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дубна

E2-87-679

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ON THE ENERGY-MOMENTUM TENSOR FOR THEORIES IN (PSEUDO)RIEMANNIAN SPACES WITH TORSION

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1987

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О тензоре энергии-импульса в /псевдо/римановых пространствах с кручением

Исследованы возможности получения обобщенных ковариантных тождеств типа Бианки для лагранжевых плотностей, зависящих от компонент ковариантных тензорных полей /конечного ранга/ и их первых и вторых ковариантных производных в /псевдо/римановых пространствах с кручением. Определены понятия канонического, обобщенного канонического и симметричного тензора энергии-импульса как локально сохраняющейся величины. Показано, что уравнения Эйлера - Лагранжа для материальной системы в /псевдо/римановом пространстве с кручением /в отличие от случая в пространстве без кручения/ не являются достаточным условием для существования симметричного тензора энергии-импульса как локально сохраняющейся величины.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1987

Manoff S. E2-87-679

On the Energy-Momentum Tensor for Theories in (Pseudo)Riemannian Spaces with Torsion

Generalized covariant Bianchi type identities are obtained and investigated for Lagrangian densities, depending on covariant tensor fields and their first and second covariant derivatives in (pseudo)Riemannian spaces with torsion (U_n - spaces). The notions of canonical, generalized canonical and symmetric energy-momentum tensor are introduced and necessary and sufficient conditions for the existence of the symmetric energy-momentum tensor as a local conserved quantity are obtained. The Euler - Lagrange equations for a material system in U_n-space (in contrast to the case in V_n-space) cannot be used as sufficient conditions for the existence of the energy-momentum tensor as a local conserved quantity.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1987

INTRODUCTION

1. In the structure of the gravitational equations in Einstein's theory of gravitation (ATG) and its variants in (pseudo)-Riemannian spaces without torsion (V_n-spaces, n:= 4), the symmetric energy-momentum tensor ${}_sT_{ij}$ (:= ${}_sT_{ji}$) of a material system takes the role of determining the field sources quantity. Its properties are connected with the characteristics of the material system as well as with the conditions, required by the gravitational equations. Among that, the condition for ${}_sT_{ij}$ to be a local conserved quantity, i.e. ${}_sT_i{}^j{}_{;j} := 0$, ${}_sT_i{}^j := g^{jk}\cdot{}_sT_{ik}$, is considered as one of the main properties of this tensor. Such a condition, on the one hand, restricts the possibilities for a choice of the gravitational equations, and, on the other, is determined by means of supplementary conditions (existence of a special type of vector fields in V_n-spaces, e.g. Killing vectors) for so-called global conservation laws for the material distribution /1,2,6/.

2. On the basis of the Lagrangian formalism and by means of the method of Lagrangians with covariant derivatives (MLCD)/5/ in V_n-spaces for Lagrangian densities, depending on the components of tensor fields and their first and second derivatives, the so-called generalized covariant Bianchi type identities (GCBI) /5/ are obtained in the form:

$$\bar{F}_k + \bar{\theta}_k{}^i{}_{;i} = 0 , \qquad (o.1)$$

$$\bar{\theta}_k{}^i - {}_s\bar{T}_k{}^i = \bar{Q}_k{}^i , \qquad (o.2)$$

where

:= means "by definition",

$$\bar{F}_k := \frac{\delta L}{\delta V_A}.V_{A;k} \ , \qquad (o.3)$$

$$\frac{\delta L}{\delta V_A} := \frac{\partial L}{\partial V_A} - \left(\frac{\partial L}{\partial V_{A;i}}\right)_{;i} + \left(\frac{\partial L}{\partial V_{A;i;j}}\right)_{;j;i} \ , \qquad (o.4)$$

$\frac{\delta L}{\delta V_A}$:= functional variation of L with respect to V_A ,

$$L := L(g_{ij},\ V_A,\ V_{A;i},\ V_{A;i;j}) \ , \qquad (o.5)$$

$$\mathcal{L} := \mathcal{L}(g_{ij},\ V_A,\ V_{A;i},\ V_{A;i;j}) := \sqrt{-g}.L \ , \qquad (o.6)$$

L := Lagrangian invariant (scalar function),

$\mathcal{L}$:= Lagrangian density,

$V_A := V_{j_1 j_2 \dots j_l}$, $A := j_1 \dots j_l$:= collective index,

$l := N < \infty$, rank $V_A := N_{V_A}$,

$g_{ij} := g_{ji}$:= components of the metric tensor , $g := \det(g_{ij}) < 0$,

$$g_{ij;k} := 0 \quad \text{(Riemannian metric)} \ , \qquad (o.6a)$$

$;i$:= covariant derivative with respect to the coordinates x^i ($i := 1, \dots, n$, special case $n := 4$),

$$V_{A;i} := V_{A,i} - \Gamma^B_{Ai}.V_B \ , \quad \Gamma^B_{Ai} := - S_{Am}{}^{Bj}.\Gamma^m_{ji} \ , \qquad (o.7)$$

$A := j_1 \dots j_l$, $B := i_1 \dots i_l$,

$$S_{Am}{}^{Bi} := - \sum_{k=1}^{l} g^i_{j_k}.g^{i_k}_m.g^{i_1}_{j_1}.g^{i_2}_{j_2}\dots g^{i_{k-1}}_{j_{k-1}}.g^{i_{k+1}}_{j_{k+1}}\dots g^{i_l}_{j_l} \ , \qquad (o.8)$$

$$S_{Am}{}^{Bi} := - g^i_A.g^B_m \ ,$$

$\Gamma^m_{ij} := \Gamma^m_{ji}$:= components of the symmetric affine connection Γ, (o.9)

$\frac{\delta L}{\delta V_A} := 0$:= Euler-Lagrange equations for the field V_A, (o.10)

$\bar{\Theta}_k{}^i$:= generalized canonical energy-momentum tensor for the Lagrangian system,

$$\bar{\Theta}_k{}^i := \bar{t}_k{}^i - \bar{W}_k{}^{il}{}_{;l} - \bar{F}_n{}^{jil}.R^n{}_{ljk} \ , \qquad (o.11)$$

$\bar{t}_k{}^i$:= canonical energy-momentum tensor for the Lagrangian system,

$$\bar{t}_k{}^i := \left[\frac{\partial L}{\partial V_{A;i}} - \left(\frac{\partial L}{\partial V_{A;i;j}} + \frac{\partial L}{\partial V_{A;j;i}}\right)_{;j}\right] V_{A;k} + \left(\frac{\partial L}{\partial V_{A;j;i}}.V_{A;k}\right)_{;j} - g^i_k.L \ , \qquad (o.12)$$

$$\bar{W}_k{}^{il} := - \bar{W}_k{}^{li} := g_{kj}(\mathcal{F}_n{}^{ij}g^{nl} - \mathcal{F}_n{}^{lj}g^{ni} - A_n{}^{il}g^{nj}) \ ,$$

$$\mathcal{F}_k{}^{il} := \mathcal{F}_k{}^{li} := \tfrac{1}{2}(\bar{V}_k{}^{il} + \bar{V}_k{}^{li}) := \bar{V}_k{}^{(il)} \ ,$$

$$A_k{}^{il} := - A_k{}^{li} := \tfrac{1}{2}(\bar{V}_k{}^{il} - \bar{V}_k{}^{li}) := \bar{V}_k{}^{[il]} \ , \qquad (o.13)$$

$$\bar{V}_k{}^{ij} := \mathcal{F}_k{}^{ij} + A_k{}^{ij} := \bar{F}_k{}^{ij} - \bar{F}_k{}^{ilj}{}_{;l} \ ,$$

$$\bar{F}_k{}^{ij} := \frac{\partial L}{\partial V_{A;i}}.S_{Ak}{}^{Bj}.V_B - \frac{\partial L}{\partial V_{A;j;i}}.V_{A;k} + \left(\frac{\partial L}{\partial V_{A;i;l}} + \frac{\partial L}{\partial V_{A;l;i}}\right)S_{Ak}{}^{Bj}.V_{B;l} \ ,$$

$$\bar{F}_k{}^{ilj} := \frac{\partial L}{\partial V_{A;i;l}}.S_{Ak}{}^{Bj}.V_B \ , \qquad \frac{\partial L}{\partial V_{A;i}} := \frac{\partial L}{\partial (V_{A;i})} \ ,$$

${}_sT_k{}^i$:= symmetric energy-momentum tensor for the Lagrangian system,

$${}_sT_k{}^i := \mathcal{T}_k{}^i - 2.\frac{\partial L}{\partial g_{il}}.g_{kl} - g^i_k.L \ , \qquad (o.14)$$

$$\mathcal{T}_k{}^i := g_{kj}\mathcal{T}^{ji} \ , \qquad \mathcal{T}^{ji} = \mathcal{T}^{ij} := \mathcal{T}^{jil}{}_{;l} \ ,$$

$$\mathcal{T}^{ij} := \mathcal{T}^{ijl}{}_{;l} := (\mathcal{F}_k{}^{il}g^{kj} + \mathcal{F}_k{}^{jl}g^{ki} - \mathcal{F}_k{}^{ij}g^{kl})_{;l} \ ,$$

$$\mathcal{T}^{ijl} := \mathcal{F}_k{}^{il}g^{kj} + \mathcal{F}_k{}^{jl}g^{ki} - \mathcal{F}_k{}^{ij}g^{kl} \ ,$$

$$\bar{Q}_k{}^i := \frac{\delta L}{\delta V_A}.S_{Ak}{}^{Bi}.V_B \ , \qquad (o.15)$$

Lagrangian system := a totality of field variables g_{ij}, V_A (with $V_{A;i}$ and $V_{A;i;j}$), characterized by means of the scalar density $\mathcal{L}$ of the type

$$\mathcal{L} := \sqrt{-g}.L(g_{ij},\ V_A,\ V_{A;i},\ V_{A;i;j}) \ , \qquad V_A :\neq g_{ij}, \qquad (o.16)$$

called a Lagrangian density /5/.

The generalized covariant Bianchi type identities (GCBI) contain the connection between the tensors of the type of the energy-momentum tensor and the field equations of the type of Euler-Lagrange equations $\delta L/\delta V_A = 0$ for the tensor field V_A. Besides, as a local conserved quantity of the type of an energy-momentum tensor for a Lagrangian system a tensor quantity is defined, who's components satisfy the condition:

$$T_k{}^i{}_{;i} := 0, \quad T_k{}^i := g_{kl}.T^{li} = g^{ij}.T_{kj} , \qquad (o.17)$$

where $T_k{}^i$ are the components of the above-mentioned energy-momentum tensors.

Under local conserved quantities for a Lagrangian system we will understand in this paper local conserved quantities of the type of an energy-momentum tensor.

3. By means of the GCBI in V_n-spaces necessary and sufficient as well as only necessary or only sufficient conditions could be found for the existence of the symmetric energy-momentum tensor as local conserved quantity. One can prove in this case /5/ that Euler-Lagrange equations are sufficient conditions for the symmetric energy-momentum tensor to be a local conserved quantity, i.e. ${}_sT_i{}^j{}_{;j} = 0$.

4. In Einstein's theory of gravitation (ATG) and its variants in V_n-spaces, a system of equations is considered containing the equations for the gravitational field and the Euler-Lagrange equations (as field equations for the material distribution). In this case the requirement ${}_sT_k{}^i$ to be a local conserved quantity with respect to the complete system of equations (for the material distrbution + the gravitational field)(s.c. selfconsistent system) is satisfied. Considering the field equations of Euler-Lagrange type for material distrbution under a given metric (and symmetric affine connection) in V_n - space, the condition ${}_sT_k{}^i{}_{;i} = 0$ is also satisfied, but in this case equations for the gravitational field of the distribution are not given (s.c. nonconsistent case for test material systems in an external gravitational field).

The generalization of ATG for (pseudo)Riemannian spaces with nonsymmetric affine connection ($\Gamma^i_{jk} \ :\neq \Gamma^i_{kj}$) and Riemannian metric ($g_{ij/k} := 0$)(s.c. (pseudo)Riemannian spaces with torsion, Riemann-Cartan spaces, U_n-spaces) imposes the introduction of the torsion tensor along with the metric tensor as field functions for the gravitational field /3,4/. In gravitational theories from the Einstein-Cartan type in U_n -spaces /3/, the problem also arises for obtaining the generalized covariant Bianchi type identities for Lagrangian systems and for finding conditions for the existence of the symmetric energy-momentum tensor as local conserved quantity by means of the MLCD.

5. In this paper the possibilities are considered for obtaining the GCBI for Lagrangian densities, depending on components of covariant tensor fields and their first and second covariant derivatives in U_n-spaces (Sec. I.). In analogy with the tensors of the type of energy-momentum tensor in V_n-spaces for a Lagrangian system, the notions as canonical, generalized canonical and symmetric energy-momentum tensor are introduced (Sec.II.). Necessary and sufficient conditions are obtained for the existence of the symmetric energy-momentum tensor as a local conserved quantity. It is shown that Euler-Lagrange equations for a material system in U_n-space in contrast to the case in V_n-space cannot be sufficient conditions for the existence of the symmetric energy-momentum tensor as a local conserved quantity (Sec.III.).

I. GENERALIZED COVARIANT BIANCHI TYPE IDENTITIES FOR LAGRANGIAN DENSITIES IN (PSEUDO)RIEMANNIAN SPACES WITH TORSION

1. Generalized covariant Bianchi type identities (GCBI) can be found by means of the MLCD in U_n-spaces in analogous way

with that in V_n-spaces by taking into account the characteristics of a U_n-space, considered as an n-dimensional C^k-differentiable manifold (k = 2,3,...) with a nonsymmetric affine connection and a Riemannian metric with respect to this connection, i.e.

$$\dim U_n := n \quad (\text{special case } n := 4), \quad \Gamma^i_{jk} :\neq \Gamma^i_{kj}, \tag{1.1}$$

Γ^i_{jk} := components of the nonsymmetric connection Γ,

$g_{ij} := g_{ji}$:= components of the metric tensor g,

$g_{ij/k} := 0$ (Riemannian metric), (1.2)

$/k$:= covariant derivative with respect to the basic vector field E_k,

$\{E_k\}_x$:= basis in p. $x \in T_x(U_n)$, $T_x(U_n)$:= tangential space in p. $x \in U_n$, (if $E_k := \partial_k$, i.e. E_k is a coordinate basis, then $/k$ means covariant derivative with respect to the coordinate x^k),

$$\mathcal{L}_u v^i := v^i{}_{/j} u^j - u^i{}_{/j} v^j - T_{jk}{}^i u^j v^k, \quad v := v^i E_i, \; u := u^k E_k,$$

$$T_{jk}{}^i := - T_{kj}{}^i := \Gamma^i_{kj} - \Gamma^i_{jk} - C_{jk}{}^i \quad (\text{in noncoordinate basis}),$$
$$:= \Gamma^i_{kj} - \Gamma^i_{jk} \quad (\text{in coordinate basis}),$$

$$\mathcal{L}_{E_j} E_k := [E_j, E_k] := E_j.E_k - E_k.E_j := C_{jk}{}^i E_i,$$

$\mathcal{L}_u v^i$:= Lie-derivative of the components of the vector field v along the vector field u.

Remark: All further investigations in this paper will be made for the components of tensor fields in any fixed basis if the opposite is not explicitly mentioned.

$$v^i{}_{/j} := v^i{}_{,j} + \Gamma^i_{kj}.v^k, \quad v^i{}_{,j} := E_j v^i \;\; (= \partial_j v^i).$$

2. The following steps could be made in obtaining the GCBI and by their representation in some useful forms:

A. Representation of the Lie-variation of the Lagrangian density $\mathfrak{L} := \sqrt{-g}.L$ along any vector field ξ by means of the Lie-derivatives of the tensor fields components and

their first and second covariant derivatives along this vector field, obtaining an identity for L.

B. Representation of the Lie-derivatives of the tensor fields components and their first and second covariant derivatives only by means of covariant derivatives and the components of the torsion tensor, using the commutation relations between Lie- and covariant derivatives and the connection between Lie- and covariant derivatives. Writing the identity for L in a form depending on the vector field components and their first covariant derivatives.

C. Obtaining the GCBI from the identity for L under the condition of arbitrariness of the vector field ξ.

◇

A. The Lie-derivative of a Lagrangian density of the type

$$\mathfrak{L} := \sqrt{-g}.L(g_{ij}, V_A, V_{A;i}, V_{A;i;j}), \tag{1.3}$$

where

$$g := \det(g_{ij}) :\neq 0, \infty, \quad -g > 0,$$
$$V_{A/i} := V_{A,i} - \Gamma^B_{Ai}.V_B := V_{A,i} + S_{Am}{}^{Bj}\Gamma^m_{ji}.V_B,$$

along an arbitrary vector field ξ can be written in the following form:

$$\mathcal{L}_\xi \mathfrak{L} = \sqrt{-g}.(L_{/i}\xi^i + \tfrac{1}{2}.L.g^{ij}\mathcal{L}_\xi g_{ij}) =$$
$$= \sqrt{-g}[(L_{/i} - T_{ki}{}^k.L)\xi^i + g^j_i.\xi^i{}_{/j}.L] = \tag{1.4}$$
$$= \sqrt{-g}[(L\xi^i)_{/i} - T_{ki}{}^k \xi^i.L],$$

$$T_{ki}{}^k := T^k_{ki} := g^k_j.T_{ki}{}^j, \quad L_{/i} = L_{,i}.$$

If one assumes that the functional variation $\delta\mathfrak{L}$ of a Lagrangian density $\mathfrak{L}$

$$\delta\mathfrak{L} := \frac{\partial\mathfrak{L}}{\partial V_A}.\delta V_A + \frac{\partial\mathfrak{L}}{\partial V_{A/i}}.\delta(V_{A/i}) + \frac{\partial\mathfrak{L}}{\partial V_{A/i/j}}.\delta(V_{A/i/j}) + \frac{\partial\mathfrak{L}}{\partial g_{ij}}.\delta g_{ij}, \tag{1.5}$$

where

$$\frac{\partial\mathfrak{L}}{\partial V_{A/i}} := \frac{\partial\mathfrak{L}}{\partial(V_{A/i})}, \quad \frac{\partial\mathfrak{L}}{\partial V_{A/i/j}} := \frac{\partial\mathfrak{L}}{\partial(V_{A/i/j})},$$

is connected with the variation of the field functions V_A (and their first and second covariant derivatives) and g_{ij} along the vector field ξ, i.e. $\delta := \pounds_\xi$, then the so-called Lie-variation $\pounds_\xi \mathfrak{L}$ of the Lagrangian density $\mathfrak{L}$, equal to the Lie-derivative of $\mathfrak{L}$ along ξ, can be written in the form:

$$\pounds_\xi \mathfrak{L} = \sqrt{-g}.(L_{/i}\xi^i + \frac{1}{2}.L.g^{ij}\pounds_\xi g_{ij}) =$$
$$= \frac{\partial \mathfrak{L}}{\partial V_A}\pounds_\xi V_A + \frac{\partial \mathfrak{L}}{\partial V_{A/i}}\pounds_\xi (V_{A/i}) + \frac{\partial \mathfrak{L}}{\partial V_{A/i/j}}\pounds_\xi (V_{A/i/j}) +$$
$$+ \frac{\partial \mathfrak{L}}{\partial g_{ij}}\pounds_\xi g_{ij} \quad . \qquad (1.6)$$

From the last identity for $\mathfrak{L}$ there follows the identity for the scalar function (invariant) L:

$$\frac{\partial L}{\partial V_A}.\pounds_\xi V_A + \frac{\partial L}{\partial V_{A/i}}.\pounds_\xi (V_{A/i}) + \frac{\partial L}{\partial V_{A/i/j}}.\pounds_\xi (V_{A/i/j}) +$$
$$+ \frac{\partial L}{\partial g_{ij}}\pounds_\xi g_{ij} - L_{/i}\xi^i = 0 \, . \qquad (1.7)$$

The variations of the field functions V_A and of the metric tensor g_{ij} along the vector field ξ are given by means of their Lie-derivatives along ξ, and the variation of the nonsymmetric affine connection (and with it also the variation of the components of the torsion tensor) is given implicitly by the Lie-derivatives of the covariant derivatives of V_A.

B. Using the explicit form of the Lie-derivative of the tensor components V_A and the commutation relations between the Lie-derivatives and the first and second covariant derivatives of V_A, the identity for L can be expressed in the form:

$$\frac{\partial L}{\partial V_A}.\pounds_\xi V_A + \frac{\partial L}{\partial V_{A/j}}.\pounds_\xi (V_{A/j}) + \frac{\partial L}{\partial V_{A/j/k}}.\pounds_\xi (V_{A/j/k}) + \frac{\partial L}{\partial g_{jk}}.\pounds_\xi g_{jk} -$$
$$- L_{/i}\xi^i = (\bar{P}_i + \bar{P}_k{}^j.T_{ij}{}^k)\xi^i + \bar{P}_i{}^j\xi^i{}_{/j} = 0 \, , \quad (1.8)$$

where

$$\pounds_\xi V_A = (V_{A/i} + T_{ki}{}^j.S_{Aj}{}^{Bk}.V_B)\xi^i - S_{Ai}{}^{Bj}.V_B.\xi^i{}_{/j} \, ,$$
$$\pounds_\xi (V_{A/j}) = (V_{A/j/i} - S_{Al}{}^{Bk}.V_{B/j}.T_{ik}{}^l + V_{A/l}.T_{ij}{}^l)\xi^i +$$
$$+ V_{A/i}\xi^i{}_{/j} - S_{Ai}{}^{Bk}.V_{B/j}.\xi^i{}_{/k} \, ,$$
$$\pounds_\xi (V_{A/j/k}) = (V_{A/j/k/i} - S_{Al}{}^{Bm}.V_{B/j/k}.T_{im}{}^l + V_{A/l/k}.T_{ij}{}^l +$$
$$+ V_{A/j/l}.T_{ik}{}^l)\xi^i - S_{Ai}{}^{Bl}.V_{B/j/k}.\xi^i{}_{/l} + V_{A/i/k}.\xi^i{}_{/j} +$$
$$+ V_{A/j/i}.\xi^i{}_{/k} \, .$$

The components $\bar{P}_i{}^j$ can be written in the following form:

$$\bar{P}_i{}^j = \bar{t}_i{}^j - \bar{V}_i{}^{kj}{}_{/k} + \bar{F}_m{}^{kjl}.R^m{}_{lik} - \frac{\partial L}{\partial V_{A/k/j}}.V_{A/l}.T_{ik}{}^l +$$
$$+ g_i^j.L + \frac{\partial L}{\partial g_{jk}}.g_{ik} - \bar{Q}_i{}^j \, , \qquad (1.9)$$

where

$$\bar{t}_i{}^j := \left[\frac{\partial L}{\partial V_{A/j}} - \left(\frac{\partial L}{\partial V_{A/j/k}} + \frac{\partial L}{\partial V_{A/k/j}}\right)_{/k}\right] V_{A/i} +$$
$$+ \left(\frac{\partial L}{\partial V_{A/k/j}}.V_{A/i}\right)_{/k} - g_i^j.L \, , \qquad (1.10)$$

$$\bar{V}_i{}^{kj} := \bar{F}_i{}^{kj} - \bar{F}_i{}^{klj}{}_{/l} \, ,$$
$$\bar{F}_i{}^{kj} := \frac{\partial L}{\partial V_{A/k}}.S_{Ai}{}^{Bj}.V_B - \frac{\partial L}{\partial V_{A/j/k}}.V_{A/i} +$$
$$+ \left(\frac{\partial L}{\partial V_{A/k/l}} + \frac{\partial L}{\partial V_{A/l/k}}\right).S_{Ai}{}^{Bj}.V_{B/l} \, ,$$

$$\bar{F}_i{}^{kjl} := \frac{\partial L}{\partial V_{A/k/j}}.S_{Ai}{}^{Bl}.V_B \, , \qquad \bar{Q}_i{}^j := \frac{\delta L}{\delta V_A}.S_{Ai}{}^{Bj}.V_B \, ,$$

$$\frac{\delta L}{\delta V_A} := \frac{\partial L}{\partial V_A} - \left(\frac{\partial L}{\partial V_{A/k}}\right)_{/k} + \left(\frac{\partial L}{\partial V_{A/k/l}}\right)_{/l/k} \, , \qquad (1.11)$$

$\frac{\delta L}{\delta V_A}$:= functional variation of the invariant function L with respect to V_A.

After transforming $\bar{V}_i^{\ kj}$ by means of a combination of symmetric and antisymmetric parts in k and j, i.e.

$$\bar{V}_i^{\ kj} := \mathcal{T}_i^{\ kj} + A_i^{\ kj} ,$$

$$\mathcal{T}_i^{\ kj} := \tfrac{1}{2}(\bar{V}_i^{\ kj} + \bar{V}_i^{\ jk}) := \bar{V}_i^{\ (kj)} ,$$

$$A_i^{\ kj} := \tfrac{1}{2}(\bar{V}_i^{\ kj} - \bar{V}_i^{\ jk}) := \bar{V}_i^{\ [kj]} ,$$

the components of $\bar{P}_i^{\ j}$ can be obtained in the form:

$$\bar{P}_i^{\ j} := \bar{\Theta}_i^{\ j} - {}_s T_i^{\ j} - \bar{Q}_i^{\ j} , \qquad (1.12)$$

where

$$\bar{\Theta}_i^{\ j} := \bar{t}_i^{\ j} - W_i^{\ jk}{}_{/k} + \bar{F}_m^{\ kjl}.R^m{}_{lik} - \frac{\partial L}{\partial V_{A/k/j}}.V_{A/l}.T_{ik}^{\ \ l} , \qquad (1.13)$$

$$\bar{V}_i^{\ kj}{}_{/k} := W_i^{\ jk}{}_{/k} + \mathcal{T}_i^{\ j} ,$$

$$W_i^{\ jk} = - W_i^{\ kj} := g_{il}(\mathcal{T}_m^{\ jl} g^{mk} - \mathcal{T}_m^{\ kl} g^{mj} - A_m^{\ jk} g^{ml}) ,$$

$${}_s T_i^{\ j} := \mathcal{T}_i^{\ j} - 2.\frac{\partial L}{\partial g_{jk}}.g_{ik} - g_i^j.L , \qquad (1.14)$$

$$\mathcal{T}_i^{\ j} := \mathcal{T}_i^{\ jk}{}_{/k} = [g_{il}(\mathcal{T}_m^{\ jk} g^{ml} + \mathcal{T}_m^{\ kl} g^{mj} - \mathcal{T}_m^{\ jl} g^{mk})]_{/k} ,$$

$$\mathcal{T}_i^{\ jk} := g_{il}(\mathcal{T}_m^{\ jk} g^{ml} + \mathcal{T}_m^{\ kl} g^{mj} - \mathcal{T}_m^{\ jl} g^{mk}) ,$$

$$\mathcal{T}_i^{\ jk} = g_{il}.\mathcal{T}^{ljk} , \qquad \mathcal{T}^{ij} := \mathcal{T}^{ijl}{}_{/l} = \mathcal{T}^{ji} ,$$

$$\mathcal{T}^{ijk} := \mathcal{T}_m^{\ ik} g^{mj} + \mathcal{T}_m^{\ jk} g^{mi} - \mathcal{T}_m^{\ ij} g^{mk} .$$

The components of $\bar{P}_i$, having the form

$$\bar{P}_i := \frac{\partial L}{\partial V_A}.V_{A/i} + \frac{\partial L}{\partial V_{A/j}}.V_{A/j/i} + \frac{\partial L}{\partial V_{A/k/l}}.V_{A/k/l/i} - L_{/i} ,$$

can be expressed by means of the terms introduced in $\bar{P}_i^{\ j}$ in the following form:

$$\bar{P}_i := \frac{\delta L}{\delta V_A}.V_{A/i} + \bar{\Theta}_i^{\ j}{}_{/j} + \bar{W}_i , \qquad (1.15)$$

$$\bar{W}_i := \bar{S}_i - \bar{S}_k^{\ j}.T_{ij}^{\ \ k} ,$$

$$\bar{S}_i := W_i^{\ jk}{}_{/k/j} + \bar{V}_k^{\ jl}.R^k{}_{lij} =$$

$$= \tfrac{1}{2}[W_i^{\ jk}{}_{/l}.T_{jk}^{\ \ l} - W_m^{\ jk}(T_{\langle ij}^{\ \ m}{}_{/k\rangle} + T_{\langle ij}^{\ \ n}.T_{nk\rangle}^{\ \ m}) -$$

$$- W_i^{\ jk}(T_{jk}^{\ \ l}{}_{/l} + T_{k/j} - T_{j/k} + T_l.T_{jk}^{\ \ l})] ,$$

$$T_{\langle ij}^{\ \ m}{}_{/k\rangle} := T_{ij}^{\ \ m}{}_{/k} + T_{ki}^{\ \ m}{}_{/j} + T_{jk}^{\ \ m}{}_{/i} ,$$

$$T_k := T_{kl}^{\ \ l} := g_l^m.T_{km}^{\ \ l} ,$$

$$\bar{S}_k^{\ j} := [\frac{\partial L}{\partial V_{A/j}} - (\frac{\partial L}{\partial V_{A/j/l}})_{/l}].V_{A/k} + \frac{\partial L}{\partial V_{A/l/j}}.V_{A/l/k} .$$

Therefore, the identity for L can be written in the form:

$$(\bar{P}_i + \bar{P}_k^{\ j}.T_{ij}^{\ \ k})\xi^i + \bar{P}_i^{\ j}.\xi^i{}_{/j} = 0 , \qquad (1.16)$$

where

$$\bar{P}_i := \frac{\delta L}{\delta V_A}.V_{A/i} + \bar{\Theta}_i^{\ j}{}_{/j} + \bar{W}_i$$

$$\bar{P}_i^{\ j} := \bar{\Theta}_i^{\ j} - {}_s T_i^{\ j} - \bar{Q}_i^{\ j} . \qquad (1.17)$$

C. The components of the vector field ξ and their first covariant derivatives $\xi^i{}_{/j}$ can be considered as arbitrary independent functions of the coordinates in U_n-space (e.g. in a coordinate system /7/, in which in a p. $x \in U_n$ $\Gamma^i_{jk}/x = 0$, $\xi^i{}_{/j}/x = \xi^i{}_{,j}$). If the identity for L is satisfied for arbitrary ξ^i and $\xi^i{}_{/j}$, then the tensor components in the expressions sitting as coefficients in front of ξ^i and $\xi^i{}_{/j}$ must identically vanish, i.e.

$$\bar{P}_i + \bar{P}_k^{\ j}.T_{ij}^{\ \ k} = 0 , \qquad (1.18)$$

$$\bar{P}_i^{\ j} = 0 . \qquad (1.19)$$

From the second condition it follows that the first condition would have the form $\bar{P}_i = 0$. In this way the generalized covariant Bianchi type identities (GCBI) for the invariant L are obtained in the form:

$$\bar{P}_i = 0 : \quad \bar{F}_i + \bar{\Theta}_i^{\ j}{}_{/j} = 0 , \qquad \bar{F}_i := \frac{\delta L}{\delta V_A}.V_{A/i} + \bar{W}_i , \qquad (1.20)$$

$$\bar{P}_i^{\ j} = 0 : \quad \bar{\Theta}_i^{\ j} - {}_s T_i^{\ j} = \bar{Q}_i^{\ j} . \qquad (1.21)$$

The components of a covariant vector field $\bar{W}_i$ depend on the components of the torsion tensor $T_{ik}{}^j$ and their covariant derivatives in such a way that in the case of $T_{ij}{}^k = 0$ there follows $\bar{W}_i = 0$, i.e. the GCBI would have the form for V_n-spaces /5/ mentioned in the introduction. The quantities $\bar{F}_i$, $\bar{\Theta}_i{}^j$, ${}_sT_i{}^j$, $\bar{Q}_i{}^j$ for $T_{ij}{}^k = 0$ are identical with those defined in V_n-spaces and these notations are retained also for the case of U_n - spaces.

II. TENSORS OF THE TYPE OF THE ENERGY-MOMENTUM TENSOR IN (PSEUDO)RIEMANNIAN SPACES WITH TORSION

1. On the basis of the obtained GCBI and in analogy with the tensors of energy-momentum tensor type, introduced for Lagrangian systems in V_n-spaces, tensors of the same type can be defined in U_n-spaces. In accordance with the structure of $\bar{\Theta}_i{}^j$, ${}_sT_i{}^j$, $\bar{t}_i{}^j$ one can introduce the following notations:

$\bar{t}_i{}^j$:= canonical energy-momentum tensor for a Lagrangian system,

$\bar{\Theta}_i{}^j$:= generalized canonical energy-momentum tensor for a Lagrangian system,

${}_sT_i{}^j$:= symmetric energy-momentum tensor for a Lagrangian system,

$\frac{\delta L}{\delta V_A} := 0$:= Euler-Lagrange equations for the fields V_A,

${}_sT_i{}^j{}_{/j} := 0$:= equations of motion for a Lagrangian system.

2. By means of GCBI and after direct computations some propositions can be proved, expressing the connection between the several kinds of energy-momentum tensors and their analogous forms in V_n-spaces /5/.

PROPOSITION 1. The necessary and sufficient condition for the generalized canonical tensor $\bar{\Theta}_i{}^j$ to be equal to the symmetric energy-momentum tensor ${}_sT_i{}^j$ is the condition

$$\bar{Q}_i{}^j := \frac{\delta L}{\delta V_A} . S_{Ai}{}^{Bj} = 0 . \qquad (2.1)$$

Proof: 1) Necessity: follows from condition $\bar{\Theta}_i{}^j = {}_sT_i{}^j$ and the identity (1.21).

2) Sufficiency: follows from the condition $\bar{Q}_i{}^j := 0$ and the identity (1.21).

PROPOSITION 1.1. If $V_A := \varphi$ is a scalar field ($N_{V_A} := 0$), then condition $\Theta_i{}^j = {}_sT_i{}^j$ is always valid.

Proof: $\bar{Q}_i{}^j = 0$ for a scalar field ($N_{V_A} = 0$), because of $S_{Ai}{}^{Bj} . V_B = 0$ for $N_{V_A} := 0$. From the identity (1.21) and $\bar{Q}_i{}^j = 0$ it follows that $\bar{\Theta}_i{}^j = {}_sT_i{}^j$.

PROPOSITION 1.2. The Euler-Lagrange equations $\delta L/\delta V_A = 0$ for the field functions V_A are a sufficient condition for the equivalence of $\bar{\Theta}_i{}^j$ and ${}_sT_i{}^j$.

Proof: follows directly from the identity (1.21).

Under the interpretation of Euler-Lagrange equations as field equations for the fields V_A (i.e. as field equations for the tensor field components V_A), both tensors $\bar{\Theta}_i{}^j$ and ${}_sT_i{}^j$ will be equal and their different forms can be used in solving the problems, which require an appropriate form of the energy-momentum tensor for founding their solutions.

III. CONDITIONS FOR THE EXISTENCE OF THE SYMMETRIC ENERGY-MOMENTUM TENSOR AS A LOCAL CONSERVED QUANTITY

By means of the GCBI the following propositions can be proved:

PROPOSITION 2. The necessary and sufficient condition for ${}_sT_i{}^j$ to be a local conserved quantities in U_n-space for a Lagrangian system with field functions V_A of a rank $N_{V_A} \neq 0$, i.e. ${}_sT_i{}^j$ to satisfy the condition

$${}_sT_i{}^j{}_{/j} = 0 , \qquad (3.1)$$

is the condition

$$\bar{F}_i + \bar{Q}_i{}^j{}_{/j} = 0: \quad \frac{\delta L}{\delta V_A} . V_{A/i} + \bar{W}_i + \bar{Q}_i{}^j{}_{/j} = 0. \qquad (3.2)$$

Proof: 1) Necessity: follows from the requirement ${}_sT_i{}^j{}_{/j} := 0$ and the identities (1.20) and (1.21).

2) Sufficiency: follows from the conditions (3.2) and the identities (1.20) and (1.21).

PROPOSITION 2.1. The necessary and sufficient condition ${}_sT_i{}^j$ to be a local conserved quantities in U_n-space for a Lagrangian system with scalar field functions $V_A = \varphi$ (of a rank $N_{V_A} := 0$), i.e. ${}_sT_i{}^j$ to satisfy the condition

$${}_sT_i{}^j{}_{/j} = 0 ,$$

is the condition

$$\bar{F}_i = 0: \quad \frac{\delta L}{\delta \varphi} . \varphi_{/i} + \bar{W}_i = 0 \qquad (3.3)$$

Proof: 1) Necessity: follows from the condition ${}_sT_i{}^j{}_{/j} = 0$, $\bar{Q}_i{}^j = 0$ (for scalar field), and the identities (1.20)-(1.21).

2) Sufficiency: follows from the conditions $\bar{F}_i := 0$, $\bar{Q}_i{}^j = 0$ (rank $V_A = 0$) and the identities (1.20)-(1.21).

PROPOSITION 3. Sufficient conditions for the existence of the symmetric energy-momentum tensor ${}_sT_i{}^j$ as a local conserved quantity, i.e. ${}_sT_i{}^j{}_{/j} = 0$, are the conditions

$$\frac{\delta L}{\delta V_A} := 0 , \quad \bar{W}_i := 0 . \qquad (3.4)$$

Proof: From conditions (3.4) and the identities (1.20)-(1.21) follows the condition for ${}_sT_i{}^j$

$${}_sT_i{}^j{}_{/j} = 0 .$$

It turns out that (in contrast to the case in V_n-space) in U_n-space the Euler-Lagrange equations cannot be sufficient conditions for the existence of ${}_sT_i{}^j$ as a local conserved quantities. The condition $\bar{W}_i = 0$ is in this case also necessary

in order ${}_sT_i{}^j{}_{/j} = 0$ to be satisfied. This points out that the torsion tensor appears as an additional tensor field, braking the condition ${}_sT_i{}^j{}_{/j} = 0$ in the case of $\delta L/\delta V_A = 0$. This fact can be interpreted as an existence of an interaction with the tensor fields V_A torsion tensor field $T_{ij}{}^k$, violating the local conservation of the symmetric energy-momentum tensor for the Lagrangian system. It could not be ignored in the construction of models, describing various interactions (including the gravitational interaction) in U_n-spaces. Some authors /3,4/ make a connection between the torsion tensor and the spin of the material distribution (appearing as a source of the torsion tensor field), but in the present considerations such a hypothesis has not its mathematical and physical ground, because in our case no equations connecting the curvature and torsion tensor of the U_n-space with the characteristics of material systems (Lagrangian density and field functions) are introduced.

CONCLUSION

1. The developing of affine-metric field theories, describing the gravitational interaction on the basis of models in (pseudo)-Riemannian spaces with non-vanishing curvature and torsion tensor, requires the specification not only of the field equations but also of the local conserved tensors of the energy-momentum type.

2. It turns out that the symmetric energy-momentum tensor for a Lagrangian system in U_n-space is not a local conserved quantity when the Euler-Lagrange equations for this system are satisfied. Therefore, if the condition for the general symmetric energy-momentum tensor of a system "material distribution + its gravitational field" is posed to be a local conserved quantity, then it must be possible for one to obtain by means of

the Lagrangian density a symmetric energy-momentum tensor for the gravitational field, which alone is not a local conserved quantity, but the sum of both tensors (of the material distribution and of the gravitational field) is a local conserved quantity. In the opposite case (when the symmetric energy-momentum for the gravitational field is a local conserved quantity or identically vanishes) the material distribution (the Lagrangian system) must obey equations different from those of Euler-Lagrange.

The search for suitable equations for the gravitational field in U_n-spaces requires a careful approach to the problems of the energy-momentum of material systems described by models in such a type of spaces.

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Received by Publishing Department
on September 10,1987.