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исследований
дубна

E2-87-883

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DEGENERATE GAUGE CONDITIONS, GENERALIZED GRIBOV'S AMBIGUITY AND BRST SYMMETRY

Submitted to "Physics Letters"

1987

Introduction

In a constrained mechanical system, some of the canonical variables cannot be varied independently of the others because of the existence of constraint functions $\phi_\alpha(q,p)$ reducing the phase space available. These constraint functions have been classified by Dirac[1] in two classes: To the first class the constraints belong that have weakly[1] vanishing Poisson brackets with the Hamiltonian as well as among themselves; to the second class all the others.

In a similar fashion, it is useful to classify gauge fixing conditions $\chi_\alpha(q,p)$ in two groups: Degenerate gauges, for which the Poisson brackets with the constraints $\{\chi_\alpha,\phi_\beta\}$ weakly vanish, preserving the I-class nature of the system; and non-degenerate gauges, for which the system is reduced to a II-class one, because $\{\chi_\alpha,\phi_\beta\}$ is not weakly equal to zero.

In other words, a gauge is degenerate provided the gauge freedom is restricted but not completely suppressed by the gauge fixing; it is non-degenerate otherwise.

Accordingly, there are two methods to deal with a constrained dynamical system: Dirac approach, that uses a non-degenerate gauge fixing to define Dirac brackets[2] that are consistent with the reduced phase space; and Batalin-Fradkin-Vilkovisky (BFV) approach[3], that instead does not require a II-class system because the residual gauge symmetry, left over by the degenerate gauge fixing, is made into a global symmetry: The BRST symmetry[4].

Whether a gauge is degenerate or not -and hence whether the BFV approach is appropriate - depends on the gauge fixing as well as on the boundary conditions for the ghost variables (or for the gauge functions, that is equivalent). For instance, the presence of a residual gauge freedom

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x) \tag{1}$$

restricted by the condition

$$\Box\alpha(x) = 0 \qquad (2)$$

is a familiar aspect of classical electrodynamics in the Lorentz gauge ($\partial\cdot A = 0$). However, such a residual gauge symmetry can be eliminated by choosing an appropriate boundary condition, i.e. that on the boundary all gauge transformations vanish. This defines a boundary problem for Eq.(2) that allows only the trivial solution $\alpha(x) = 0$ everywhere. Therefore, the Lorentz gauge is non-degenerate for these boundary conditions, whereas it would be degenerate for other choices.

It is interesting to examine what happens to the BRST symmetry when the boundary conditions reduce the degeneracy of a gauge. In electrodynamics the BRST symmetry is obtained by replacing the gauge function $\alpha(x)$ by $\omega\eta(x)$, where ω is a constant and $\eta(x)$ a new dynamical field: The ghost. Eq.(1) becomes the system (Feynman gauge)

$$\delta A_\mu(x) = \omega\partial_\mu\eta(x) \;;\; \delta\eta(x) = \omega\partial_\mu A^\mu(x) \qquad (3)$$

that defines the BRST transformation for the gauge fixed Lagrangian

$$L = -(1/4)F_{\mu\nu}F^{\mu\nu} - (1/2)(\partial\cdot A)^2 + (1/2)\partial_\mu\eta\partial^\mu\eta. \qquad (4)$$

The equation of motion for the ghost coincides with Eq.(2), the condition on the residual gauge symmetry. As expected, BRST transformations are exactly this residual symmetry of the degenerate Lorentz gauge[5]. Boundary conditions on the gauge function are now conditions on the ghost field. If the standard boundary conditions (vanishing of fields) were applied, the on-shell ghost field would vanish everywhere and there would be no BRST symmetry. This is another expected feature: the reduction of degeneracy in a gauge condition may break down BRST symmetry.

One is therefore lead either to Dirac approach - in the non-degenerate gauge defined by the standard boundary conditions above - or to choose boundary conditions appropriate for a dynamical field instead, namely incoming and outgoing plane waves. In this case, the boundary problem of Eq.(2) admits non-trivial solutions, e.g. progressive plane waves, and the gauge is degenerate. The BFV approach can thus be used.

Boundary Conditions

The discussion in the Introduction suggests that no a priori boundary condition should be enforced on the ghost fields. In particular, a vanishing ghost field on the boundary may endanger BRST symmetry. This feature is not a peculiarity of electrodynamics but rather a general property of the BFV approach. In fact, from the BFV action[3]

$$S_{BFV} = \int_1^2 dt\,[\dot{q}_i p^i + \dot{\lambda}_\alpha \pi^\alpha + \dot{\eta}_\alpha \mathbb{P}^\alpha + \dot{\bar{\eta}}_\alpha \bar{\mathbb{P}}^\alpha - H_0 + \{\psi, \Omega\}] \qquad (5)$$

(where

$$\Omega = \pi_\alpha \bar{\mathbb{P}}^\alpha + \phi_\alpha \eta^\alpha - (1/2)\eta^b\eta^c c^a_{bc}\mathbb{P}_a \qquad (6)$$

is the BRST charge, c^a_{bc} the structure constant of the group generated by the constraints ϕ_α and π_α, and ψ is an arbitrary function containing the gauge fixing condition) the ghost equations of motion are in the form (see Appendix)

$$\ddot{\eta}_\alpha + F(t)\dot{\eta}_\alpha + G(t)\eta_\alpha = 0, \qquad (7)$$

where F and G are two functions containing the other canonical variables. A complete discussion of Eq.(7) requires a specific system and a definite gauge choice. Nevertheless, Eq.(7), when re-written as

$$\ddot{V} = f(t)V, \qquad (8)$$

can be solved perturbatively (the non-perturbative case is considered below) in the couplings g_j between the ghosts and the other variables. For

$$V = V^{(0)} + g_j V^{(1)} + \dots \qquad (9)$$

and imaginary time, Eq.(8) leads to a Direchlet problem

$$\ddot{V}^{(0)} = 0 \;;\; \ddot{V}^{(1)} = g_j f^j(t) V^{(0)}; \;\dots \qquad (10)$$
$$V^{(i)}(1) = V^{(i)}(2) = 0$$

that has only vanishing solutions.

The case of an infinite number of degrees of freedom is similar, with Eq.(7) replaced by an hyperbolic second-order differential equation.

Since Eq.(7) is by definition also the Poisson brackets between the gauge conditions and the constraints, the boundary condition requiring vanishing ghost fields at the end points leads to a non-degenerate gauge, in contradiction with the BFV approach.

It is now possible to consider the boundary problem in a quite general manner. The boundary condition on the ghost variables must be such as to insure the BRST invariance of the BFV action (5). Since $\{H_0,\Omega\}$ and $\{\{\Psi,\Omega\},\Omega\}$ weakly vanish, the only change in the action caused, by a canonical transformation generated by Ω is

$$(\partial\Omega/\partial\sigma_k)\,\sigma_k - \Omega\,\Big|_1^2 = 0 \tag{11}$$

σ_k's being the momenta.

Ordinarily (see, e.g., the review in ref. [6]), the vanishing of (11) is attained by requiring that both terms in the equation be independently equal to zero. By inspection, there are then three sets of boundary conditions that make (11) vanish

$$\begin{aligned} &1.\quad \pi_\alpha\Big|_1^2 = \Phi_\alpha\Big|_1^2 = \mathcal{P}_\alpha\Big|_1^1 = 0 \\ &2.\quad \eta_\alpha\Big|_1^2 = \bar{\mathcal{P}}_\alpha\Big|_1^2 = 0 \\ &3.\quad \eta_\alpha\Big|_1^1 = \pi_\alpha\Big|_1^2 = 0. \end{aligned} \tag{12}$$

However, the previous discussion suggests that a further criterion is the degeneracy of the gauge, that must be preserved. Conditions 2. and 3. do not satisfy this criterion because they require vanishing ghost fields. Set 1 also is not completely acceptable because of the vanishing momenta.

Therefore, it appears that the independent vanishing of Ω and $\partial\Omega/\partial\sigma_k$ is too strong a condition to be enforced. Such a condition is sufficient only in the trivial sense that it produces, by the conservation law, a BRST charge that is, at any time, strongly zero on-shell. On the other hand, that such a condition is not necessary can be seen already in the simple case of a free relativistic particle, where the vanishing of Eq.(11) is a consequence of the equations of motion solely (ref. [7] notwithstanding). Conditions (12) are also particularly worrisome on physical ground. The initial and final states are physical because of the presence of ghost fields and the resultant quartet decoupling[8] of all un-physical degrees of freedom. Imposing the vanishing of these fields to make a state physical is therefore wrong.

Unfortunately, one is thus left without a comprehensive prescription on the boundary conditions. This is a rather unsatisfactory situation: the vanishing of (11) has to be verified in each new system; if necessary, boundary conditions can be imposed, but being careful not to spoil BRST symmetry.

Gribov's Ambiguity

An interesting by-product of the preceding discussion is a new angle on a problem of Yang-Mills theories: Gribov's ambiguity. Gribov[9] pointed out that in the non-Abelian case the boundary conditions considered in the Introduction for the Abelian case, namely the vanishing of the gauge function $\alpha(x)$ at infinity, may not be sufficient to make the Lorentz gauge (as well as the Coulomb one) non-degenerate.

In fact, the non-Abelian nature of the fields introduces in Eq. (2) an additional term:

$$\Box\alpha(x) + g A_\mu(x)\partial^\mu\alpha(x) = 0. \tag{13}$$

For g large, Eq.(13) admits a non-trivial solution vanishing at infinity. The consequent ambiguity in the definition of the potentials makes the Lorentz gauge degenerate. Although the non-perturbative nature of this ambiguity has made it possible to dodge it in all current application, in principle, in the non-Abelian case, the Lorentz gauge (and Coulomb one) are always degenerate, and Dirac approach should not be used.

Moreover, a similar ambiguity could exist for Eq.(7) as well. This suggests a generalized Gribov-like ambiguity for all gauge theories with a non-Abelian algebra.

The non-perturbative degeneracy, originated by these special solutions, cannot be used to define BRST symmetry because this requires the existence of infinitesimal transformations. Therefore, the perturbative result of Eq.(10) still holds.

At the same time, the previous discussion makes clear that, in the BFV approach, these Gribov-like ambiguities, as well as Gribov's original ambiguity for Yang-Mills theories, are just a particular BRST transformation and no additional care is required in dealing with them.

Appendix

In this Appendix, Eq.(7) is obtained from the action (5). In the BFV method, the gauge fixing is in the generally covariant form

$$\dot{\lambda}_\alpha + \chi_\alpha(q,p,\lambda) = 0. \tag{14}$$

In order to integrate out the ghost momenta, it is convenient to choose

$$\Psi = \bar{\eta}_\alpha\chi^\alpha + \mathcal{P}_\alpha\lambda^\alpha \tag{15}$$

to make the functional integral

$$Z = \int d\mu \, e^{-S_{BFV}} \tag{16}$$

Gaussian in $\mathbb{P}$ and $\bar{\mathbb{P}}$.

After integration in these momenta,

$$S'_{ghost} = \int_1^2 dt \, \dot{\bar{\eta}}_\alpha \left[\dot{\eta}^\alpha + C^\alpha_{rt} \lambda^r \eta^t + \{ \chi^\alpha, \phi^\beta \} \eta_\beta \right] . \tag{17}$$

Eq.(7) is obtained by using the additional condition $\bar{\eta}_\alpha |_1^2 = 0$, that is BRST invariant.

Note that Eq.(17) can also be written as

$$S'_{ghost} = \int_1^2 dt \, \dot{\bar{\eta}}_\alpha \delta_\eta (\dot{\lambda}_\alpha + \chi_\alpha) \tag{18}$$

because $\delta_\eta \lambda_\alpha = \dot{\eta}_\alpha + C^\alpha_{t d} \lambda^t \eta^d$, by definition.

In this form,

$$\delta_\eta (\dot{\lambda}_\alpha + \chi_\alpha) = \{ \dot{\lambda}_\alpha + \chi_\alpha , \phi_\beta \} = 0 \tag{19}$$

the equations of motion for the ghosts are identical to the condition defining a degenerate gauge.

Only systems with a closed algebra have been considered. The condition $\{ H_c, \phi_\alpha \} = 0$, while not necessary, has been used to simplify the notation.

I would like to thank A.T. Filippov for helpful conversations and the Laboratory of Theoretical Physics of the JINR for kind hospitality.

References

1. P.A.M. Dirac, Proc. Royal Soc. 246 (1958) 326; Lectures on Quantum Mechanics, Yeshiva University (Academic Press, N.Y., 1967).
2. A.J. Hanson, T.Regge and C.Teitelboim, Constrained Hamiltonian Systems (Accademia Nazionale dei Lincei, Rome 1976).

3. E.S. Fradkin and G.A. Vilkovisky, Phys.Lett. 55B (1975) 224; I.A. Batalin and G.A. Vilkovisky, Phys.Lett. 69B (1977) 309.
4. C. Becchi, A. Rouet and R. Stora, Phys.Lett. 52B (1974) 344; I.V. Tyutin, Lebedev Institute preprint 39 (1975).
5. R.A. Brandt, Nucl.Phys. B116 (1976) 413.
6. M. Henneaux, Phys.Rep. 126 (1985) 1.
7. M. Henneaux and C. Teitelboim, Ann. of Phys. 143 (1982) 127.
8. T. Kugo and I. Ojima, Supp. Prog.Theor.Phys. 66 (1979).
9. V.N. Gribov, Nucl.Phys. B139 (1978) 1.

Received by Publishing Department on December 17, 1987.

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Фаббрикези М.Е. E2-87-883
Вырожденные калибровочные условия, обобщенные неоднозначности Грибова и БРСТ-симметрия

Рассматривается БФВ-БРСТ подход к калибровочным теориям. Утверждается, что обычно используемые БРСТ-инвариантные граничные условия не обладают необходимым вырождением при фиксации калибровки. В соответствии с этим, предлагается существование обобщенных неоднозначностей Грибова. Тем не менее показано, что эти неоднозначности представляют собой частный случай БРСТ-преобразований.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1987

Fabbrichesi M.E. E2-87-883
Degenerate Gauge Conditions, Generalized Gribov's Ambiguity and BRST Symmetry

The BFS-BRST approach to gauge theories is considered. It is argued that the BRST-invariant boundary conditions ordinarily used do not maintain the necessary degeneracy in the gauge fixing. As a by-product of this discussion, the existence of a generalized Gribov-like ambiguity is suggested. This ambiguity is however shown to be just a particular BRST transformation.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1987