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ELIMINATION OF GAUGE FREEDOM IN SINGULAR THEORIES

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1. INTRODUCTION

The field theory description of realistic models of elementary particle interactions is mainly based on singular Lagrangians. Usually, singularity of the Lagrangian is thought about by invariance of action with respect to transformations of field functions containing arbitrary coordinate and time functions. These transformations, defined in tangent bundle, are often called the gauge transforms and the corresponding theories are the gauge theories.

A general method to obtain Hamiltonian dynamics for singular Lagrangians was described by Dirac [1].

The motion equation for an arbitrary dynamic variable g has the following form in Dirac's approach [1]:

$$\dot{g} = \{g, H_T\}, \quad H_T = H_c + u^1_k \phi^1_k, \quad k = 1, \ldots, m. \tag{1}$$

Here H_c is the canonical Hamiltonian, u^1_k are the arbitrary multipliers, ϕ^1_k are the primary constraints of the 1st type. Here and on the repeating indices mean summation.

The function H_T is called the total Hamiltonian. Note that in this paper we are interested in gauge degrees of freedom and in problems related to gauge fixation; so, we shall assume that there are only constraints of the 1st type in the theory. This assumption simplifies some formulae but the interpretation remains general.

Primary constraints of the 1st type produce gauge transformations in the phase space. Secondary constraints of the 1st type can also produce gauge transformations. Dirac assumed that all constraints of the 1 st type produced gauge transformations [1], and proposed to replace H_T by the generalised Hamiltonian:

$$H_E = H_T + u^{m_k}_k \phi^{m_k}_k, \quad m_k = 2, \ldots, M_k, \quad k = 1, \ldots, m, \tag{2}$$

where $u^{m_k}_k$ are the arbitrary factors, $\phi^{m_k}_k$ are the secondary constraints, $M_k - 1$ is the maximum number of secondary relations obtained on the basis of required time stationarity of the k-th primary constraint.

Generally speaking, Dirac's assumption was wrong. There are examples where secondary constraints of the 1 st type do not produce gauge transformations [2,3].

Dirac's iteration procedure provides no reasons for adding secondary constraints to the total Hamiltonian. The global and geometric generalization of Dirac's approach throws no light on this problem. Being a result of these algorithms, the total Hamiltonian describes dynamics of system but it does not contain all gauge degrees of freedom. Therefore, it is often more convenient to employ the generalized Hamiltonian [1] in order to eliminate nonphysical degrees of freedom from the theory by using additional or gauge conditions.

A general method of applying gauge conditions within singular theories was proposed by Dirac [4]. Later on this method was reproduced many times (e.g., see ref. [5]). New limits are imposed on the coordinates q and momenta p

$$\chi_i(q,p) \approx 0, \quad i=1,\dots,\sum_{k=1}^{m} M_k \equiv I \tag{3}$$

and the function $\chi_i(q,p)$ must obey the following conditions:

$$\det\|\{\chi_i, \phi_k^{m_k}\}\| \neq 0, \quad k=1,\dots,m, \quad m_k=1,\dots,M_k, \tag{4}$$

$$\{\chi_i, \chi_{i'}\} = 0, \quad i,i'=1,\dots,I. \tag{5}$$

Note that for the functions χ_i these conditions are necessary but insufficient for being gauge conditions. This is due to the fact that relations (3) together with motion equations can lead to new relations of dynamic variables and more degrees of freedom will be lost. Such examples are considered in ref. [6].

It is easy to establish a relation between the functions $\chi_i(q,p)$ and Lagrange factors. The required stationarity of gauge conditions (3) yields

$$\dot{\chi}_i = \{\chi_i, H_c\} + u_k^{m_k}\{\chi_i, \phi_k^{m_k}\}. \tag{6}$$

Owing to condition (4), eq. (6) allows determination of Lagrange factors.

The gauge freedom occurring in Hamiltonian motion equations with the generalized Hamiltonian H_E is known to be wider than the gauge freedom in Lagrange's formalism.

Using generators of gauge transformations from ref. [7] and the method of construction of finite transformations for quasigroups [8] for singular systems, we shall obtain Hamilton motion equations with a complete gauge freedom.

New constraints will be obtained for the functions χ_i, the constraints and relations (4) will be the necessary and sufficient conditions for elimination of gauge freedom.

The paper is organized as follows: in the next section we obtain finite gauge transformations for the given singular Lagrangian. Section 3 formulates the sufficient conditions for elimination of gauge freedom. Examples are considered in section 4.

2. CONSTRUCTION OF FINITE GAUGE TRANSFORMATIONS

A given singular Lagrangian allows construction [7] of infinitesimal transformations of coordinates q and momenta p, which are not related to a change in the physical state, in the co-tangent bundle. The operators $\tilde{\Phi}$ for these transformations

$$q'(t) = (1+\tilde{\Phi})q(t), \quad p'(t) = (1+\tilde{\Phi})p(t) \tag{7}$$

are given by the expression

$$\tilde{\Phi} = \{\epsilon_k^{m_k}\phi_k^{m_k}, \quad \}. \tag{8}$$

We use the following notation $\{A, \}B = \{A, B\}$. In expression (7) $\phi_k^{m_k}$, $k=1,\dots,m$, $m_k = 1,\dots,M_k$, are the complete set of the 1 st type occurring within generalized Hamiltonian formalism for the physical system under consideration. The coefficients $\epsilon_k^{m_k}$ must obey [7] the following equation:

$$\dot{\epsilon}_k^{m_k} - \epsilon_{k'}^{m_{k'}} g_{k'k}^{m_{k'}m_k} \doteq 0, \quad m_k > 1, \tag{9}$$

where a point over ϵ means the total time derivative, and the functions $g_{k'k}^{m_{k'}m_k}$ are defined by the relation

$$\{H, \phi_{k'}^{m_{k'}}\} = g_{k'k}^{m_{k'}m_k}\phi_k^{m_k} \equiv \Psi_{k'}^{m_{k'}}. \tag{10}$$

For each value of the index k we parametrize the factors $\epsilon_k^{M_k}$ through an optional infinitesimal function $\delta\lambda_k(t)$: $\epsilon_k^{M_k} = \delta\lambda_k(t)$. Then, on the basis of equation (9) all other $\epsilon_k^{m_k}$ will be expressed through $\delta\lambda_k(t)$, q and p. Note that the number of optional functions $\delta\lambda_k(t)$ is exactly the same as the number of primary constraints of the first type. Thus, operators (8) will include both optional functions $\delta\lambda_k(t)$ and their time derivatives of the M_k-th order.

Now we rewrite formula (8) in a form more suitable for our purpose. To do this, we use the identity

$$\delta\lambda^{(k)}(t) = (-1)^k \int \delta\lambda(t')\,\partial_{t'}^{(k)}\delta(t'-t)\,dt', \quad \delta\lambda^{(k)}(t) \equiv \frac{d}{dt^k}\delta\lambda(t). \tag{11}$$

Then, we find the following expression for the operator $\tilde{\Phi}$ from formulae (8) and (11) with allowance for notation (10):

$$\tilde{\Phi} = (-1)^{M_k - m_k} \int \delta\lambda_{k'}(t') \{ \Psi_{k'}^{m_{k'}}, \quad \} \partial_{t'}^{(M_k - m_k)} \delta(t' - t)\, dt\, dt'. \tag{12}$$

We substitute this operator into formulae (7) and find increments of coordinates and momenta:

$$\delta q_j(t) = \int \delta\lambda_k(t') Q_{ki}(t', t'') \frac{\delta}{\delta q_i(t'')} dt' dt''\, q_j(t) = \int Q_{kj}\, \delta\lambda_k\, dt'$$
$$\delta p_j(t) = \int \delta\lambda_k(t') P_{ki}(t', t'') \frac{\delta}{\delta p_i(t'')} dt' dt''\, p_j(t) = \int P_{kj}\, \delta\lambda_k\, dt'. \tag{13}$$

The following notation is introduced here

$$Q_{ki}(t', t'') = (-1)^{M_k - m_k + 1} \frac{\partial \Psi_k^{m_k}}{\partial p_i} \partial_{t'}^{(M_k - m_k)} \delta(t' - t''),$$
$$P_{ki}(t', t'') = (-1)^{M_k - m_k} \frac{\partial \Psi_k^{m_k}}{\partial q_i} \partial_{t'}^{(M_k - m_k)} \delta(t' - t''). \tag{14}$$

Actually, the operators Q_{ki} and P_{ki} are the generators of gauge transformations. Using these generators on the basis of the results obtained in ref.[9] for quasigroups, one can (in many ways) reconstruct finite gauge transformations. These transformations may be formally written as

$$q_j'(t) = G q_j(t), \quad p_j'(t) = G p_j(t)$$
$$G = \exp\{ \int \lambda_k(t') [Q_{ki}(t', t'') \frac{\delta}{\delta q_i(t'')} + P_{ki}(t', t'') \frac{\delta}{\delta p_i(t'')}] dt' dt'' \}. \tag{15}$$

Actually, this solves the problem of constructing finite gauge transformations at the given singular Lagrangian.

3. ELIMINATION OF GAUGE FREEDOM

To simplify further writing, we shall stick to the following notation (as in formulae (1, 2)):

$$\phi_k^1 \equiv \phi_k, \quad \phi_k^{m_k} \equiv \Phi_j, \quad k = 1, \dots, m, \quad m_k = 2, \dots, M_k,$$
$$j = 1, \dots, n = \sum_{k=1}^{m} (M_k - k), \quad u_k^1 = \alpha_k, \quad u_k^{m_k} = \beta_j.$$

Let us consider time evolution of the system using generalized Hamiltonian (2). We shall take an optional dynamic variable g and see how it is expressed at the moment $t + \delta t$ assuming that $g(t)$ has a definite value.

According to Dirac [1] we have

$$g(t + \delta t) = g(t) + \dot{g}(t)\,\delta t = g(t) + \{g, H_E\}\delta t =$$
$$= g(t) + (\{g, H_c\} + \alpha_k \{g, \phi_k\} + \beta_j \{g, \Phi_j\})\,\delta t. \tag{16}$$

Let us take some other values for the coefficients α_k and β_j, e.g., α_k' and β_j'. This results in another value of $g(t + \delta t)$. Noteworthy is the following fact. We want to keep all the gauge freedom which is present in the theory. If we fix the initial conditions at the moment t before imposing gauge constraints, we shall limit gauge freedom, because gauge constraints must not contradict the initial conditions. Besides, we think it is incorrect to set initial conditions for nonphysical (or a combination of physical and nonphysical) degrees of freedom. Therefore, we assume that the difference between the same dynamic variables in two different gauge obeys the general law (15) at the moment t as well.

These remarks taken into account, we find the gauge variation of the dynamic variable at the moment

$$\Delta g(t + \delta t) = \Delta g(t) + \delta t[(\alpha_k - \alpha_k')\{g, \phi_k\} + (\beta_j - \beta_j')\{g, \Phi_j\}]. \tag{17}$$

On the other hand, we may choose a certain trajectory for the dynamic variable $g(t + \delta t)$ and act on it by the operator G from (15). This will mean that the dynamic variable goes from one gauge to another (optional). Subtracting the variable g from Gg, we obtain the gauge variation of the dynamic variable

$$\Delta g(t + \delta t) = G g(t - \delta t) - g(t + \delta t). \tag{18}$$

On the basis of (17) and (18) we obtain the equation

$$(G - 1)\, g(t + \delta t) - (G - 1)\, g(t) = \delta t[(\alpha_k - \alpha_k')\{g, \phi_k\} + (\beta_j - \beta_j')\{g, \Phi_j\}]. \tag{19}$$

Now let us discuss how one can use eq. (19) in a general case. Then, we shall give examples to illustrate the general discussion. Naturally, we can always take the generalized coordinate q as g. Since the constraints of ϕ_k and Φ_j are linearly and functionally independent, we can always find a situation when ϕ_k will contain at least one momentum variable, e.g. p_ℓ, which does not enter into Φ_j. Then, the term $\{q_\ell, \Phi_j\}$ in (19) reduces to zero for the variable q_ℓ. So we find the functional interdependence between gauge transformation parameters entering into formulae (15) and functions $\alpha_k - \alpha'_k$. Eq. (19) for the coordinate whose conjugated momentum is in Φ_j will bind the parameters λ_k from (15) with functions $\beta_j - \beta'_j$, $\alpha_k - \alpha'_k$ can also be included. Finally, we obtain that in a general case arbitrary fixation of the factors α_k and β_j in the generalised Hamiltonian may fail to correspond to any gauge. In other words, when choosing gauge constraints and using eq. (6) for fixation of the factor $u_k^{m_k}$, we must not break the relations between these factors as established in eq. (19). This is the only case when conditions for χ_i will be the gauge constraints which, on the one hand, fix the whole gauge freedom and, on the other hand, do not lead — together with the motion equation — to new constraints (relations). Thus, we have found the sufficient condition which allows functions χ_i obeying conditions (4) and (5) to be regarded as gauge functions.

4. EXAMPLES

To make it all clear, let us consider some simple examples. The first example is chargeless electrodynamics.

The electrodynamics Lagrangian has the form:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

This theory has one primary constraint $\phi \equiv \pi_0 \approx 0$ and one secondary constraint $\Phi \equiv \partial_i \pi^i \approx 0$ of the 1 st type. The generalised Hamiltonian is defined by the following expression:

$$H_E = H_c + \int d^3\vec{x} (\alpha \pi^\circ + \beta \partial_i \pi_i),$$

where H_c is the canonical Hamiltonian.

On the basis of the general method for obtaining infinitesimal gauge transformations for the given singular Lagrangian [7] we find

$$\delta A_\mu(x) = \partial_\mu \epsilon(x),$$

where $\epsilon(x)$ is the infinitesimal arbitrary function. Now we construct an operator Q defined by the relation

$$Q_\mu(y, z) = \partial^y_\mu \delta(y - z).$$

Then, we find the operator

$$G = \exp\{-\int d^4y\, d^4z\, \lambda(y)\, \partial^4_\mu \delta(y - z) \frac{\delta}{\delta A_\nu(z)}\}$$

and finite gauge transformations

$$A'_\mu(x) = GA_\mu(x) = A_\mu(x) + \partial_\mu \lambda(x).$$

Replacing g in (19) by A_0 and then by A_i, we find the following relations:

$$\partial_0^2 \lambda(x)\, \delta t = [\alpha'(x) - \alpha(x)]\, \delta t,$$

$$\partial_i \partial_0 \lambda(x)\, \delta t = \partial_i [\beta'(x) - \beta(x)]\, \delta t.$$

Finally

$$\partial_0 (\beta'(x) - \beta(x)) = \alpha'(x) - \alpha(x).$$

Now let us consider the model Lagrangian proposed in ref. [9,10]

$$\mathcal{L} = \frac{1}{2} [(\frac{d}{dt} - yT)\vec{x}]^2 - V(\vec{x}^2),$$

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

A two-dimensional vector $\vec{x}$ and y are the generalized coordinates here. This model has one primary $p_y \approx 0$ and one secondary constraint $\vec{p}T\vec{x}$. It is easy to find the operator of finite gauge transformations

$$G = \exp\{\int dt'\, dt''\, \lambda(t') [\delta(t' - t'') x_1(t'') \frac{\delta}{\delta x_2(t'')} - \delta(t' - t'') x_2(t'') \frac{\delta}{\delta x_1(t'')} - \delta'_{t'}(t' - t'') \frac{\delta}{\delta y(t'')}\}.$$

Using this operator, we find gauge transformations

$$x'_1(t) = x_1(t) \cos\lambda(t) - x_2(t) \sin\lambda(t),$$

$$x'_2(t) = x_1(t) \sin\lambda(t) + x_2(t) \cos\lambda(t),$$

$$y'(t) = y(t) + \dot{\lambda}(t).$$

Formula (19), binding the coefficients in the generalized Hamiltonian, has the following form in this model:

$$\frac{d}{dt} [x_1 \cos\lambda - x_2 \sin\lambda] - \dot{x}_1 = -(\beta' - \beta) x_2,$$

$$\frac{d}{dt}[x_2\cos\lambda + x_1\sin\lambda] - \dot{x}_2 = (\beta' - \beta)x_1, \quad \ddot{\lambda} = \alpha' - \alpha.$$

These formulae allow determination of relations between the coefficients α and β.

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Устранение калибровочного произвола в сингулярных теориях

Найдены достаточные условия на калибровочные функции. Получены гамильтоновы уравнения движения, которые содержат правильный калибровочный произвол. Показано, что полученные результаты в рамках конечномерной механики легко обобщаются на случай теории поля. Рассмотрены примеры.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Elimination of Gauge Freedom in Singular Theories

The sufficient conditions for gauge functions are found. The Hamilton equations containing the real gauge freedom are obtained. It is shown that the obtained point mechanical results can be easily generalized to field theory. Examples are analysed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1987