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A RELATIVISTIC GAUGE MODEL DESCRIBING N PARTICLES BOUND BY HARMONIC FORCES

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The recent development of the superstring theory has led to reconsidering some basic ideas concerning the relation between relativistic particle theory and local quantum field theory. In the modern approach (see, e.g. refs. /1,2/) one starts from a gauge-like formulation of the relativistic particle theory /3,4/ in which constraints, such as $p^2+m^2=0$, $p\xi=0$, etc., play a role of (super)gauge symmetry generators (in our notaion Greek/Latin characters are used for Fermi/Bose variables, $q^\mu(t)$ are coordinates; $p^\mu(t)$, momenta; $\xi^\mu(t)$, anticommuting space-time vectors, t is an evolution parameter). In the Lorentz invariant approach to quantizing such a theory one uses the FV /5/ gauge fixing conditions, $\dot{\ell}\equiv d\ell/dt=0, \dot{\lambda}=0$, where $\ell(t)\,/\,\lambda(t)$ are the Lagrange multipliers corresponding to the Fermi/Bose constraints. These constraints are included in the Lagrangian by adding the "kinetic" terms $\dot{\ell}k, i\dot{\lambda}\varkappa$, i.e. the Lagrange multipliers $k,\varkappa$ play the role of the conjugate momenta to ℓ,λ. To cancel all unphysical degrees of freedom, a further extension of the phase space is enforced by adding the Faddeev-Popov ghosts, in conjunction with the Parisi-Sourlas supersymmetry (see, e.g. /1,2/). In this formalism all variables have equal dynamical status (all variables are "equal"), and one can use the standard Liouville measure in the extended phase space. In the second-quantized theory this fact is reflected in the dependence of the field variables on all coordinates: q, ℓ, λ, ghost coordinates.

The meaning of the extra variables is however quite different, and some of them are more "equal" than others. The main characters in the described scenario are the constraints and the corresponding Lagrange multipliers $\ell(t), \lambda(t)$. The constraints define gauge-like transformations and the multipliers transform similarly to gauge potentials. For example, in the theory of the scalar particle,

$$L_0 = p_\mu \dot{q}^\mu - \frac{1}{2}\ell_1(t)(p^2+m^2), \quad 0\leq t\leq 1, \tag{1}$$

the constraint $g_1\equiv\frac{1}{2}(p^2+m^2)$ generates the abelian transformation through the Poisson brackets

$$\delta p=[f_1(t)g_1, p]_{P.B}=0, \quad \delta q=[f_1(t)g_1, q]_{P.B.}=f_1(t)p. \tag{2}$$

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The action for the Lagrangian (I) is invariant if

$$\delta \ell_1 = \dot{f}_1 , \quad f_1(0) = f_1(1) = 0. \tag{3}$$

It follows that $\ell_1^{(0)} \equiv \int_0^1 dt\, \ell_1(t)$ is invariant under the gauge transformation (3). The numbers $\ell_1^{(0)} \geq 0$ enumerate the inequivalent classes of the gauge group and are similar to the Teichmüller parameters in the string theory.

Taking into account the existence of the gauge invariants as well as the dynamical equality of the variables ℓ, λ with the basic variables p, q, ξ, one is tempted to interpret ℓ, λ as real gauge potentials. However, to fully support this point of view a Lagrangian is required with some rigid symmetry giving the gauge Lagrangian (like eq. (I)) by the standard procedure of gauging symmetries. The extremely simple model of eq. (1) is easy to rewrite in the standard gauge form /6/

$$L = \frac{1}{2} \overset{T}{\Psi} C (\partial_t - A) \Psi , \tag{4}$$

where

$$\Psi^\mu = \begin{pmatrix} p^\mu \\ q^\mu \end{pmatrix} , \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad A = \begin{pmatrix} 0 & 0 \\ \ell_1 & 0 \end{pmatrix} . \tag{5}$$

The transformations (2),(3) can also be presented in the standard form

$$\delta \Psi = F \Psi , \quad \delta A = \dot{F} + [F, A] \equiv \dot{F} + (FA - AF), \tag{6}$$

where the matrix F is obtained from A by simply substituting $f(t)$ for $\ell(t)$. The Lagrangian (4) can be obtained by gauging linear canonical symmetries of the following simple (rudimentary) Lagrangian /6/

$$L_0 = \frac{1}{2} \overset{T}{\Psi} C (\partial_t - H_0) \Psi + \Delta_B , \tag{7}$$

where

$$H_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \quad \dot{\Delta}_B = \frac{1}{2} (pq)^{\dot{}} .$$

One can easily convince oneself that the most general (not touching space-time indices) linear canonical transformation, that leaves (7) invariant, is given by eq. (2). The Lagrangian (7) is the simplest Poincare-invariant bilinear form, $L_0 = \frac{1}{2} \dot{q}^2$, written in the first order Lagrangian formalism. The boundary term Δ_B

influences only boundary conditions for $f(t)$, it also has to be added to eq. (4); usually we leave it aside.

This simple idea can further be exploited to obtain more complicated models /6/. First, for m=0 the Lagrangian (1) has an additional global symmetry of the Weyl type

$$\delta p = - f_2 p , \quad \delta q = f_2 q , \quad \delta \ell_1 = 2 f_2 \ell_1 .$$

Gauging this symmetry gives a new gauge theory which is obtained from (1) by simply substituting

$$F = \begin{pmatrix} -f_2 & 0 \\ f_1 & f_2 \end{pmatrix} , \quad A = \begin{pmatrix} -\ell_2 & 0 \\ \ell_1 & \ell_2 \end{pmatrix}$$

into (6) and (4). The particle described by this theory is massless due to the non-abelian nature of the gauge transformation for ℓ_1:

$$\delta \ell_1 = \dot{f}_1 + 2 f_2 \ell_1 - 2 f_1 \ell_2 , \quad \delta \ell_2 = \dot{f}_2 .$$

Presumably, this model gives a particle description of the dilaton.

The other extension of the idea consists in using rudimentary Lagrangians depending on anticommuting variables ξ and describing spinning particles /7,8,3,4/. Such Lagrangians have some supercanonical symmetries by gauging which one obtains, in an extremely simple and transparent manner, general supergauge theories (one-dimensional supergravities) of particles with different spins described by supercoordinates ξ_k^μ, $k = 1, \dots, K$. For K=1,2 such theories were first obtained in ref./3/ by using a rather complicated superspace formalism; to other values of K the results were extended in /9/.

These observations reveal a rather general principle of gauging linear (super)canonical symmetries of bilinear rudimentary Lagrangians. Employing this principle allows one to construct in a transparent and unified manner all known models of relativistic particles as well as gauge formulations of bosonic and fermionic string theory /10/. In addition, quite new theories can be derived. A nontrivial example has been given in ref./10/ - a relativistic gauge theory of 2 and 3 scalar particles bound by linear (harmonic) forces. As pointed out in /10/ the approach can be used to obtain the N-particle theory, however, the identification of the relevant N-particle gauge group, given in/10/ is incorrect. Here, a general relativistic theory of N particles bound by harmonic forces is given. It can be applied to hadrons, strings, membranes, etc.

First we present a rather general formulation of our approach to gauging canonical symmetries. Extending the ideas of refs. /7,8,3,6/ , consider the following rudimentary Lagrangian

$$L_0 = g_{\mu\nu} p_i^{\mu} \dot{q}_i^{\nu} - \frac{i}{2} h_{\alpha\beta} \xi^{\alpha} \dot{\xi}^{\beta} - \mathcal{H}_0(p, q, \xi), \tag{8}$$

where the index $i = 1, \ldots, N$ enumerates the particles. The constant matrices $g_{\mu\nu}$ and $h_{\alpha\beta}$ can be diagonalized by suitable linear transformations of canonical variables. Neglecting the new ones corresponding to zero eigenvalues, we obtain $g_{\mu\nu} = (-1, \ldots, -1, +1, \ldots, +1)$. It can be shown that the quantum theory of the gauge invariant Lagrangian (I) (and of its generalizations) is consistent only for the Minkovski signatire, i.e. $g_{\mu\nu} = (-1, +1, \ldots, +1)$, otherwise the Hilbert space of the system has indefinite metric . In that sense, the gauge principle implies relativistic invariance (for quantizing the theory one can employ the Dirac method /3,7–9/ or modern approaches /5,4/). In what follows we use the Minkovski metric $g_{\mu\nu}$ and suppress all contracted space-time indices μ, ν; Lorentz invariance is trivially satisfied everywhere. The anti-commuting variables ξ may be chosen, to some extent, arbitrarily, and this allows one to describe spin and internal degrees of freedom (e.g., adding to (I) the term $-\frac{i}{2}\xi^{\mu}\dot{\xi}_{\mu}$ gives the spin 1/2 massless particle Lagrangian, adding to that $-\frac{i}{2}\xi^{D}\dot{\xi}^{D}$ gives the theory of the Dirac particle) . The Lagrangian (8) can easily be written in the standard form (I). Its rigid supercanonical symmetries, $\delta\Psi = F(f, \varphi)\Psi$, satisfy the conditions

$$\overset{T}{F} C + CF = 0, \quad [F, H_0] \equiv FH_0 - H_0 F = 0 \tag{9}$$

(remember that the transposed supermatrix, $\overset{T}{F}$, is defined so as to preserve the relation $(F\Psi)^T = \overset{T}{\Psi}\overset{T}{F}$ with due respect to anti-commutativity). Now the gauged Lagrangian, L, that is invariant under the local transformations, $\delta\Psi = F(f(t), \varphi(t))\Psi$, can be presented in the form (4), where the supermatrix $A(\ell, \lambda)$ is obtained from $F(f, \varphi)$ simply by substituting $f \to \ell$, $\varphi \to \lambda$. The gauge transformations of A are defined by the standard formula (6).

To derive the boundary conditions for the gauge parameters $f(t)$, $\varphi(t)$ one has to calculate the variation of the boundary term Δ_B . This completes formulating our gauge construction.

A more practical approach to determining the rigid symmetry group of the rudimentary Lagrangians as well as to constructing the corresponding gauge theory is based on using, instead of the super-matrices F, the generating function of the supercanonical transformations

$$G(p, q, \xi) = \sum_a f_a g_a + \sum_{\alpha} \varphi_{\alpha} \gamma_{\alpha}, \quad \delta X = [G, X]_{P.B.} \tag{10}$$

$$\delta p = -\frac{\partial G}{\partial q}, \quad \delta q = \frac{\partial G}{\partial p}, \quad \delta\xi = i\frac{\partial^L G}{\partial \xi}. \tag{11}$$

Under local symmetry transformations, $[G, H_0]_{P.B.} = 0$, and H_0 is unchanged, while the variation of Lagrangian (8) is

$$\delta L_0 = \frac{d}{dt}\left[p\frac{\partial G}{\partial p} + \frac{1}{2}\xi\frac{\partial^L G}{\partial \xi} - G\right] + \dot{f}\frac{\partial G}{\partial f} + \dot{\varphi}\frac{\partial^L G}{\partial \varphi}. \tag{12}$$

The first term defines the boundary conditions for $f(t), \varphi(t)$, and other terms are cancelled by adding to (8) the obvious compensating terms

$$-\frac{1}{2}\overset{T}{\Psi} C A \Psi = -\sum_a \ell_a(t) g_a(p, q, \xi) - \sum_{\alpha} \lambda_{\alpha}(t)\gamma_{\alpha}(p, q, \xi), \tag{13}$$

where $\overset{T}{\Psi} = (p_i, q_i, \xi_{\alpha})$. The transformation law for the gauge potentials ℓ, λ can be derived either from eq. (6) or directly by applying to the new Lagrangian the requirement of gauge invariance (remember that the superalgebra of the generators is closed with respect to the Poisson brackets, due to the condition $[G, H_0]_{P.B} = 0$).

Now we apply the general approach to constructing relativistic gauge models for N particles bound by harmonic forces. To simplify the presentation we only treat here the spinless particles. Then, the rudimentary Lagrangian is

$$L_0 = p_i \dot{q}_i - \frac{1}{2} p_i p_i - \frac{1}{4} v_{ij} (q_i - q_j)^2, \quad v_{ij} = v_{ji}, \quad v_{ii} = 0. \tag{14}$$

The most general linear canonical transformation is defined by the generating function

$$G = \frac{1}{2} a_{ij} p_i p_j + b_{ij} p_i q_j + \frac{1}{2} c_{ij} q_i q_j \equiv \frac{1}{2}\overset{T}{\Psi}\begin{pmatrix} a & b \\ b^T & c \end{pmatrix}\Psi, \tag{15}$$

where $a_{ij} = a_{ji}$, $c_{ij} = c_{ji}$ (remind that we are not considering the Lorentz transformations and all indices μ, ν are contracted).

The Lagrangian (14) is invariant under the transformations (11)(or, $\delta\Psi = c^{-1}\,\partial^L G/\partial\Psi$) if and only if

$$[V,a] = 0 = [V,b] , \quad b^T = -b , \quad c = -Va , \tag{16}$$

where V is the following NxN matrix

$$V_{ii} = -\sum_{j=1}^{N} v_{ij} , \quad V_{ij} = v_{ij} , \quad i \neq j . \tag{17}$$

Equations (16) leave in G not less than N commuting generators which are some linear combinations of the bilinear Lorentz invariants $p_i p_j, p_i q_j, q_i q_j$. Therefore, the time components of the coordinates and momenta can always be excluded by solving $\geqslant N$ constraints together with the same number of gauge fixing conditions.

The physics content of the gauge Lagrangian corresponding to the rudimentary Lagrangian (14) crucially depends on the coupling parameters v_{ij} . If $v_{ij} \equiv v_0$, for all i,j , the Lagrangian describes the system of N identical particles with pair harmonic coupling. The gauge group in that case is $T_1 \otimes U_1 \otimes SU_{N-1}$. This can be shown with the aid of the general formulae (15)-(17). To see this more directly we introduce new canonical coordinates. Define center-of-mass coordinates and momenta

$$Q = \frac{1}{\sqrt{N}} \sum q_i , \quad \mathcal{P} = \frac{1}{\sqrt{N}} \sum p_i ,$$

and choose other coordinates y_i and momenta z_i $(i = 1,\dots, N-1)$ so as to diagonalize the Lagrangian :

$$L_0 = \mathcal{P}\dot{Q} - \frac{1}{2}\mathcal{P}^2 + z_i \dot{y}_i - \frac{1}{2} z_i z_i - \frac{1}{2} y_i y_i \tag{18}$$

(the parameter v_0 is absorbed in coordinates, with due rescaling of t). Applying our general construction we arrive at the gauge Lagrangian

$$L = \mathcal{P}\dot{Q} + z_i \dot{y}_i - \frac{1}{2}\ell_0(\mathcal{P}^2 + M^2) - \frac{1}{2}\ell_1(z_i z_i + y_i y_i - \mathcal{P}^2 - m^2) - \frac{1}{2}\ell^s_{ij}(z_i z_j + y_i y_j) - \frac{1}{2}\ell^a_{ij}(z_i y_j - z_j y_i) , \tag{19}$$

where

$$\ell^s_{ij} = \ell^s_{ji} , \quad \sum_{i=1}^{N-1} \ell^s_{ii} = 0 , \quad \ell^a_{ij} = -\ell^a_{ji} . \tag{20}$$

Here the constraint coupled to ℓ_0 generates the translations T_1, the one coupled to ℓ_1 generates U_1, and the others give the algebra of SU_{N-1} . (The constraints coupled to ℓ^s_{ii} generate its Cartan subalgebra). In writing eq. (19) we have used the abelian nature of the T_1 and U_1 generators which allows one to add the mass parameters M^2, m^2 without destroying the gauge symmetry (likewise, the term $-\mathcal{P}^2$ in the U_1 generator, commuting with all generators, can be removed or multiplied by an arbitrary number). If the pair couplings are not identical, i.e. v_{ij} depend on i,j the SU_{N-1} group will be broken. Note that the gauge group for N=2 is $T_1 \otimes U_1$. To obtain the corresponding Lagrangian from eq. (19) one simply has to set $z_i = y_i = 0$, $i \geqslant 2$, and to keep the first two constraints.

A most natural approach to quantizing this theory is that described in the introductory paragraph of this letter. Hopefully, the application of modern methods /1,2/ will allow one to develop both a relativistic quantum theory of free composite particles and an effective quantum field theory describing their interactions.

To obtain a theory of discrete "strings", i.e. of linear chains of particles bound by harmonic forces, we choose $v_{ij} = \delta_{|i-j|,1}$, for open strings, and $v_{ij} = \delta_{|i-j|,1} + \delta_{iN}\delta_{j1} + \delta_{i1}\delta_{jN}$, for closed ones, and employ the general formulae (16),(17). The detailed derivation will be presented elsewhere and here we only calculate the number of the gauge parameters. The equations for b_{ij} are easy to solve. For the open string $b_{ij} \equiv 0$, and for the closed one the condition $[V,b] = 0$ is equivalent to the relations

$$b_{ij} = b_{j-i} , \; i<j \; ; \quad b_{ij} = b_{N-|i-j|} , \quad i>j .$$

Together with $b_{ij} = -b_{ji}$, this leaves $[(N-1)/2]$ independent parameters $b_1, \dots, b_{[(N-1)/2]}$, where the square brackets denote the integer part of the enclosed number. The most difficult to solve are the equations $[V,a] = 0$. For the open string there are N independent parameters a_{ij} , as

$$a_{ij} = \sum_{\ell=j-i+1}^{j+i-1} a_{1\ell}(-1)^{\ell-j+i-1} , \quad i \leqslant j \leqslant N - i + 1 ;$$

$$a_{ij} = a_{N-j+1,N-i+1} , \quad i+j > N+1 .$$

For the closed string the equations $[V,a]=0$ are rather complicated due to periodicity conditions. However, the number of independent parameters a_{ij} is easy to calculate, it is $[(3N-1)/2]$. The total number of independent gauge parameters for the closed string is $2N-2$, N even; $2N-1$, N odd.

The generators correspond to the Virasoro generators for the closed string. The detailed derivation of the discrete string gauge algebra will be presented elsewhere. Note that the closed "string" with N=3 is described by eq. (19).

In conclusion we mention some possible extensions and applications of our results. By adding suitable Grassmann variables one can describe the bound states of N spinning particles having internal degrees of freedom. Similarly one can construct discrete strings of different sorts, e.g. compactified on tori or orbifolds. The theory of N-particle bound states can be applied to the quark model of hadrons; while the theory of discrete strings, to an approximate description of massless string states in realistic models. A quantum field theory of discrete strings is possibly simpler than that of continual ones. Finally, the scheme proposed here can in principle be applied to constructing other relativistic discrete theories, e.g., to membranes (i.e. two-dimensional lattices of particles with nearest-neighbour harmonic couplings). To find the gauge group in that case is a more complicated technical problem.

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Релятивистская калибровочная модель N частиц, связанных гармоническими силами

Применение принципа локализации к линейным каноническим симметриям простейших /рудиментарных/ билинейных лагранжианов позволяет получить релятивистскую версию лагранжиана для частиц, связанных гармоническими силами. Для попарно связанных тождественных частиц калибровочная группа есть $T_1 \otimes U_1 \otimes SU_{N-1}$. Указано также как строится модель линейной цепочки N частиц /дискретная релятивистская струна/. Предложенные калибровочные теории частиц можно квантовать стандартными методами.

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A Relativistic Gauge Model Describing N Particles Bound by Harmonic Forces

Application of the principle of gauging to linear canonical sammetries of simplest /rudimentary/ bilinear lagrangians is shown to produce a relativistic version of the Lagrangian describing N particles bound by harmonic forces. For pairwise coupled identical particles the gauge group is $T_1 \otimes U_1 \otimes SU_{N-1}$. A model for the relativistic discrete string /a chain of N particles/ is also discussed. All these gauge theories of particles can be quantized by standard methods.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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