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ON THE HIGHER-LEVEL BETHE ANSATZ

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The Bethe ansatz equations in their simplest but, nevertheless, typical form read[1-3]

$$e^{iLp(\lambda_j)} = (-)^{M-1} \prod_{k=1}^{M} e^{i\theta(\lambda_j - \lambda_k)} , \quad j = 1, \ldots, M , \tag{1}$$

with $p(\lambda)$ and $\theta(\lambda)$ being the momentum and two-body phase shift of elementary excitations, respectively ($p(-\lambda) = -p(\lambda)$, $\theta(-\lambda) = -\theta(\lambda)$). If the vacuum state of the model corresponds to $L, M \to \infty$ with a density $n = M/L$ fixed (and finite), then the parameters of physical excitations happen to obey an analogous system[4]

$$e^{iLp_\alpha} = (-)^{\nu} \prod_{\beta} e^{iF(\alpha-\beta)} , \tag{2}$$

where the momentum p_α and phase shift $F(\alpha-\beta)$ are now the physical ones. It is natural to call eqs.(2) the higher-level Bethe ansatz equations.

In this note we try to retrace the origin of the relations of such a type in terms of general functions $p(\lambda)$ and $\theta(\lambda)$. The ideas of each individual step within our approach are not at all new. However, the program as a whole has not been carried out in the literature; so, we hope that the present note would be instructive.

At first, we describe in our terms the standard procedure[1,5,6] of obtaining the integral Bethe-ansatz equations from the discrete ones (1). For our purposes it will suffice to consider the solutions $\{\lambda_j\}$ with all λ_j real. Taking a logarithm of (1) results in

$$Lp(\lambda_j) = \sum_{k=1}^{M} \theta(\lambda_j - \lambda_k) + 2\pi Q_j , \quad j = 1, \ldots, M , \tag{3}$$

where Q_j are integers or half-integers (it depends on M being odd or even). A vacuum configuration corresponds to the set $\{Q_j\}$ with Q_j as closely spaced as possible, $Q_{j+1} = Q_j + 1$, from $-\frac{M-1}{2}$ to $\frac{M-1}{2}$. The vacuum root density $\rho(\lambda)$,

$$\rho(\lambda_j) = \lim_{L\to\infty} \frac{1}{L(\lambda_{j+1} - \lambda_j)} , \quad \text{i.e. } \Delta Q = L\rho(\lambda)\Delta\lambda \tag{4}$$

enables one to approximate the sums over roots by integrals:

$$\frac{1}{L}\sum_k \Phi(\lambda_k) \simeq \int_{-\Lambda}^{\Lambda} d\lambda\, \rho(\lambda)\,\Phi(\lambda). \tag{5}$$

Subtracting eqs.(3) for adjacent j's , we obtain the integral equation for $\rho(\lambda)$,

$$2\pi\rho(\lambda) = h(\lambda) - \int_{-\Lambda}^{\Lambda} dx\, a(\lambda-x)\rho(x)\,, \tag{6}$$

where $a(\lambda)=\theta'(\lambda)$, $h(\lambda)=p'(\lambda)$, and $\pm\Lambda$ is the Fermi surface. A value of Λ is to be found from the normalization condition

$$\int_{-\Lambda}^{\Lambda} d\lambda\, \rho(\lambda) = n \tag{7}$$

and can prove, in general, finite as well as infinite.

A physical excited state from the class treated in this paper is parametrized by a set $\{\tilde{Q}_j\}$ which is precisely the vacuum set $\{Q_j\}$ with several numbers Q_α removed and (or) several extra numbers $Q_{\bar\alpha}$ added. We shall use the notation $\sum' \equiv \sum_\alpha - \sum_{\bar\alpha}$ that means taking into account the removed Q's (holes) with plus and extra ones (particles) with minus sign. The roots $\tilde\lambda_j$ of the excited state satisfy the following system of equations:

$$L\, p(\tilde\lambda_j) = \sum_{k=1}^{M} \theta(\tilde\lambda_j - \tilde\lambda_k) - \sum_\alpha{}' \theta(\tilde\lambda_j - \tilde\lambda_\alpha) + 2\pi \tilde{Q}_j\,. \tag{8}$$

Here j numbers only roots, not holes. Parameters $\tilde\lambda_\alpha$ related to the holes are not defined yet; moreover, they, in fact, drop out from (8). However, we choose to keep formally $\tilde\lambda_\alpha$ in (8), anticipating the replacement of (8) by an integral. An explicit definition of $\tilde\lambda_\alpha$ will be given later; for brevity, we write $\tilde\lambda_\alpha \equiv \alpha$.

Now our goal is to derive an analog of (5) for excited states. Let us introduce the function $f(\lambda)$,

$$f(\lambda_j) = \lim_{L\to\infty} \frac{\tilde\lambda_j - \lambda_j}{\lambda_{j+1} - \lambda_j}\,, \tag{9}$$

which obeys, due to (8), the equation [5]

$$2\pi f(\lambda) = -\int_{-\Lambda}^{\Lambda} dx\, a(\lambda-x) f(x) - \sum_\alpha{}' \theta(\lambda-\alpha)\,. \tag{10}$$

It is natural to represent $f(\lambda)$ as a sum in α ,

$$f(\lambda) = \sum_\alpha{}' f(\lambda,\alpha)\,, \tag{11}$$

where $f(\lambda,\mu)$ is given by

$$2\pi f(\lambda,\mu) = -\int_{-\Lambda}^{\Lambda} dx\, a(\lambda-x) f(x,\mu) - \theta(\lambda-\mu)\,. \tag{12}$$

Note that (12) does not involve parameters of physical excitations α .

Let $\tilde\Phi$ be some (additive in elementary excitations) quantity related to the excited state. Subtracting the corresponding vacuum quantity Φ we get

$$\tilde\Phi - \Phi = \sum_k \left[\Phi(\tilde\lambda_k) - \Phi(\lambda_k)\right] - \sum_\alpha{}' \Phi(\alpha) \simeq \int_{-\Lambda}^{\Lambda} d\lambda\, f(\lambda)\,\Phi'(\lambda) - \sum_\alpha{}' \Phi(\alpha)\,. \tag{13}$$

Because of (11), this expression is, in turn, additive in physical excitations:

$$\tilde\Phi - \Phi \simeq \sum{}' \Phi_\alpha\,,$$
$$\Phi_\alpha = -\Phi(\alpha) + \int_{-\Lambda}^{\Lambda} dx\, f(x,\alpha)\,\Phi'(x)\,. \tag{14}$$

The integral in (14) is a backflow (i.e. reaction of the sea) due to insertions of holes and particles.

Now we use (13) to perform the following transformations:

$$p(\lambda) - \frac{1}{L}\sum_k \theta(\lambda-\tilde\lambda_k) + \frac{1}{L}\sum_\alpha{}'\theta(\lambda-\alpha) = p(\lambda) - \frac{1}{L}\sum_k \theta(\lambda-\lambda_k) -$$
$$\frac{1}{L}\sum_k\left[\theta(\lambda-\tilde\lambda_k) - \theta(\lambda-\lambda_k)\right] + \frac{1}{L}\sum_\alpha{}'\theta(\lambda-\alpha) \simeq p(\lambda) - \int_{-\Lambda}^{\Lambda} dx\,\rho(x)\,\theta(\lambda-x) + \tag{15}$$
$$\frac{1}{L}\int_{-\Lambda}^{\Lambda} dx\, f(x)\, a(\lambda-x) + \frac{1}{L}\sum_\alpha{}'\theta(\lambda-\alpha) = p(\lambda) - \int_{-\Lambda}^{\Lambda} dx\,\rho(x)\,\theta(\lambda-x) - \frac{2\pi}{L} f(\lambda)\,.$$

Denoting the r.h.s. of (15) by $\tilde\varphi(\lambda)$,

$$\tilde\varphi(\lambda) = p(\lambda) - \int_{-\Lambda}^{\Lambda} dx\,\rho(x)\,\theta(\lambda-x) - \frac{2\pi}{L} f(\lambda)\,, \tag{16}$$

we see from comparing (8) and (15) that $\tilde\varphi(\tilde\lambda_j) \simeq \frac{2\pi}{L}\tilde{Q}_j$ for the roots. So, it is only natural to take condition

$$\tilde\varphi(\alpha) = \frac{2\pi}{L} Q_\alpha \tag{17}$$

as a definition of the hole positions α as well (note that the equality (17) is exact). As a result, the eqs.(6),(7),(11),(12), and (17) form a complete set of relations which enables one, in principle, to determine α's in an explicit form.

To transform (17) into the higher-level ansatz form (2), one has to do more calculations. Consider the total momentum of the excited state. From (14) it follows that

$$\tilde{P} - P \simeq \sum_\alpha{}' p_\alpha\,, \quad p_\alpha = -p(\alpha) + \int_{-\Lambda}^{\Lambda} dx\, f(x,\alpha)\, h(x)\,. \tag{18}$$

Transform the integral using (6) and (12):

$$\int_{-\Lambda}^{\Lambda} dx\, f(x,\alpha)\, h(x) = \int_{-\Lambda}^{\Lambda} dx\, dy\, f(x,\alpha)\left[2\pi\delta(x-y) + a(x-y)\right]\rho(y) = \tag{19}$$

$$-\int_{-\Lambda}^{\Lambda} dy\, \rho(y)\, \theta(y-\alpha) = \int_{-\Lambda}^{\Lambda} dx\, \rho(x)\, \theta(\alpha - x).$$

We now see that

$$p_\alpha = -p(\alpha) + \int_{-\Lambda}^{\Lambda} dx\, \rho(x)\, \theta(\alpha - x), \tag{20}$$

$$\widetilde{\varphi}(\lambda) = -p_\lambda - \frac{2\pi}{L}\sum_\beta{}' f(\lambda,\beta). \tag{21}$$

Substituting $\lambda = \alpha$ we obtain

$$L p_\alpha = -2\pi \sum_\beta{}' f(\alpha,\beta) - 2\pi Q_\alpha \tag{22}$$

or, in the exponentional form,

$$e^{iLp_\alpha} = (-)^\nu \prod_\beta{}' e^{-2\pi i f(\alpha,\beta)}, \quad (-)^\nu = e^{-2\pi i Q_\alpha} = \pm 1. \tag{23}$$

This is the higher-level ansatz equation for the parameters α of physical excitations. The "dressed" momentum p_α appears in the l.h.s. of (23) owing to the algebraic structure of the principal eqs.(6) and (12). This fact has first been established in[6]; our proof (19) is simpler and, in a sense, minimal.

To prove that $f(\alpha,\beta)$ in the r.h.s. of (23) can be viewed as a two-body phase shift of physical excitations, we have to use the direct method proposed by Korepin[7]. If we consider the x-space structure of the Bethe wave function proper to (8) and evaluate the total phase shift for a root $\widetilde{\lambda}_j$ nearest to a hole α ($\widetilde{\lambda}_j \simeq \alpha$) on the full interval $[-L/2, L/2]$, we get

$$\exp i\Delta\varphi = \exp i\left[L p(\alpha) + \sum_k \theta(\widetilde{\lambda}_k - \alpha) - \sum_\beta{}' \theta(\beta-\alpha)\right] \simeq \tag{24}$$

$$\exp iL\widetilde{\varphi}(\alpha) = \pm \exp i\left[-Lp_\alpha - 2\pi\sum_\beta{}' f(\alpha,\beta)\right].$$

From here the interpretation of $f(\alpha,\beta)$ as a physical phase shift becomes evident.

In conclusion, we shall make two remarks. The first is that when dealing with arbitrary functions $p(\lambda)$ and $\theta(\lambda)$ we have no control over the accuracy of our approximation procedure. In concrete models, as a rule, the neglected terms in (5),(13) etc. are of the

order $O(1/L)$. Note that (21)-(23) are exact by definition. It enables us to rewrite eq.(18) in the exact form too

$$\widetilde{P} - P = -\frac{2\pi}{L}\sum_\alpha{}' Q_\alpha = \sum_\alpha{}' p_\alpha + \frac{2\pi}{L}\sum_{\alpha\beta}{}' f(\alpha,\beta) \tag{25}$$

(the first equality results from (8)). We see that the total momentum $\widetilde{P} - P$ is discrete whereas the sum of dressed momenta $\sum_\alpha{}' p_\alpha$ is not unless a quantity $\sum_{\alpha\beta}{}' f(\alpha,\beta)$ equals zero. The latter can be transformed as follows:

$$\sum_{\alpha\beta}{}' f(\alpha,\beta) = \sum_\alpha{}' f(\alpha) = -\frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} dx\, f(x) \sum_\alpha{}' a(\alpha - x) =$$

$$\int_{-\Lambda}^{\Lambda} dx\, f(x)\left[f'(x) + \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} dy\, f(y)\, a'(x-y)\right] = \tag{26}$$

$$\int_{-\Lambda}^{\Lambda} dx\, f(x) f'(x) = \frac{1}{2}\left[f^2(\Lambda) - f^2(-\Lambda)\right].$$

For finite Λ this expression may perfectly well be nonzero. However, for $\Lambda = \infty$ the l.h.s. of (26) is necessarily zero, and the sum of dressed momenta turns out to be discrete:

$$\sum_\alpha{}' p_\alpha = -\frac{2\pi}{L}\sum_\alpha{}' Q_\alpha. \tag{27}$$

Really, if $\Lambda = \infty$ we derive from (10) that

$$f(\lambda) = \sum_\alpha{}' F(\lambda - \alpha), \quad \text{i.e.} \quad f(\lambda,\mu) = F(\lambda - \mu), \tag{28}$$

$$2\pi F(\lambda) = -\int_{-\infty}^{\infty} dx\, a(\lambda - x) F(x) - \theta(\lambda). \tag{29}$$

Obviously, $F(-\lambda) = -F(\lambda)$, and we arrive at $\sum_{\alpha\beta}{}' F(\alpha - \beta) = 0$.

The second remark concerns the $\Lambda = \infty$ case. Here, one can use, in complete analogy with the vacuum state, the root density $\widetilde{\rho}(\lambda)$ from the very beginning[2,8]. The root density for excited states,

$$\widetilde{\rho}(\lambda) = \rho(\lambda) + \frac{1}{L}\sum_\alpha{}' \sigma(\lambda - \alpha), \tag{30}$$

is defined through the integral equation

$$2\pi\widetilde{\rho}(\lambda) = h(\lambda) - \int_{-\infty}^{\infty} dx\, a(\lambda - x)\widetilde{\rho}(x) + \frac{1}{L}\sum_\alpha{}' a(\lambda - \alpha) \tag{31}$$

which approximates (8). For $\sigma(\lambda)$ it follows that

$$2\pi\sigma(\lambda) = a(\lambda) - \int_{-\infty}^{\infty} dx\, a(\lambda - x)\, \sigma(x), \tag{32}$$

and it is easy to deduce that

$$\sigma(\lambda) = -F'(\lambda). \tag{33}$$

Integration by parts shows that the F - and σ - approaches are consistent. The key formula (14) assumes the form

$$\Phi_\alpha = -\Phi(\alpha) + \int_{-\infty}^{\infty} d\lambda\, \Phi(\lambda)\, \sigma(\lambda-\alpha). \tag{34}$$

A simple algebraic nature of the mechanism observed supports our confidence in its relevance to a much richer class of solutions than that of purely real solutions of (1) directly considered in this paper.

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REFERENCES

1. Thacker H.B. - Rev.Mod.Phys.,53(1981) 253.
2. Lowenstein J. H.-In: Recent Advances in Field Theory and Statistical Mechanics (Les Houches 1982 Summer School), Amsterdam, North Holland, 1984.
3. Tarassov V.O., Takhtajan L.A. and Faddeev L.D. - TMF,57(1983) 163.
4. Destri C. and Lowenstein J.H. - Nucl.Phys.,B205 (1982) 369; Babelon O., de Vega H.J. and Viallet C.M. - Nucl.Phys.,B220 (1983) 13; Japaridze G.I., Nersesyan A.A. and Wiegmann P.B. - Phys.Scripta, 27 (1983) 5.
5. Lieb E.H. - Phys.Rev., 130 (1963) 1616.
6. Yang C.N. and Yang C.P. - J.Math.Phys., 10(1969) 1115.
7. Korepin V.E. - TMF, 41 (1979) 169.
8. de Cloizeaux J. and Pearson J.J. - Phys. Rev., 128 (1962) 2131; Takhtajan L.A. and Faddeev L.D. - Zap.Nauch.Semin.LOMI, 109 (1981) 134.

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К вопросу о вторичном анзаце Бете

Механизм получения уравнений "вторичного анзаца" из интегральных соотношений анзаца Бете прослежен в общем виде для широкого круга моделей.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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On the Higher-Level Bethe Ansatz

The origin of the higher-level Bethe ansatz is studied for a large class of integrable models.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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