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# THE DEVIATION EQUATION AS AN EQUATION OF MOTION

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## 1. INTRODUCTION

In this work we are going to look on the generalized deviation equation from a dynamical point of view. This is very natural because of its definition (see [3], sect. 1.3 and [4], sect. IV.3) because it is intuitively clear that the equations (laws) of the (relative) motion of two point particles in spaces $L_n$ with the affine connection could be considered also as deviation equations.

Ideas analogous to ours have been considered in more special cases in [1,2,5,6]. In [6] the deviation equation is derived of two infinitesimally near interacting point particles in the space-time of general relativity. In [5] the influence of small perturbative forces of dissipative and periodical kinds in the right-hand side of the usual equation of geodesic deviation in a Riemannian space $V_n$ ( $n = 4$) is treated. Paper [1] is an investigation of the relative dynamics of geodesies in spaces with an affine connection without torsion in terms of absolute two-point derivatives of a geodesic radius-vector. The closest to our work is [2], where exact and approximate equations of relative motion of trial point particles in the field of external forces in a Riemannian space are obtained in terms of absolute two-point derivatives of a geodesic radius-vector.

The purpose of the present work, in which the notation is the same as in [3,4], is to show that in the general case the equation of motion of two point particles in the field of external force in space with the affine connection is a special case of the generalized deviation equation. After some preliminary considerations and one simple example (sections 2 and 3) we derive in section 4 in the general case the deviation equation in the form of an equation of motion. In sections 5 and 6 we consider two examples illustrating the developed general theory.

## 2. THE BASIC PRELIMINARY CONSTRUCTION

Let us define the curve $x:[s',s'']\to L_n$, $L_n$ is the space with the affine connection as unique solution of the following initial-value problem:

$$\frac{Du(s)}{ds} = F(s, x(s), u(s)), \quad u^{\alpha} := u^{\alpha}(s) := \frac{dx^{\alpha}(s)}{ds}, \tag{2.1}$$

$$x(s_o) = x_o \in L_n, \; u(s_o) = u_o \in T_{x_o}(L_n), \; s, s_o \in [s', s''], \tag{2.2}$$

where $D/ds$ is the covariant derivative with respect to $s$ along $x(s)$, and $F$ is a continuous function of its arguments.

Physically, we shall interpret $x$ as a trajectory (world line) of an observer who is affected by the force field $F$ having a meaning of a force per unit mass and whose trajectory passes through the point $x_o$ with a velocity $u_o$.

Let us define the family of curves $\gamma_s : [r', r''] \longrightarrow L_n$, $s \in [s', s'']$ as the unique solution of the following initial-value problem:

$$\frac{D}{ds}\Big|_{\gamma_s} (\gamma'_s(r)) = F_s(r) := F_s(r, \gamma_s(r), \gamma'_s(r)) \in T_{\gamma_s(r)}(L_n),$$

$$\gamma'^{\alpha}_s(r) := \partial \gamma^{\alpha}_s(r)/\partial s, \tag{2.3}$$

$$\gamma_s(r)|_{s=s_o} = \chi(r) \in L_n, \; \gamma'_s(r)|_{s=s_o} = \chi'(r) \in T_{\chi(r)}(L_n), \tag{2.4}$$

where $F_s$, $\chi$ and $\chi'$ are given smooth functions of their arguments, $r \in [r', r'']$, $s \in [s', s'']$, the number $s_o \in [s', s'']$ is the same as in (2.2) and $D/ds|_{\gamma_s}$ means the covariant derivative with respect to $s$ along the $r$-curve $\gamma_s(r)$, $r = \text{const}$ (by $D/ds$ we denote the covariant derivative with respect to $s$ along the curve $x(s)$, cf. (2.1)).

Let the curves $x_a : [s'_a, s''_a] \longrightarrow L_n$, $a = 1, 2$ be defined by

$$x_1(\tau_1(s)) := \gamma_s(r'), \; x_2(\tau_2(s)) := \gamma_s(r''), \tag{2.5}$$

where $s \in [s', s'']$ and $\tau_a : [s', s''] \longrightarrow [s'_a, s''_a]$, $a = 1, 2$ are some given smooth mapping. So, due to (2.4) the curves $x_1$ and $x_2$ pass through the points $\chi(r')$ and $\chi(r'')$, respectively, and have at them tangent vectors $\chi'(r')$ and $\chi'(r'')$, respectively.

Physically, we shall interpret $F_s$ as a force (per unit mass) acting in the 2-dimensional submanifold (surface) $\{\gamma_s(r) | s \in [s', s''], r \in [r', r'']\}$. (It is clear that $F_s$ and $F$ are analogues of the Minkowski four-force per unit mass). The curves $x_1$ and $x_2$ shall be considered as trajectories (world lines) of point particles observed from the observer defined above whose trajectory is defined by (2.1) and (2.2).



At the end, as has been done in [4], we shall define a family of curves $\eta_s : [\rho', \rho''] \longrightarrow L_n$, such that $\eta_s(\rho') := x_1(\tau_1(s))$ and $\eta_s(\rho'') := x(s))$, $s \in [s', s'']$.

In the next sections we shall consider the problem of finding the deviation equation of $x_2$ with respect to $x_1$ as it is observed from $x$ at some point $x(s)$ (for the corresponding definitions see [3,4]). This means to express the relative (deviation) acceleration $D^2 h_{12}(s, x)/ds^2$ ($h_{12}(s, x)$ is the corresponding deviation vector) through the defined above quantities, and first of all, through the force field $F_s$, which practically means to write down the equation of relative motion of $x_2$ with respect to $x_1$ relatively to $x$. In this sense, the equation of motion is a special case of the generalized deviation equation.

## 3. THE DEVIATION EQUATION AS AN EQUATION OF MOTION: EUCLIDEAN CASE

To derive the deviation equation we have to do the following: using the definition of the deviation vector (see [4], sect. IV.1), we have to compute its second covariant derivative $D^2 h_{12}(s, x)/ds^2$ (along $x(s)$) and next to substitute in the obtained expression the formulae (2.1)-(2.4).

To make these ideas more clear we shall consider at first the most simple possible case, which we shall call an "Euclidean case": let us take $L_n$ to be an $n$-dimensional Euclidean space $E_n$ and $I^{\gamma}$ to be a parallel transport along $\gamma$. Then (see [4], sect. III.3.1 and [4], sect. IV.1) we find the deviation vector $h_{12}(s, x)$ to be with the components

$$h^{\alpha}_{12}(s, x) = x^{\alpha}_2(\tau_2(s)) - x^{\alpha}_1(\tau_1(s)), \tag{3.1}$$

so the deviation equation is simply

$$\frac{D^2 h_{12}(s, x)}{ds^2} = F_2 - F_1, \tag{3.2}$$

where

$$\begin{aligned} F_1 &:= F_s(r', x_1(\tau_1(s)), dx_1(\tau_1(s))/ds), \\ F_2 &:= F_s(r'', x_2(\tau_2(s)), dx_2(\tau_2(s))/ds). \end{aligned} \tag{3.3}$$

are the forces per unit mass acting on the particles 1 and 2, respectively.

Evidently, eq. (3.2) is nothing else but the second Newton law of dynamics, i.e. the equation of the relative motion of the particle 2 with respect to the particle 1 in the force field $F_s$.

## 4. THE DEVIATION EQUATION AS AN EQUATION OF MOTION: GENERAL CASE

We shall begin this section with a mathematical definition necessary for the following considerations:

Let $A^{i_1\dots i_p}_{j_1\dots j_q}(z_1(s),\dots,z_{p+q}(s))$ be components of the $(p+q)$-point $C^1$-tensor $A$ from the space

$$T_{z_1(s)}(L_n)\otimes\dots\otimes T_{z_p(s)}(L_n)\otimes T^*_{z_{p+1}(s)}(L_n)\otimes\dots\otimes T^*_{z_{p+q}(s)}(L_n),$$

where $z_a : [s', s''] \longrightarrow L_n$, $a = 1,\dots,p+q$ are some $C^1$-maps and $s \in [s', s'']$. Then, by definition

$$\frac{D}{ds}A^{i_1\dots i_p}_{j_1\dots j_q}(z_1(s),\dots,z_{p+q}(s)) :=$$

$$:= \frac{d}{ds}A^{i_1\dots i_p}_{j_1\dots j_q}(z_1(s),\dots,z_{p+q}(s)) + \sum_{b=1}^{p}\Gamma^{i_b}_{\cdot k\ell}(z_b(s)) \times$$

$$\times A^{i_1\dots i_{b-1}k\,i_{b+1}\dots i_p}_{j_1\dots j_q}(z_1(s),\dots,z_{p+q}(s))\,z'^{\ell}_b(s) - \sum_{b=1}^{q}\Gamma^{k}_{\cdot j_b\ell}(z_{p+b}(s))\times$$

$$\times A^{i_1\dots i_p}_{j_1\dots j_{b-1}k\,j_{b+1}\dots j_q}(z_1(s),\dots,z_{p+q}(s))\,z'^{\ell}_{p+b}(s), \qquad (4.1)$$

where $z'^{\alpha}_a(s) := dz^{\alpha}_a(s)/ds$, $a = 1,\dots,p+q$ and $\Gamma^i_{\cdot jk}(y)$ are the coefficients of the affine connection at $y \in L_n$.

It is not difficult to see that (4.1) are components of a many-point tensor of the same tensor space as the initial tensor $A$.

From this definition it is easy to check that the operator $D/ds$ is a differentiation with respect to the tensor products of (many-points) tensors, i.e. the usual Liebniz rule holds for tensor products ($D(A\otimes B)/ds = (DA/ds)\otimes B + A\otimes(DB/ds)$) and $D/ds$ commutes with the contraction operator, and as a consequence of this, $D/ds$ commutes also with the contracted tensor products (e.g.,

$$D(A^j_{\cdot j}(z_1(s), z_2(s))B^j(z_2(s))/ds =$$

$$= \sum_{j=1}^{n}\{[D(A^j_{\cdot j}(z_1(s), z_2(s))B^k(z_3(s)))/ds]\Big|_{\substack{z_3=z_2\\ k=j}}\}).$$

Let us now recall the definition of the deviation vector $h_{12}(s,x)$ of $x_2$ with respect to $x_1$ relatively to $x$ at the point $x(s)$ (see [4], sect. IV.1):

$$h_{12}(s,x) := I^{\eta_s}_{x_1(\tau_1(s))\to x(s)}\int_{r'}^{r''} I^{\gamma_s}_{\gamma_s(r)\to x_1(\tau_1(s))}\dot{\gamma}_s(r)\,dr, \qquad (4.2)$$

or (see [3], sect. II.2.4)

$$h^k_{12}(s,x) = H^k_{\cdot p}\int_{r'}^{r''}\Lambda^p_{\cdot q}\,\dot{\gamma}^q_s(r)\,dr, \qquad (4.3)$$

where $\dot{\gamma}^{\alpha}_s(r) := \partial\gamma^{\alpha}_s(r)/\partial r$, $I^{\eta_s}_{\dots}$ and $I^{\gamma_s}_{\dots}$ are the generalized transports along $\eta_s$ and $\gamma_s$, respectively, defined in [3], sect. II.1 and

$$H^k_{\cdot p} := H^k_{\cdot p}(x(s), \gamma_s(r')) := (I^{\eta_s}_{\gamma_s(r')\to x(s)})^k_{\cdot p},$$

$$(H^{-1})^p_{\cdot q} := H^p_{\cdot q}(\gamma_s(r'), x(s)) = [(I^{\eta_s}_{\gamma_s(r')\to x(s)})^{-1}]^p_{\cdot q}, \qquad (4.4)$$

$$\Lambda^p_{\cdot q} := \Lambda^p_{\cdot q}(\gamma_s(r'), \gamma_s(r)) := (I^{\gamma_s}_{\gamma_s(r)\to\gamma_s(r')})^p_{\cdot q}.$$

are the components of the two-point tensors representing the corresponding linear mapping $I^{\dots}_{\dots}$ in a given basis (see [3], sect. II.2.4).

Now we are ready to derive the deviation equation of $x_2$ with respect to $x_1$ relatively to $x$ at $x(s)$:

For some time, for the sake of shortness, we shall suppress the indices and the arguments of all quantities. Thus, we can rewrite (4.3) symbolically in the form

$$h = H\cdot\int\Lambda\cdot\dot{\gamma}, \qquad (4.5)$$

where the dot ($\cdot$) means a contracted tensor product (e.g., $\Lambda\cdot\dot{\gamma}$ is a vector with components $(\Lambda\cdot\dot{\gamma})^j := \Lambda^j_{.k}\dot{\gamma}^k = \Lambda^j_{.k}(\gamma_s(r'), \gamma_s(r))\dot{\gamma}^k_s(r)$, $\int\Lambda\cdot\dot{\gamma} := \int\Lambda\cdot\dot{\gamma}\,dr$ and so on).

So, using (4.1) and (4.5), we get:

$$\frac{D^2}{ds^2}\Big|_x h := \frac{D}{ds}\Big|_x\left(\frac{D}{ds}\Big|_x h\right) = \frac{D^2 H}{ds^2}\cdot H^{-1}\cdot h +$$

$$+ 2\frac{DH}{ds}\cdot\int\left(\frac{D\Lambda}{ds}\cdot\dot{\gamma} + \Lambda\cdot\frac{D\dot{\gamma}}{ds}\right) + H\cdot\int\left(\frac{D^2\Lambda}{ds^2}\cdot\dot{\gamma} + 2\frac{D\Lambda}{ds}\cdot\frac{D\dot{\gamma}}{ds} + \Lambda\cdot\frac{D^2\dot{\gamma}}{ds^2}\right). \tag{4.6}$$

By a direct calculation one can check the following equality:

$$\frac{D^2\dot{\gamma}^p}{ds^2} = \left(R^p_{.ijk}\gamma'^i\gamma'^j - T^p_{.jk}\frac{D\gamma'^j}{ds} + \frac{DT^p_{.jk}}{ds}\gamma'^j\right)\dot{\gamma}^k +$$

$$+ [\dot{\gamma}''^p + 2\Gamma^p_{.ij}\dot{\gamma}'^i\dot{\gamma}'^j + \Gamma^p_{.ij}\frac{D\gamma'^i}{ds}\dot{\gamma}^j + (\Gamma^p_{.ij,k} + T^p_{.iq}\Gamma^q_{.kj})\gamma'^i\gamma'^j\dot{\gamma}^k], \tag{4.7}$$

where $R^p_{.ijk}$ and $T^p_{.jk}$ are the components of the curvature and torsion tensors, respectively, $\gamma'^\alpha := \gamma'^\alpha_s(r) := \partial\gamma^\alpha_s(r)/\partial s$, $\Gamma^\alpha_{.\beta\gamma,\delta} := \partial\Gamma^\alpha_{.\beta\gamma}/\partial x^\delta$.

It is not difficult to show that the sum in square brackets in (4.7) is

$$[\ldots] = \frac{D}{ds}\left(\frac{D\gamma'^p}{ds}\right) + T^p_{.jk}\gamma'^j\frac{D\dot{\gamma}^k}{ds}. \tag{4.8}$$

Substituting (4.8) into (4.7), then the so-obtained equality in (4.6) and using (2.1)-(2.4), we finally get the following deviation equation:

$$\frac{D^2}{ds^2}\Big|_x h^k_{12}(s,x) = \frac{D^2H^k_{.p}}{ds^2}(H^{-1})^p_{.q}\, h^q_{12}(s,x) +$$

$$+ 2\frac{DH^k_{.p}}{ds}\int_{r'}^{r''} dr\left(\frac{D\Lambda^p_{.q}}{ds}\dot{\gamma}^q_s(r) + \Lambda^p_{.q}\frac{D\dot{\gamma}^q_s(r)}{ds}\right) +$$

$$+ H^k_{.p}\int_{r'}^{r''} dr\,\{\frac{D^2\Lambda^p_{.q}}{ds^2}\dot{\gamma}^q_s(r) + 2\frac{D\Lambda^p_{.q}}{ds}\frac{D\dot{\gamma}^q_s(r)}{ds} +$$



$$+ \Lambda^p_{.q}[(R^q_{.ij\ell}(\gamma_s(r))\gamma'^i_s(r)\gamma'^j_s(r) + T^q_{.j\ell}(\gamma_s(r))F^j_s(r) +$$

$$+ \frac{DT^q_{.j\ell}(\gamma_s(r))}{ds}\gamma'^j_s(r))\dot{\gamma}^\ell_s(r) +$$

$$+ T^q_{.j\ell}(\gamma_s(r))\gamma'^j_s(r)\frac{D\dot{\gamma}^\ell_s(r)}{ds} + \frac{DF^q_s(r)}{dr}]\}. \tag{4.9}$$

We want to remark that the force F acting on the observer is also presented in (4.9), but implicitly it is hidden in the second derivative $D^2H^k_{.p}/ds^2$.

Thus, we see that the deviation equation (4.9) really has a meaning of an equation of motion of the particle 2 with respect to the particle 1 as it is observed from an observer with a trajectory x.

From a dynamical point of view, the most important terms in the deviation equation (4.9) are those in which the force $F_s(r)$ appears, viz.

$$H^k_{.p}\int_{r'}^{r''}\Lambda^p_{.q}(T^q_{.j\ell}(\gamma_s(r))F^j_s(r)\dot{\gamma}^\ell_s(r) + DF^q_s(r)/dr)\,dr =$$

$$= [H^k_{.p}(\Lambda^p_{.j}(\gamma_s(r'),\gamma_s(r''))F^j_s(r'') - F^p_s(r'))] +$$

$$+ H^k_{.p}\int_{r'}^{r''}\left(\Lambda^p_{.q}T^q_{.j\ell}(\gamma_s(r))\dot{\gamma}^\ell_s(r) - \frac{D\Lambda^p_{.j}(\gamma_s(r'),\gamma_s(r))}{dr}\right)F^j_s(r)\,dr, \tag{4.10}$$

where we have done an evident integration by parts of the term $\Lambda^p_{.q}DF^q_s(r)/dr$.

From (4.10) we see that the only independent of $\gamma$ terms in (4.9) are those in the square brackets in (4.10), which are equal to the defined by means of the generalized transports $I^{\gamma_s}$.. and $I^{\eta_s}$.. difference of the forces acting on the observed particles.

## 5. EXAMPLE: EUCLIDEAN CASE

The most simple application of the general theory from the previous section is the derivation of the Euclidean equation of motion (3.2).

In the Euclidean case (see section 3), we replace $L_n$ with the $E_n$ space and $I^{\gamma}$ with a parallel transport along $\gamma$. So, working in a global coordinate basis in which $\Gamma^{\alpha}_{.\beta\gamma} \equiv 0$, we have $H^{\alpha}_{.\beta} = \Lambda^{\alpha}_{.\beta} = \delta^{\alpha}_{\beta}$ ($\delta^{\alpha}_{\beta} = 1$ for $\alpha = \beta$ and $\delta^{\alpha}_{\beta} = 0$ for $\alpha \neq \beta$). Thus, in this case due to (4.9) the deviation equation is

$$\frac{D^2}{ds^2}\Big|_x h_{12}(s,x) = 0 + 0 + \int_{r'}^{r''} dr\,(0 + \dots 0 + \frac{dF_s(r)}{dr}) = F_s(r'') - F_s(r'), \tag{5.1}$$

which, because of (2.3) and (3.3), is identical with (3.2).

## 6. EXAMPLE: THE FIRST ORDER DEVIATION EQUATION (EQUATION OF MOTION)

In this section we shall derive the first approximation to the exact deviation equation (equation of motion) (4.9), or more precisely, we shall write down eq. (4.9) within terms of an order of $O(\rho'' - \rho')$ and $O((r'' - r')^2)$ with respect to the parameters of the families $\eta_s : [\rho', \rho''] \longrightarrow L_n$ and $\gamma_s : [r', r''] \longrightarrow L_n$, respectively (see section 2). As we shall see below, this approximation is independent of the concrete choice of the generalized transports $I^{\eta_s}$ and $I^{\gamma_s}$.

Having in mind that by definition (see section 2) $\eta_s(\rho') := x_1(\tau_1(s)) := \gamma_s(r')$, $\eta_s(\rho'') := x(s)$, $\gamma_s(r'') =: x_2(\tau_2(s))$ and $r \in [r', r'']$ and using eq. (2.6) from [3], sect. II.2.4 (which means that $I^{\dots}_{y \to y} := \mathrm{id}$ for any $y \in L_n$ – see [3], sect. II.1), we immediately derive from (4.4)

$$H^k_{.p} = \delta^k_p + O(\rho'' - \rho'), \qquad (H^{-1})^p_{.q} = \delta^p_q + O(\rho'' - \rho'),$$

$$\Lambda^p_{.q} = \delta^p_q + O(r'' - r'). \tag{6.1}$$

On differentiating these two-point tensors with respect to $s \in [s', s'']$, we get due to (4.1)

$$DH^k_{.p}/ds = D^2 H^k_{.p}/ds^2 = O(\rho'' - \rho'),$$

$$D\Lambda^p_{.q}/ds = D^2\Lambda^p_{.q}/ds^2 = O(r'' - r'). \tag{6.2}$$

By vertue of eq. (3.3) from [3], sect. II.3 we have



$$\int_{r'}^{r''} \Lambda^p_{.q}\,\dot\gamma^q_s(r)\,dr = \dot\gamma^p_s(r')(r'' - r') + O((r'' - r')^2)$$ (this result is a direct consequence of (6.1) and (6.6) (see below)); thus from (4.3) and (6.1), we find

$$h_{12}(s,x) = \xi + O((r'' - r')^2), \tag{6.3}$$

where the vector

$$\xi := \xi(s) := \dot\gamma_s(r')(r'' - r') \tag{6.4}$$

is called an infinitesimal (local) deviation vector and evidently has an order of $O(r'' - r')$.

Now substituting (6.1)-(6.3) into (4.9), we get

$$\frac{D^2}{ds^2}\Big| \xi^k = O(\rho'' - \rho') + O((r'' - r')^2) + \int_{r'}^{r''} dr\,[\,O(r' - r) +$$

$$+ R^k_{.ij\ell}(\gamma_s(r))\gamma'^i_s(r)\gamma'^j_s(r)\dot\gamma^\ell_s(r) + T^k_{.j\ell}(\gamma_s(r))F^j_s(r)\dot\gamma^\ell_s(r) +$$

$$+ DF^k_s(r)/dr + \gamma'^j_s(r)D(T^k_{.j\ell}(\gamma_s(r))\dot\gamma^\ell_s(r))/ds]. \tag{6.5}$$

But for any smooth function $f : [r', r''] \longrightarrow \mathbb{R}$ the following equality is fulfilled:

$$\int_{r'}^{r''} f(r)\,dr = f(r')(r'' - r') + O((r'' - r')^2). \tag{6.6}$$

(Proof: expand the integral in (6.6) into a Taylor series with respect to the difference $(r'' - r')$ at the point $r'$).

Thus, applying (6.6) to the integral in (6.5) and using (6.4), we find the following final result:

$$\frac{D^2}{ds^2}\Big|_x \xi^k = R^k_{.ij\ell}(\gamma_s(r'))\gamma'^i_s(r')\gamma'^j_s(r')\xi^\ell +$$

$$+ \frac{DF^k_s(r)}{dr}\Big|_{r=r'}(r'' - r') + T^k_{.j\ell}(\gamma_s(r'))F^j_s(r')\xi^\ell +$$

$$+ \gamma'^j_s(r')\frac{D(T^k_{.j\ell}(\gamma_s(r'))\xi^\ell)}{ds} + O(\rho'' - \rho') + O((r'' - r')^2). \tag{6.7}$$

If we neglect here the terms of an order of $O(\rho''-\rho')$ and $O((r''-r')^2)$, we shall get the first order deviation equation (or all the same the first order equation of motion) whose solutions (with respect to $\xi$) are the first approximation to the solutions of the general equation (4.9).

From (6.7) one can easily get the derived in [6] equation of relative motion (deviation equation) of two point particles which are moving along the curves $x(s,v)$ and $x(s,v+dv)$ ($s$ parametrizes their trajectories) of the family of $C^2$-curves $x(\sigma,\tau)$. Defining $u^\alpha := \partial x^\alpha(s,v)/\partial s$, substituting $x(s,v)$ for $\gamma_s(r)$, $v$ for $r'$, $v+dv$ for $r''$, $u^i$ for $\gamma_s'^{\,i}(r)$ and putting $\rho''=\rho'$, we get from (6.7):

$$\frac{D^2\xi^k}{\partial s^2} = R^k_{\cdot ij\ell}\, u^i u^j \xi^\ell + \left(\frac{DF^k_s(v)}{\partial v}\right) dv + T^k_{\cdot j\ell}\, F^j_s(v)\xi^\ell + u^j\,\frac{D(T^k_{\cdot j\ell}\,\xi^\ell)}{\partial s} + O((dv)^2), \tag{6.8}$$

where

$$\xi^\alpha = \frac{\partial x^\alpha(s,v)}{\partial v}dv, \qquad F^k_s(v) = \frac{Du^k}{\partial s}. \tag{6.9}$$

To get from (6.8) the deviation equation derived in [6], one has to neglect in (6.8) the terms $O((dv)^2)$ and to put $T^k_{\cdot ij}=0$ because in [6] the usual space-time of general relativity without torsion is used.

## 7. CONCLUSION

In this article we have shown that the equations of relative motion of two point particles in the field of external force $F_s(r)$, as they are observed from a third point particle (observer) which is influenced by another force field $F(r)$, are a special case of the generalized deviation equation.

One can give many examples, e.g., for the force field $F_s(r)$ (cf. [2]). For instance, if the observed particles are in an electromagnetic field described by a tensor $F_{ij}$, we have to put

$$F^j_s(r') = \frac{e'}{m'}\, g^{jk}(\gamma_s(r'))F_{k\ell}(\gamma_s(r'))\,\gamma_s'^{\,\ell}(r'),$$
$$F^j_s(r'') = \frac{e''}{m''}\, g^{jk}(\gamma_s(r''))F_{k\ell}(\gamma_s(r''))\,\gamma_s'^{\,\ell}(r''), \tag{7.1}$$

where $e', m'$ and $e''$ and $m''$ are the electric charges and the masses of the particles 1 and 2, respectively, and we have admitted that $L_n$ is endowed (may be independent of the connection) also with a metric tensor with contravariant components $g^{jk}$.

The values of $F_s(r)$ for $r \in (r',r'')$ have to be chosen from physical considerations conserning the choice of the curves $\gamma_s(r)$, whose physical meaning was mentioned in [4], sect. IV.2.

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Уравнение девиации как уравнение движения


На основе понятия о неинфинитезимальном векторе девиации дан новый вывод в пространствах аффинной связности обобщенного уравнения девиации в случае неявного задания /посредством дифференциальных уравнений второго порядка/участвующих в его определении кривых, которые могут быть как геодезические, так и негеодезические. Показано, что частным случаем этого уравнения являются уравнения относительного движения двух точечных частиц в пространствах аффинной связности в поле внешних сил. В качестве примеров рассмотрены "евклидовый случай" и общее инфинитезимальное /локальное/ уравнение девиации.



Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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The Deviation Equation as an Equation of Motion


On the basis of a noninfinitesimal deviation vector a new derivation (in spaces with the affine connection) of a generalized deviation equation is given for the case when the curves appearing in it, which can be geodesic as well as nongeodesic, are given implicitly (by second order differential equations). It is shown that the equations of relative motion of two point particles in spaces with the affine connection in the field of external forces are special cases of the generalized deviation equation. As examples the "Euclidean case" and the general infinitesimal (local) deviation equation are considered.



The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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