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A.B.Pestov

ON DYNAMIC EQUATIONS FOR INTERACTION OF THE AFFINOR FIELD WITH AFFINE CONNECTION

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INTRODUCTION

Of fundamental importance for the theory of strong, electromagnetic, and gravitational interactions are such tensor fields as a scalar field, covariant vector field, covariant skew-symmetric tensor field of the second rank, covariant symmetric tensor field of the second rank; this list of the simplest tensor fields of rank not higher than the second is incomplete since it does not contain a mixed tensor field of the second rank that is called the affinor field. If one recalls that general relativity is essentially the theory of a symmetric tensor field of the second rank, it is quite natural to assume that the theory of the affinor field is not only of a pure mathematical interest.

In this report devoted to the theory of affinor field I consider a model of two interacting fields, one of which is an affinor field and the other is a non-tensor field called the affine connection.

1. LAGRANGIAN OF THE AFFINOR FIELD

Out of two affinors S^α_β and T^α_β an affinor may be constructed called their product

$$P^\alpha_\beta = S^\alpha_\sigma T^\sigma_\beta. \tag{1}$$

If $|S^\alpha_\beta| \neq 0$, there exists an affinor $S^{-1\alpha}_{\beta}$ inverse to S^α_β, $S^\alpha_\sigma S^{-1\sigma}_\beta = S^{-1\alpha}_\sigma S^\sigma_\beta = \delta^\alpha_\beta$. With the operation of multiplication thus defined, the set of affinors with a nonzero determinant forms a group, denoted by G_i, which plays a fundamental rôle in the theory of affinor field as the group of internal symmetry. If the fundamental affinor field $\Psi^\alpha_\beta(x)$ is transformed under the action of the group G_i by the law $\Psi^\alpha_\beta \Rightarrow S^{-1\alpha}_\sigma \Psi^\sigma_\tau S^\tau_\beta$, then a scalar Ψ^α_α is obviously an invariant of the group G_i. It is known from the theory of linear spaces that there also exist other invariants among which I will choose a quadratic G_i - invariant function $\Psi^\alpha_\sigma \Psi^\sigma_\alpha$ of the affinor field.

A fundamental method of constructing new tensor fields from the given fields is the operation of covariant differentiation, a basic differential operation of the tensor analysis. A covariant derivative makes a tensor field of type (p,q) correspond to a tensor field of type (p,q+1). In particular, for the affinor field we have

$$\nabla_\mu \Psi^\alpha_\beta = \partial_\mu \Psi^\alpha_\beta + \Gamma^\alpha_{\mu\sigma}\Psi^\sigma_\beta - \Psi^\alpha_\sigma \Gamma^\sigma_{\mu\beta}. \qquad (2)$$

A quite arbitrary system of functions $\Gamma^\alpha_{\mu\beta}(x)$ in (2) specifies an object called the affine connection /1-3/ .

For simplicity indices related to the transformations of the group G_1 will be omitted, and the operator notation

$$\Psi^\alpha_\beta \Rightarrow \Psi, \quad \Psi^\alpha_\alpha = \mathrm{Tr}\Psi, \quad \Psi^\alpha_\sigma \Phi^\sigma_\beta \Rightarrow \Psi\Phi,$$

$$\mathrm{Tr}(\Psi\Phi) = \mathrm{Tr}(\Phi\Psi) = \Psi^\alpha_\sigma \Phi^\sigma_\alpha, \quad \Gamma^\alpha_{\mu\beta} \Rightarrow \Gamma_\mu$$

will be used, in which formula (2) reads

$$\nabla_\mu \Psi = \partial_\mu \Psi + [\Gamma_\mu, \Psi]. \qquad (3)$$

Relation (3) allows one to connect the covariant derivative with the group G_1. Denote the covariant derivative in the connection $'\Gamma^\alpha_{\mu\beta}$ by $'\nabla_\mu$; then, according to (3), we have

$$'\nabla_\mu \Psi = \nabla_\mu \Psi + [\delta\Gamma_\mu, \Psi], \qquad (4)$$

where

$$\delta\Gamma_\mu = {}'\Gamma_\mu - \Gamma_\mu \qquad (5)$$

is the affine-deformation tensor /7/ . Substituting into (4) the affinor $'\Psi = S^{-1}\Psi S$, instead of Ψ , we get

$$'\nabla_\mu {}'\Psi = S^{-1}(\nabla_\mu \Psi)S + [\delta\Gamma_\mu - S^{-1}\nabla_\mu S, \Psi]. \qquad (6)$$

From (6) it follows that under the condition

$$\delta\Gamma_\mu = S^{-1}\nabla_\mu S \qquad (7)$$

we have

$$'\nabla {}'\Psi = S^{-1}(\nabla_\mu \Psi)S. \qquad (8)$$

From (5) and (7) we obtain the relation

$$'\Gamma_\mu = \Gamma_\mu + S^{-1}\nabla_\mu S \qquad (9)$$

to be accomplished by the formula

$$'\Psi = S^{-1}\Psi S. \qquad (10)$$

Now I shall assume that under the action of the group the affinor field is transformed by the law (10); and the Γ-field, by the law (9) and construct a G_1-invariant theory of the interaction of those fields. To start with, an important concept is to be introduced, that of the affinor derivative. The affinor derivative D_μ is characterized by that it does not change the transformation law of an object transformating as a true affinor under the action of the group G_1. However, the tensor nature of the operator D_μ requires in each separate case a special consideration. For the affinor it is obvious since, according to (2),

$$D_\mu \Psi = \nabla_\mu \Psi \qquad (11)$$

but $D_\mu D_\nu \Psi$ is no longer tensor field, because $D_\mu D_\nu \Psi = \nabla_\mu \nabla_\nu \Psi + \Gamma^\sigma_{\mu\nu}\nabla_\sigma \Psi$. Setting $S^\alpha_\beta = e^\lambda \delta^\alpha_\beta$ in (9) we obtain the so-called λ-transformation of the affine connection:

$$'\Gamma^\alpha_{\mu\beta} = \Gamma^\alpha_{\mu\beta} + \partial_\mu \lambda \delta^\alpha_\beta,$$

considered in ref. /4/ .

Consider the Lagrangian of the affinor field

$$\mathfrak{L}_\Psi = -\frac{1}{2}\mathrm{Tr}(g^{\mu\nu} D_\mu \Psi D_\nu \Psi + m^2 \Psi\Psi). \qquad (12)$$

From (8)-(11) it follows that under the transformation $\Psi \Rightarrow S^{-1}\Psi S$, $\Gamma_\mu \Rightarrow \Gamma_\mu + S^{-1}D_\mu S$, $g^{\mu\nu} \Rightarrow g^{\mu\nu}$ the Lagrangian (12) is not changed, and consequently, the action $\mathfrak{A} = \int \sqrt{|g|}\,\mathfrak{L}_\Psi d^4x$ is G_1-invariant. In what follows it is to be taken into consideration that the operation of raising and lowering indices does not commute with the derivative D_μ since

$$D_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta}.$$

By variation from (12) we obtain the following equations of second order for the affinor feild

$$\frac{1}{\sqrt{|g|}} D_\sigma (\sqrt{|g|} D^\sigma \Psi) - m^2 \Psi = 0, \qquad (13)$$

where m = const. Equations (13) are G_i-covariant, which means that substituting into them the affine connection $'\Gamma_\mu = \Gamma_\mu + S^{-1} D_\mu S$ instead of Γ_μ we obtain equations equivalent to the initial ones.

Consider an infinitesimal transformation of the group G_i setting $S^\alpha_\beta = \delta^\alpha_\beta + \Omega^\alpha_\beta$; then

$$\delta\Psi = [\Psi, \Omega], \quad \delta\Gamma_\mu = D_\mu \Omega.$$

If $D_\mu \Omega = 0$, then $\delta\Gamma_\mu = 0$. For those transformations $\delta \mathfrak{L}_\Psi = 0$ and hence the conservation law

$$j^\sigma_{;\sigma} = \frac{1}{\sqrt{|g|}} \partial_\sigma (\sqrt{|g|}\, j^\sigma) = 0,$$

follows, where the vector j^σ is given by the expression

$$j^\sigma = \mathrm{Tr}([D^\sigma \Psi, \Psi] \Omega).$$

Direct calculations show that the current j^σ is conserved provided that Ψ obeys equations (13) and the affinor Ω is a covariant constant in the connection Γ_μ, $D_\mu \Omega = 0$.

Varying the action $\mathfrak{A} = \int \sqrt{|g|}\, \mathfrak{L}_\Psi\, d^4x$ over $g^{\mu\nu}$ we obtain the energy momentum symmetric tensor

$$\Theta_{\mu\nu} = \mathrm{Tr}(D_\mu \Psi D_\nu \Psi) + g_{\mu\nu} \mathfrak{L}_\Psi. \tag{14}$$

If $g_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric tensor, then

$$\Theta_{00} = \frac{1}{2} \mathrm{Tr}\left(\sum_{\nu=0}^{3} D_\nu \Psi D_\nu \Psi + m^2 \Psi\Psi \right).$$

Thus, in a free case the energy density will be positive provided that $\mathrm{Tr}(\Psi\Psi) > 0$. The states that do not obey that condition will, generally, be unstable.

2. EQUATIONS OF THE AFFINE CONNECTION

Let us show that $[D_\mu, D_\nu]\Psi$, unlike $D_\mu D_\nu \Psi$, is a tensor field of the type (1,3). Since

$$D_\mu D_\nu \Psi = [\partial_\mu \Gamma_\nu, \Psi] + [\Gamma_\mu, [\Gamma_\nu, \Psi]] + \partial_\mu \partial_\nu \Psi + [\Gamma_\mu, \partial_\nu \Psi] + [\Gamma_\nu, \partial_\mu \Psi],$$

then

$$[D_\mu, D_\nu]\Psi = [R_{\mu\nu}, \Psi], \tag{15}$$

where

$$R_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu] \tag{16}$$

is the Riemann tensor of the affine connection $\Gamma^\alpha_{\mu\beta}$. Thus, $[D_\mu, D_\nu]\Psi$ is really a tensor of the type (1,3). From (15) it follows that under transformations of the group G_i the tensor field $R_{\mu\nu}$ transforms as a true affinor:

$$R_{\mu\nu} \Rightarrow {}'R_{\mu\nu} = S^{-1} R_{\mu\nu} S. \tag{17}$$

The transformation law (17) may directly be derived from (9) and (16).

The corresponding invariant Lagrangian for the field $\Gamma^\alpha_{\mu\beta}(x)$ is of the form

$$\mathfrak{L}_\Gamma = -\frac{1}{4} \mathrm{Tr}(g^{\mu\alpha} g^{\nu\beta} R_{\mu\nu} R_{\alpha\beta}) = -\frac{1}{4} \mathrm{Tr}(R_{\mu\nu} R^{\mu\nu}). \tag{18}$$

Varying the action $\mathfrak{A} = \int \sqrt{|g|}\, \mathfrak{L}_\Gamma\, d^4x$ over $g^{\mu\nu}$ with the Lagrangian (18), we obtain the energy-momentum tensor of the affine connection in the form

$$\overset{\Gamma}{\Theta}_{\mu\nu} = \mathrm{Tr}(R_{\mu\sigma} R^\sigma_\nu) + g_{\mu\nu} \mathfrak{L}_\Gamma. \tag{19}$$

The total Lagrangian of the interaction of an affinor field and affine connection

$$\mathfrak{L} = -\frac{1}{2} \mathrm{Tr}(D_\mu \Psi D^\mu \Psi + m^2 \Psi\Psi) - \frac{1}{4} \mathrm{Tr}(R_{\mu\nu} R^{\mu\nu}), \tag{20}$$

is invariant under the transformations

$$\Psi \Rightarrow S^{-1}\Psi S, \quad \Gamma_\mu \Rightarrow \Gamma_\mu + S^{-1} D_\mu S, \quad g^{\mu\nu} \Rightarrow g^{\mu\nu},$$

and, consequently, the pairs (Ψ, Γ_μ) and $('\Psi, '\Gamma_\mu)$ correspond to the same internal state of the system described by the fields Ψ and Γ_μ.

Varying the Lagrangian (20) over $\Gamma^\alpha_{\mu\beta}$ we arrive at the following manifestly G_i-invariant form of the equations for the affine connection

$$\frac{1}{\sqrt{|g|}} D_\mu (\sqrt{|g|}\, R^{\mu\nu}) = J^\nu, \tag{21}$$

where

$$J^{\nu} = [\Psi, D^{\nu}\Psi]. \tag{22}$$

The tensor current J^{ν} should obey the equation

$$D_{\nu}(\sqrt{|g|}\, J^{\nu}) = 0 \tag{23}$$

following from the identities

$$D_{\mu} D_{\nu} R^{\mu\nu} = 0.$$

From (13) and (22) it follows that the tensor current does obey equation (23); thus, the system of equations (13) and (21) is consistent.

Let us now find divergences $\Theta^{\mu\nu}{}_{;\mu}$ and $\overset{\Gamma}{\Theta}{}^{\mu\nu}{}_{;\mu}$. If ψ and Γ_{μ} obey the equations of motion, then

$$\Theta^{\mu\nu}{}_{;\mu} = \mathrm{Tr}(R^{\mu\nu} J_{\mu}), \qquad \overset{\Gamma}{\Theta}{}^{\mu\nu}{}_{;\mu} = \mathrm{Tr}(R^{\nu\mu} J_{\mu}).$$

Hence it follows that the energy-momentum tensor $T_{\mu\nu} = \Theta_{\mu\nu} + \overset{\Gamma}{\Theta}_{\mu\nu}$ of interacting fields obeys the equation

$$T^{\mu\nu}{}_{;\mu} = 0.$$

From the last equation we see that eqs. (13) and (21) are general covariant.

As it is known, the affine connection has always played a fundamental role in the development of general relativity from the very start of its creation /2,3,5/ and /6/ . The conclusion drawn in this report that the affine connection has a conserved energy-momentum tensor and therefore may be a source of the gravitational field radically changes the views on that object and the laws of its interaction with other fields.

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Пестов А.Б. E2-87-826
О динамических уравнениях, описывающих
взаимодействие аффинорного поля и
аффинной связности

Установлен лагранжиан, описывающий взаимодействия аффинорного поля и аффинной связности. Выведены уравнения движения и законы сохранения. Показано, что существует симметричный, сохраняющийся тензор энергии-импульса аффинной связности.

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Pestov A.B. E2-87-826
On Dynamic Equations for Interaction
of the Affinor Field with Affine Connection

The Lagrangian of interaction of an affinor field with an affine connection is constructed and the equations of motion and conservation laws are derived. It is shown that there exists a symmetric conserved tensor of the affine-connection energy-momentum.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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