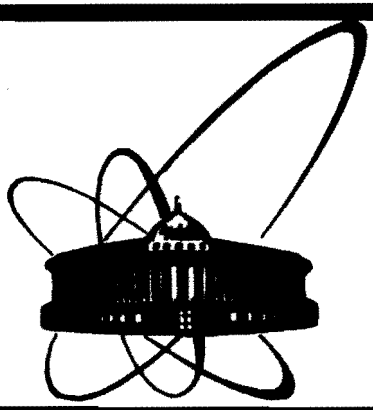

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исследований
дубна

E2-87-675

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A SYSTEMATIC ITERATIVE APPROACH TO THE EQUATIONS OF LOW TYPE

1987

1. INTRODUCTION

Integral equations of the Low type

$$F = V + FGF^{+} \tag{1.1}$$

may be encountered, e.g., in the three-dimensional inverse-scattering formalism [1] . In accord with Ref. [2] , the most important and promising application of these equations lies in the relativistic description of π–N interactions. In such a context, the precision must be high and the realistic computations may easily exceed the capacity of computers. A non-linear structure of (1.1) leads to a number of difficulties as follows.

(a) We need a mathematically well-founded conversion of integral equation into a sequence of its numerically solvable approximate forms. In particular, a standard replacement of integrations by the finite summations

$$\int du\, f(u) = \sum_{i=1}^{N} f(v_i) w_i \qquad N \to \infty \tag{1.2}$$

with the "grid points" v_i and the "weight coefficients" w_i [3] must be done properly.

(b) The resulting finite (N-dimensional) matrix equation with the quadratic nonlinearity remains still fairly complicated. An iterative method of its solution is currently employed. Unfortunately, the sequence of approximants of the type of power series, e.g.,

$$F[1] = V, \quad F[2] = V + VGV^{+}, \ldots \tag{1.3}$$

is often divergent and necessitates a Pade resummation [4-7] .

(c) Of course, a many times repeated generation of approximants (1.3) is a difficult task even on the computer, especially due to the rather complicated structure of the realistic kernels G. For illustration, we shall consider here the "homework" example of the π–N kernel

$$G = S + C, \qquad (1.4)$$

where S is singular and C mediates a coupling to all the other partial waves [2,8].

In the pionic physics context, the whole power-series - Pade approach to (1.1) is not entirely satisfactory. Indeed, the relativistic and sometimes quite small corrections are evaluated, but the computational errors remain rather large [6]. New and more effective methods are also needed for a very feasibility of the fully consistent calculations incorporating, e.g., the complete crossing symmetry, etc. [2].

In the present paper, a new non-power-series method will be proposed and discussed in some detail. It is intended to proceed in an iterative manner which facilitates a control of precision in each step.

2. THE EFFECTIVE-INTERACTION ITERATIONS

2.1. The Mixing of Partial Waves

In accord with the homework-problem structure (1.4) of the kernel, our integral equation

$$F = V + FCF^{+} + FSF^{+}, \qquad (2.1)$$

has a "small" but rather complicated component FCF^{+}. Its omission or incorporation into a modified or "effective" interaction

$$W = V + FCF^{+} \qquad (2.2)$$

is therefore a reasonable approximation. The new form of eq. (2.1)

$$F = W + FSF^{+} \qquad (2.3)$$

becomes also easier to solve. Thus, after an arbitrary choice of an initial approximation (say, F=0, W=V), our overall computational algorithm may proceed in an iterative manner:

(a) solve (2.3)
(b) improve W (2.2)
(c) return to (a) if necessary.

The same perturbative strategy may also be applied to the general Low-type equation with the structure

$$F = V + FC(1)F^{+} + \ldots + FC(t)F^{+} \qquad (2.4)$$

and with a decreasing influence of the separate components on the final result.

2.2. A Singularity of the Kernel

In the realistic π-N problem, the partial-wave mixing (1.4) is important and must be calculated by the above-mentioned iterations. In each step, we have to solve (2.1) reduced to an integral equation

$$F(x,y) = V(x,y) + \int_0^\infty u^2 du\, v^2 dv\, F(x,u)\, S(u,v)\, F^{*}(y,v) \qquad (2.5)$$

with the standard y-dependent and diagonal singular kernel of the type

$$S(u,v) = u^{-2}\delta(u-v)S(u), \quad S(u) = 1/[E(u) - E(y) - i\epsilon]. \qquad (2.6)$$

Here, the symbol

$$E(z) = (M_N^2 + z^2)^{1/2} + (M_\pi^2 + z^2)^{1/2}$$

denotes the energies [2].

We may decompose also the kernel (2.6),

$$S(u) = s(u) + PS(u), \quad s(u) = i\delta(E(u) - E(y)). \qquad (2.7)$$

This is the well-known formula where the first term (delta-function) is complemented by the component which requires a special, so-called principal-value integration.

Now, we are prepared to repeat the trick of the preceding paragraph and include temporarily also the second term of (2.7) into an effective interaction,

$$V^{eff}(x,y) = V(x,y) + P\int_0^\infty F(x,u)(E(u) - E(y))^{-1} F^{*}(y,u) u^2 du. \qquad (2.8)$$

With this effective potential and with the delta-function kernel, we may re-write finally our integral Low equation as an algebraic one,

$$F(x,y) = V^{eff}(x,y) + F(x,y)\, s(y)\, F^{+}(y,y). \qquad (2.9)$$

This equation possesses an easy solution for any discrete set of the integration points

$$x = x_i^{(o)} \quad \text{and} \quad y = y_j^{(o)}, \quad i,j = 1,2,\ldots,N_o \tag{2.10}$$

entering, say, the definition of the cross sections.

3. QUADRATIC NONLINEARITY AND THE CONTINUED-FRACTIONAL ITERATIONS

3.1. On-Shell Amplitudes and the Analytic Continued Fractions

Provided that we choose $x = y$ and put $a = F(y,y) = b^+$ and $v = V^{eff}(y,y)$ in (2.5) and in its hermitean conjugate, the "on-shell amplitude" $F = a$ becomes defined by the exactly solvable pair of relations (2.9),

$$a = v + igab, \qquad b = v^+ - igab. \tag{3.1}$$

We may formulate immediately the following simple observations.

(a) In general, the solution is not unique. In (3.1), the input interaction $v = iw$ leads to the pair of exact roots

$$a\{1\} = i(1 + \mathrm{sqrt}(1 - 4gw)/(2g)$$
$$a\{2\} = 2iw/(1 + \mathrm{sqrt}(1 - 4gw)). \tag{3.2}$$

Only the second one disappears in the limit of zero interaction and may be considered physical.

(b) The iterations (1.3) remove the ambiguity in general. Here, they reproduce correctly the second root, $F(x,x) = a\{2\}$, at least for the sufficiently small interactions V. This represents a physical support for the use of iterations in spite of their actual divergence [5].

(c) The divergence cured by the Pade technique (cf. also [4,6,7]) is obviously related to the magnitude of G, V, or $|gw| > 1$. In our particular example, the problem may easily be circumvented by a transition to the well-known analytic continued fractional form of the physical root $a\{2\}$,

$$F(x,x) = iw/(1 - gw/(1 - gw/(\ldots))). \tag{3.3}$$

This is a rather trivial special case of a general Pade approximation. Nevertheless, its specific iterative structure may

be generalized in various ways. One of them is to be described here.

3.2. The Matrix Equation and Continued Fractions

A straightforward derivation of the continued-fractional formula (3.3) may be based on a quasilinear re-interpretation

$$F(1 - GF^+) = V. \tag{3.4}$$

of our original equation. From its combination with its hermitean conjugate, i.e., from the pair of relations

$$F = V/(1 - GF^+) \qquad F^+ = (1 - FG^+)^{-1} V^+ \tag{3.5}$$

we may formally eliminate F^+ and obtain

$$F = V/(1 - G(1 - FG^+)V^+. \tag{3.6}$$

The iterations imply

$$F = V\,\cfrac{1}{1 - G\,\cfrac{1}{1 - V\,\cfrac{1}{1 - G\,\cfrac{1}{1 - V\,\cfrac{1}{1 - \ldots}\,G^+}\,V^+}\,G^+}\,V^+} \tag{3.7}$$

which resembles formula (3.3) and forms the so-called matrix continued fraction (MCF [9]). By its definition, it is merely a repeated mapping

$$(F, F^+) \to (F', F^{+\prime}) \to \ldots \tag{3.8}$$

or iterative algorithm

$$F' = V/(1 - H), \quad H = GF^+, \quad F^{+\prime} = (1 - H^+)^{-1} V^+, \quad H^+ = FG^+ \tag{3.9}$$

with the trivial initial choice of (F,F^+) or $(H,H^+) = (0,0)$.

4. SINGULAR KERNEL AND THE MODIFIED MCF ITERATIONS

Due to the definition of the principal value [3], the approximate integration in (2.8) must proceed via a summation over points which are distributed symmetrically around y.

Thus, we are forced to replace our original mesh of grid points (2.10) by another set

$$x_i^{(1)} \ (\neq x_j^{(o)}, \quad j = 1,2,\ldots,N_o), \qquad i = 1,2,\ldots,N_1 \tag{4.1}$$

which depends on y in general. This new set defines our auxiliary effective interaction matrix (2.8) as a sum

$$V_{ij}^{eff} = V(x_i^{(o)}, y_j^{(o)}) + \sum_{m=1}^{N_1} F(x_i^{(o)}, u_m^{(1)}) w_m^{(1)} \tag{4.2}$$
$$\times S(u_m^{(1)}) F^*(y_j^{(o)}, u_m^{(1)}), \qquad i,j = 1, 2, \ldots, N_o,$$

where the "new" matrices F(s,t) contain arguments s and t belonging to the two different discrete sets (4.1).

In accord with our preceding discussion, the principal-value grid points u in (4.2) depend on the external index j. Hence, the actual discrete form of our C = 0 equation (2.5) reads

$$F_{ij} = V_{ij} + \sum_{m=1}^{N_{01}} F_{im} S_m(j) P_m^{(j)} F_{mj}^+, \quad N_{01} = N_0 \times (N_1 + 1), \tag{4.3}$$

where a formal unit (j-dependent projector $P^{(j)}$) is introduced to compensate an extension of the summation to a union of all the (j-dependent) subsets of the grid points (4.1), complemented also by the N_o "external" points y themselves.

In the power-series iterative approaches, the latter construction leads usually to a rapid increase of the dimensions. Hence, it is reasonable to keep it under control by a careful specification of the suitable interpolation technique. Let it be mediated by some y - independent $N_{01} \times N_0$ - dimensional matrix Z transforming a function at points $x^{(0)}$ into its values at the union $\{ x^{(0)}, u^{(1,1)}, \ldots, u^{(1,N_o)} \}$. Then, our Low eq. (4.3) will contain the kernel-term summation

$$\sum_{m=1}^{N_{01}} F_{im} S_m(j) F_{jm}^* = \sum_{k=1}^{N_0} \sum_{n=1}^{N_0} F_{ik} G_{kn}^{(j)} F_{jn}^* \tag{4.4}$$

$$G_{kn}^{(j)} = \sum_{m=1}^{N_{01}} Z_{mk} S_m(j) P_m^{(j)} Z_{mn}.$$

We may summarise that the singularity and interpolation modify merely our propagator G (to a non-diagonal form) and give our equation its final matrix form

$$F = V + FH, \tag{4.5}$$

where

$$H_{ij} = \sum_{k=1}^{N_0} G_{ik}^{(j)} F_{jk}^*. \tag{4.6}$$

The latter expression differs from its matrix-product predecessor of sect. 3. It is an element-by-element product formed directly from the matrix elements of the matrix F and of the set of matrices G.

It remains for us to use the hermitean conjugation

$$H_{ij}^+ = \sum_{k=1}^{N_0} F_{ik} G_{jk}^{*(i)} \tag{4.7}$$

and recall the MCF iterations. Alternatively, we may employ also a MCF definition with the slightly modified recurrences

$$F' = V/(1 - H), \qquad F^{*\prime} = V^*/(1 - H^*)$$
$$H_{ij} = \sum_{k=1}^{N_0} G_{ik}^{(j)} F_{jk}^*, \qquad H_{ij}^* = \sum_{k=1}^{N_0} G_{ik}^{(j)*} F_{jk} \tag{4.8}$$

i.e., an iterated mapping $(F_{ij}, F_{k\ell}^*) \to (F'_{mn}, F_{op}^*) \to \ldots$ of the complex conjugate pairs of matrices.

5. AN ILLUSTRATIVE EXAMPLE

In accord with all our preceding considerations and with eq. (4.4) in particular, we may write our general Low equation (1.1) in the NxN - dimensional matrix form with the j-dependent kernel,

$$F_{mj} = V_{mj} + \sum_{k,n=1}^{N} F_{mk} G_{kn}^{(j)} F_{nj}^+, \qquad m,j = 1,2,\ldots,N. \tag{5.1}$$

For the separable potentials $V_{mj} = \langle m|a\rangle g \langle a|j\rangle$ and amplitudes

$$F_{mj} = \langle m|a\rangle f_j \langle a|j\rangle \tag{5.2}$$

we may reduce the matrix equation (5.1) to a "vectorial" one,

$$f_j = g + \sum_{k,n=1}^{N} f_k M^{(j)}_{kn} f^+_n , \qquad j = 1,2,\ldots,N \tag{5.3}$$

This is suitable for numerical tests.

For the sake of definitness, let us consider now a schematic model characterised by the particular two-dimensional and one-parametric family of kernels

$$M^{(1)}_{\bullet} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \qquad M^{(2)} = zM^{(1)}, \qquad z \in (-\infty, \infty). \tag{5.4}$$

After such a choice, our eq. (5.3) acquires the form

$$f_1 = g + Q(f_1, f_2) \qquad f_2 = g + zQ(f_1, f_2), \; Q(x,y) = x^2 + xy + y^2 \tag{5.5}$$

the solutions of which may be understood very easily.

5.1. The Power-Series Iterations

Our very way of writing equation (5.5) is a direct inspiration of the standard power-series iterative algorithm

$$x' = g + Q(x,y) \qquad y' = g + zQ(x,y) \tag{5.6}$$

which is usually initialised by $(x,y) = (0,0)$[4-7]. For the sufficiently small couplings $g \to 0$, it should generate a convergent power series in g.

Due to the simplicity of our example, the rigorous analysis of its convergence is in fact very easy. Indeed, our mapping $(x,y) \to (x',y')$ becomes one-dimensional after a transition to the new variables Q and $Q'(=Q(x',y'))$,

$$Q' = 3g^2 + 3(1+z)gQ + (1 + z + z^2)Q^2. \tag{5.7}$$

This formula may be complemented by the definitions (5.6).

When we put $Q' = Q$ in (5.7), we obtain the two solutions $(Q_o^{(+/-)})$ of (5.5) (or fixed points of the mapping (5.6)) from the quadratic equation. Both these roots remain real for couplings in an interval

$$g \in (g^{(0)}_{-}(z), \; g^{(0)}_{+}(z)), \quad g^{(0)}_{e} = (1-z)^{-2}(-1 - z + eD^{1/2}), \; e = \pm 1,$$
$$D = 4(1 + z + z^2)/3 \tag{5.8}$$

which contains also the point $g = 0$ in its interior.

Outside the interval (5.8), the complex values of solutions Q_o cannot be reached by the power-series iterations since all these approximants remain real. We obtain at least an oscillatory divergence, unless we start from some complex initialization.

The mapping $Q \to Q'$ (5.7) has an obvious necessary condition of convergence $|\partial Q'/\partial Q| < 1$ and a sufficient condition of divergence $|\partial Q'/\partial Q| > 1$. After an insertion of the explicit formulae and restriction of the couplings to the "permitted interval" (5.8), this implies that the power-series iterations never converge to the unphysical fixed point and they

(a) diverge for $z \geq 0$ and for the couplings lying in the interval

$$g \in (g^{(1)}_{-}(z), \; g^{(1)}_{+}(z)), \quad g^{(1)}_{e} = -(1 + e\sqrt{z})^{-2}, \quad e = -1 \text{ or } +1, \tag{5.9}$$

(b) may converge to the physical solution otherwise (i.e., for the couplings in the interval (5.8) and out of interval (5.9)).

5.2. The Present Method

The "divergence interval" (5.9) is negative and lies always in the interior of interval (5.8). Both these intervals become infinite for a disappearing j-dependence ($z \to 1$, $g^{(1)}_{-} \to \infty$, $g^{(1)}_{+} \to -1/4$) in our model kernels (5.4). Thus, we may conclude that the power-series method definitely fails for most values of the "permitted" couplings. Moreover, numerical tests indicate that even an incidence of couplings g in the "second permitted" interval $(g^{(0)}_{-}, g^{(1)}_{-})$ leads always to a quick divergence (caused by our bad choice of the initial values. In a comparison, our MCF method seems a priori less dependent on both the couplings and initial values.

An application of our general MCF prescriptions (3.9)+ +(4.6-7) or (4.8) to the particular model (5.3) is very straightforward and gives the prescription

$$x' = \{[1 + (1-z)a]g - q\}/(1-b-za), \quad y' = \{[1-(1-z)b]g - zq\}/(1-b-za)$$
$$q = (e-1)Q, \quad a = \frac{e}{2}(x + 2y), \quad b = \frac{e}{2}(y + 2x), \tag{5.10}$$

where we have to put $e = 1$ (this type of notation enables us to return to eq. (5.6) by using simply $e = 0$ here).

In spite of a highly schematic character of our present example, its numerical behaviour is extremely illuminating. It does not only illustrate the typical convergence pattern

of the power-series iterations (Table 1, the first column), but it exhibits also nicely both the re-summation role and efficiency of the present MCF algorithm (Table 1, the second column). For the sake of definitness, we have chosen here g = = 0.05 from within the "convergent" interval (-0.084, +0.023) of couplings. The Table displays the relative errors | f - - f(exact)| /| f(exact)| of the N-th approximant, with f(exact)= = (-0.0460, -0.0260) for z = 6. A similar picture could be obtained also for the various other choices of the parameters.

Table 1

A typical convergence pattern for the power-series (PS) and MCF iterations

N	PS	MCF
2	0.40	0.11
4	0.18	0.007 4
6	0.085	0.000 51
8	0.042	0.000 035
10	0.021	0.000 002 4
12	0.010	0.000 000 14

5.3. A Comparison with the Newton Method

In the numerical practice, nonlinear equations of the type of our pair of equations (5.5),

$$F_1(x,y) = 0, \quad F_2(x,y) = 0 \tag{5.11}$$

are often treated by the generalised Newton method. Here, the corresponding iterative prescription reads

$$x' = \{[1 + (1-z)A]g - Q\}/(1 - B - zA), \quad y' = \{[1-(1-z)B]g - zQ\}/(1 - B - zA)$$
$$Q = x^2 + xy + y^2, \quad A = x + 2y, \quad B = y + 2x, \tag{5.12}$$

In accord with the theorem of Kantorovich (cf. [3] , paragraph 20.2-8), the convergence of this mapping is quadratic (this is a reward for its rather complicated derivation) and may be guaranteed under very weak assumptions. In the present context, the underlying quasilinearization of the nonlinear mapping is in fact simulated in a simplified manner (formulae

of the type (5.12) will hardly be available in the realistic calculations).

In accord with the numerical example as given in Table 2 (computed at g = -0.5), the precision of our quasilinearization is not bad. Indeed, the first few MCF and NKA iterations produce a quite comparable precision. Moreover, the NKA eq. (5.12) coincides with our MCF eq. (5.10), provided only that we modify it slightly and choose e = 2 there.

Table 2

The similar comparison of the MCF results with the Newton - Kantorovich algorithm (NKA) (in the domain where the power-series diverges)

N	MCF	NKA
2	0.10	0.68
3	0.038	0.24
4	0.010	0.052
5	0.003 0	0.003 8
6	0.000 84	0.000 024
7	0.000 24	0.000 000 11
8	0.000 069	0.000 000 078

We may summarise that equation (5.10) is a common form of the above three different iterative prescriptions. A priori, we may expect therefore that the properties of our MCF algorithm will lie somewhere in between the above two extremes and "interpolate" somehow between the "very simple" power-series iterations and the quadratically convergent Newton-Kantorovich prescription.

In practice, the number of iterations will always remain restricted. Then, our MCF method may prove useful due to its combination of efficiency and universality with simplicity. In fact, even our simple example enables us to notice that at a fixed and finite number of iterations N, an optimal value of e (giving the best results) remains often close to one for small N and moves towards two slowly. This is illustrated in Table 3 and represents our last argument here. Of course, the final evaluation of the MCF technique may only be made after its future use in the realistic computations.

Table 3

The N-dependence of the optimal values of e in eq.(5.10)

(a) z = 1.1 and g = -420.0										
N	2	4	6	8	10	12	14	16	18	20
e	1.0	1.0	1.1	1.2	1.4	1.6	1.9	2.0	2.0	2.0
(b) z = 0.1 and g = -2.0										
N	2	3	4	5	6	7	8	9	10	
e	1.2	1.5	1.8	1.8	2.0	2.0	2.0	2.0	2.0	

ACKNOWLEDGEMENT

I acknowledge with thanks the discussions with A.I.Machavariani during his short stay in Dubna - they made me aware of importance of the Low type equations, and his "homework example" (1.4) or (2.6) motivated my present work, indeed.

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Received by Publishing Department
on September 8, 1987.

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E2-87-675

Систематический итерационный подход к уравнениям типа Лоу

Нелинейные сингулярные интегральные уравнения типа Лоу появляются при описании π - N амплитуд рассеяния при релятивистских энергиях. Обыкновенное итерационное решение расходится и не дает достаточно точных результатов даже при применении аппроксимации Падэ. В статье предлагается новый подход, поставленный на приблизительной линеаризации уравнения и ведущий к развитию амплитуд в форме матричных цепных дробей. Простой пример показывает, что новый метод улучшает сходимость старшего подхода и значительно расширяет область его сходимости. С другой стороны показывается его не эквивалентность более сложному методу Ньютона - Канторовича. В будущих, более реалистических применениях метода возможно ожидать повышение надежности результатов.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1987

Znojil M.
E2-87-675

A Systematic Iterative Approach to the Equations of Low Type

In the formalism described, e.g., by Kopaletshvili and Machavariani (Ann. Phys. (N.Y.) 175, 1 (1987)), the relativistic pion - nucleon scattering amplitude may be specified by means of a nonlinear and singular integral equation of the Low type. In the present paper, a new iterative approach to its solution is proposed. Its essence lies in a repeated formal simplification of the equation (via an introduction of the effective interaction) accompanied by a representation of the simplified amplitude in a generalized continued-fractional form. In this way, the precision of the approximations may always be kept under control. Presumably, the method will be able to improve or even replace the standard iteration procedures based on the use of Pade approximants - this is demonstrated on a simple schematic example numerically.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1987