the Lipatov form. The coefficients in (3.9) are the Hamburger moments for all k

$$\frac{(-1)^k \Gamma(2k+\frac{1}{2})}{\Gamma(k+1)} = \int_{\infty}^{\infty} x^k \left[\frac{1}{\pi} \left(-\frac{x}{8}\right)^{-\frac{1}{2}} \exp\left(\frac{x}{8}\right) K_{\frac{1}{4}}\left(-\frac{x}{8}\right) \theta(-x) \right] dx \tag{3.10}$$

and $Y(z^2)$ in upper and lower half-planes is given by the Hilbert transform of the measure in (3.10)

$$Y(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\left(-\frac{x}{8}\right)^{-\frac{1}{2}} \exp\left(\frac{x}{8}\right) K_{\frac{1}{4}}\left(-\frac{x}{8}\right) \theta(-x)\, dx}{1 + xz} . \tag{3.11}$$

To give it the meaning for $z \geqslant 0$, i.e., $z = g^2$ we have to calculate its principal value

$$Y(g^2) = \lim_{\varepsilon \to 0} \frac{1}{2} \left(Y(g^2 + i\varepsilon) + Y(g^2 - i\varepsilon) \right) = \tag{3.12}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2} \left(\frac{1}{\sqrt{-4g^2 - i\varepsilon}} e^{-\frac{1}{32g^2 + i\varepsilon}} K_{\frac{1}{4}}\left(-\frac{1}{32g^2 + i\varepsilon}\right) + \frac{1}{\sqrt{-4g^2 + i\varepsilon}} e^{-\frac{1}{32g^2 - i\varepsilon}} K_{\frac{1}{4}}\left(-\frac{1}{32g^2 - i\varepsilon}\right) \right) =$$

$$= \frac{\pi}{\sqrt{16g^2}} e^{-\frac{1}{32g^2}} \left(I_{\frac{1}{4}}\left(\frac{1}{32g^2}\right) + I_{-\frac{1}{4}}\left(\frac{1}{32g^2}\right)\right),$$

where we used the transformation properties of Mac Donald's functions with respect to the analytic continuation

$$K_{\frac{1}{4}}(-z) = \frac{\pi}{2}\left(I_{-\frac{1}{4}}(z) - I_{\frac{1}{4}}(z)\right) - \frac{i\pi}{2}\left(I_{-\frac{1}{4}}(z) + I_{\frac{1}{4}}(z)\right) \tag{3.13}$$

$$K_{\frac{1}{4}}(z^*) = K_{\frac{1}{4}}^*(z)$$

and the reality of the functions $I_{\frac{1}{4}}(z)$ for real argument. For the Hamburger problem of moments Carleman's uniqueness condition is satisfied if the series $\sum_{n=0}^{\infty} (f_n)^{-\frac{1}{n}}$ is divergent. Since it is fulfilled for the case considered

$$\left(\frac{\Gamma(2k+\frac{1}{2})}{\Gamma(k+1)}\right)^{-\frac{1}{k}} > \frac{c}{k+\frac{1}{2}} \qquad (3.14)$$

the construction of the function $f(g^2)$ is unique.

The analysis of the nonalternating perturbation series is far more difficult then the previous analysis of the alternating ones. No universal method to find Hamburger measure in (2.15) is known. Also the Borel method is not applicable /11/ because the Watson theorem is not valid. The only possibility left is Padé summation. The suitable version of the N.-H. theorem gives that the sequence of the Padé approximations of (2.13) $[N-1,N]$, $N=1,2,3,$ contains the subsequence $[\tilde{N}-1,\tilde{N}]$, $\tilde{N}=n_1,n_2,n_3,\dots$, converging in $D(\Lambda)\ \{z\ |z|<R,\ \mathrm{Im}\, z>0\}$ to the real symmetric, analytic in $D(\Lambda)$ function $f^{(k)}(z)$

$$\lim_{\tilde{N}\to\infty}[\tilde{N}-1,\tilde{N}]\,(z)^k f^{(k)}(z) = (-z)^k\int_{-\infty}^{\infty}\frac{d\mu_k(u)}{1+uz} \qquad (3.15)$$

admitting (2.13) as its asymptotic expansion for $z\to 0$ in $S=\{z:\ \varepsilon<|\arg z|<\pi-\varepsilon\}$

$$f^{(k)}(z) = \int_{-\infty}^{\infty}\frac{d\mu_k(u)}{1+uz} \underset{|z|\to 0}{\sim} \sum_{j=0}^{\infty} f_{k+j}(-z)^j. \qquad (3.16)$$

To our knowledge, the Padé summation was not applied in the analysis of the nonalternating series. Except the difficulties with the computation of the coefficients in (2.13), there are serious restrictions given by the Carleman criterion. The sufficient uniqueness condition is $|f_n| \le C^n\Gamma(n+\frac{1}{2})$ and is not fulfilled even for such simple model like quantum mechanical double well where coefficients of the perturbation expansion of the ground state energy have the asymptotical form /16/

$$E_j^0 \underset{j\to\infty}{\sim} -\frac{1}{\pi}\,3^{j+1}\Gamma(j+1)\left(1-\frac{103}{36j}+O\left(\frac{1}{j^2}\right)\right). \qquad (3.17)$$

Indeed, it may be proved /1/ that the ground state energy in this model is unstable, i.e. there exist two eigenvalues of the full Hamiltonian, differing from the function $\exp(-\alpha/g^2)$ and obeying the same asymptotics for $g\to 0$. For such problems and the relevant ones, appearing in the non-Abelian gauge field theories further development of the formalism (especially the methods of the indefinite moment problem) is necessary.

4. Conclusions

Up to now we have considered the problem of the interpretation of the given perturbation series and we have not analysed the question: is the given series asymptotic expansion of the function for which we have started to obtain it. In mathematics the answer is known in some special cases like the functions analytic in $S=\{z: |z|<R,\ \alpha<\arg z<\beta\}$ and obeying any asymptotic expansion for $S\ni z\to 0$. For these functions the existence of the series $\sum_{n=0}^{\infty} f_n z^n$ such that $f(z)\sim\sum_{n=0}^{\infty} f_n z^n$, $S\ni z\to 0$ guarantees that $f_n=\frac{1}{n!}\lim_{S\ni z\to 0} f^{(n)}(z)$, i.e. the asymptotic character of the Taylor-Maclaurin expansion /6/. In physical applications the problem is fully understood only in the constructive quantum field theory, for the models mentioned in the introduction. It is obvious that if the answer is negative the perturbation series is not connected with the original function and useless as a computational method. But for a positive answer (suggested by successful applications) the perturbation expansions have to be understood and summed properly especially when the theoretical calculations and experimental results are very exact - for example in QED /17/. For such problems the interpretation of the perturbative results as the approximation by polynomials is false and taking into consideration the "tail" of perturbation series is necessary.

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Хожеля А. E2-87-888
Классическая проблема моментов
и значение пертурбационных разложений в квантовой физике

При использовании методов классической теории моментов рассмотрены вопросы, значения и интерпретации расходящихся рядов теории возмущений. Доказано, что ряды с коэффициентами, асимптотически обладающими факториальным ростом, имеют строго определенный математический смысл, и можно их суммировать методами Бореля и Паде /в случае знакопеременных рядов/ и Паде /в случае знакопостоянных рядов/. Рассмотрены вопросы однозначности методов реконструкции и даны условия на коэффициенты рядов, для которых использованные методы дают однозначный ответ.

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Horzela A. E2-87-888
The Problem of Moments and the Meaning
of Perturbation Expansions in Quantum Physics

The methods of the classical moment theory are used to give the interpretation of the divergent perturbation expansions. The expansions with factorially growing coefficients are proved to be mathematically well defined objects which may be resummed using the Borel and Pade methods. The problem of uniqueness of the resummation is analysed and some conditions of the uniqueness are obtained.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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