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SCHWARZSHILD METRIC AND DE DONDER CONDITION

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The exact static spherically-symmetric solution of general relativity equations for "the point mass" found by Schwarzschild and Droste in 1916 is usually called the (external) Schwarzschild metric. General relativity theory is invariant under general nonsingular, sufficiently smooth coordinate transformations. Therefore for several decades a lot of various forms have been found for the Schwarzschild metric including nonstatic ones. One may mention such impressible results as Birkhoff's theorem and particularly nonstatic Kruscal-Szekeres [1] form. The latter represents a minimally complete extension of the Schwarzschild space-time, more precisely, the minimal causal factorised part of the universal covering metric [2]. All other forms are clearly not so complete as the Kruskal one and coordinate transformations between them are not always nonsingular. But this circumstance does not confuse physicists very much, though pushes them sometimes to reservations, such as the transformations "are not equivalent and the corresponding metrics may describe different physical situations" [3].

In the "rectilinear, nearly Cartesian" coordinates (Schrödinger [4]) $\bar{x}^0 = ct,\ \bar{x}^1,\ \bar{x}^2,\ \bar{x}^3$ the Schwarzschild metric is

$$ds^2 = \bar{g}_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu = \left(1 - \frac{2\alpha}{\rho}\right)(c\,dt)^2 - \left[\delta_{ik} + \left(\frac{\rho}{\rho - 2\alpha} - 1\right)\frac{\bar{x}^i \bar{x}^k}{\rho^2}\right] d\bar{x}^i d\bar{x}^k \quad (1)$$

here $\rho \equiv \sqrt{\bar{x}^i \bar{x}^i}$, $i, k = 1,2,3$, $\mu, \nu = 0,1,2,3$. Actually ρ is the "radial coordinate", so that $2\pi\rho$ is the circumference (in 3-space) having the origin as the centre and crossing a given point $\bar{x}^i$. The condition of correspondence with Newton's theory as $\rho \to \infty$ gives $\alpha = GM/c^2$; here G is the Newtonian gravitational constant, M is the mass of the central body, c is the velocity of light. It is obvious, that one may use this system of coordinate only if $\rho > 2\alpha = 2\,GM/c^2 = r_g$, i.e. not reaching the well-known coordinate "Schwarzschild singularity". Now, a transformation is possible, so that new coordinates of the same sort are $x^0,\ x^1,\ x^2,\ x^3$ and

$$\bar{x}^0 = x^0 = ct, \quad \bar{x}^i = x^i \frac{\rho(r)}{r}, \quad r \equiv \sqrt{x^i x^i}. \quad (2)$$

The interval (1) takes the form

$$ds^2 = \left(1 - \frac{2\alpha}{\rho(r)}\right)(c\,dt)^2 - \frac{\rho^2(r)}{r^2}\left[\delta_{ik} + \left(\frac{\rho'^2 r^2}{\rho^2 - 2\alpha\rho} - 1\right)\frac{x^i x^k}{r^2}\right] dx^i dx^k \quad (3)$$

$$g \equiv \det g_{\mu\nu} = -\frac{\rho^4 \rho'^2}{r^4}, \quad \rho' \equiv \frac{d\rho(r)}{dr}.$$

For the correspondence with Newtonian theory one must demand as $r \to \infty$

$$\lim \frac{\rho(r)}{r} = 1. \quad (4)$$

Lanczos [5] and later Rosen [6] and Fock [7] have suggested to make this transformation so that the new coordinates satisfy the De Donder condition

$$\frac{1}{\sqrt{-\bar{g}}} \frac{\partial}{\partial \bar{x}^\beta}\left(\sqrt{-\bar{g}}\ \bar{g}^{\alpha\beta} \frac{\partial x^\mu(\bar{x}^\sigma)}{\partial \bar{x}^\alpha}\right) = 0 \quad (5)$$

Fock in 1939 called coordinates of that type "harmonic" (in fact with some additional conditions). We see from (5) that $x^0 = \bar{x}^0$ is harmonic itself. The De Donder condition at $\mu = 1$, or 2,3 leads to the Legendre equation for the function $r = r(\rho)$, the inverse of $\rho(r)$:

$$\frac{d}{d\rho}\left[(\rho^2 - 2\alpha\rho)\frac{dr}{d\rho}\right] - 2r = 0. \quad (6)$$

The general solution of this equation is

$$r = C_1\left(\frac{\rho}{\alpha} - 1\right) + C_2\, Q\left(\frac{\rho}{\alpha} - 1\right), \quad Q(z) \equiv \frac{z}{2}\ln\frac{z+1}{z-1} - 1, \quad (7)$$

where C_1 and C_2 are integration constants. As the function $Q(\rho/\alpha - 1) \approx \frac{1}{3}(\alpha/\rho)^2 + \ldots \to 0$ when $\rho \to \infty$, the choice $C_1 = \alpha$ (and C_2 is arbitrary) leads to the condition (4) being satisfied. Beginning from Lanczos himself one has usually put $C_2 = 0$. Then we have the Lanczos form, later used by Rosen [6], Fock [7] and other authors

$$\rho(r) = r + \alpha. \quad (8)$$

The argumentation for putting $C_2 = 0$ is as follows. Lanczos pays attention to that for $C_2 \neq 0$, in the limit

$\alpha = GM/c^2 \to 0$ we do not arrive at the Minkowski interval $ds_M^2 = (c\,dt)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$. This situation may be avoided putting $C_2 = -\mathcal{K}\alpha$, where $\mathcal{K}$ is a dimensionless constant. Fock points to the function $Q(\rho/\alpha - 1) \sim -\frac{1}{2}\ln(\rho - 2\alpha) \to \infty$ as $\rho \to 2\alpha$. Chou [8] speaks about the metric singularity too.

Existence of the coordinate singularity for $C_2 \neq 0$ results in that the "initial" Schwarzschild 4-space would be somewhat narrowed. In particular, it would not be possible to come close to the Schwarzschild singularity at $\rho = 2\alpha$. But these harmonic coordinates in their range of definition are admissible. No doubt, it may be possible to use them for the description of an external gravitational field of a body, when the singular sphere does not appear. But the only example of using them I know is the Balek work [9] on a sort of the gravitation theory "with privileged harmonic coordinates".

So, putting $C_1 = \alpha$, $C_2 = -\mathcal{K}\alpha$, we have from (7)

$$r = \rho(r) - \alpha - \mathcal{K}\alpha\left(\frac{\rho - \alpha}{2\alpha}\ln\frac{\rho}{\rho - 2\alpha} - 1\right). \qquad (9)$$

Now, the magnitude and sign of constant $\mathcal{K}$ remains still undefined. Let us estimate the admissible regions of new coordinates for various $\mathcal{K}$. If $\mathcal{K} < 0$, we have the coordinate singularity when $\rho' \to \infty$ and then $g = -\rho^4\rho'^2/r^4 \to -\infty$. Denote ρ at those points by $\rho_s(r_s, \mathcal{K})$. The metric (3) is consistent if $\rho_s < \rho < \infty$. Say, when $\mathcal{K} = -1$, we have from (9) $r_s \cong 1{,}6\alpha$; $\rho_s \cong 2{,}3\alpha$, i.e. the admissible region is somewhat smaller than the initial one $2\alpha < \rho < \infty$.

When $|\mathcal{K}|$ is very large, namely, $|\mathcal{K}| \gg 10^3$, we have approximately

$$\rho_s \approx \left(-\tfrac{2}{3}\mathcal{K}\right)^{1/3}\alpha, \quad r_s \approx \left(-\tfrac{9}{4}\mathcal{K}\right)^{1/3}\alpha.$$

If $\mathcal{K} > 0$, the "radial coordinate" r may accept any positive value, the singularity appears when $r \to 0$. Then $g \to -\infty$, as $\rho(0, \mathcal{K}) > 0$, $\rho' > 0$. Say, if $\mathcal{K} = 1$, then $\rho(0,1) \cong 2{,}04\,\alpha$, so the admissible region $\rho(0,1) < \rho < \infty$ is a slightly smaller than the initial Schwarzschildian.

When $\mathcal{K} \gg 10^3$, $\rho(0, \mathcal{K}) \approx (\mathcal{K}/3)^{1/3}\alpha$. At the space infinity $r, \rho(r, \mathcal{K}) \to \infty$ these coordinates rapidly tend to the Lanczos ones

$$\rho(r, \mathcal{K}) = r + \alpha + \frac{\mathcal{K}\alpha}{3}\left(\frac{\alpha}{r}\right)^2 + \ldots, \quad r = \rho - \alpha - \frac{\mathcal{K}\alpha}{3}\left(\frac{\alpha}{\rho}\right)^2 + \ldots$$

In other words, at least if $0 \leq |\mathcal{K}| \lesssim 10$ such systems of harmonic coordinates might be used for the description of the external gravitational field of some known celestial bodies.

Now we consider the energy density of this gravitational field. Since outside the body the material tensor $T_{\mu\nu} = 0$, the Einstein energy-momentum pseudotensor for harmonic coordinates may be written as [7]

$$t_\mu^{\nu} = \frac{c^4}{16\pi G}\frac{\partial}{\partial x^\sigma}\left[\frac{g_{\mu\tau}}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(g g^{\alpha\nu} g^{\sigma\tau} - g g^{\alpha\sigma} g^{\nu\tau}\right)\right].$$

The energy density in the case of metric (3) is

$$t_0^0 = \frac{c^4}{16\pi G}\cdot\frac{2}{r^2\rho\mathcal{D}}\left[2\rho'^2 - \rho^2 r^{-2} - 4r\rho\rho'^2\mathcal{D} + 3\rho^2\rho'\mathcal{D} - 2\rho\rho'' + r^2\rho^2\rho''\mathcal{D}\right],$$

where $\mathcal{D} \equiv \rho'(\rho^2 - 2\alpha\rho)^{-1}$.

It is sufficient for us to find the asymptotic result when $\frac{\alpha}{r} \to \infty$, namely,

$$t_0^0 = \frac{c^4}{16\pi G}\cdot\frac{2\alpha^2}{r^4}\left(1 + \frac{2}{3}\mathcal{K}\frac{\alpha}{r} + \dots\right)$$

and we see that it depends on $\mathcal{K}$. Thus, using different harmonic systems of coordinates, we get different energy-density values for the same physical 4-point.

If we take the symmetrical Landau-Lifshitz-Fock energy-momentum pseudotensor adapted for harmonic coordinates [7]

$$\mathcal{U}^{\mu\nu} = \frac{c^4}{16\pi G}\left(\hat{g}^{\alpha\beta}\frac{\partial^2\hat{g}^{\mu\nu}}{\partial x^\alpha\partial x^\beta} - \frac{\partial\hat{g}^{\alpha\mu}}{\partial x^\beta}\cdot\frac{\partial\hat{g}^{\beta\nu}}{\partial x^\alpha}\right),\quad \hat{g}^{\mu\nu} \equiv \sqrt{-g}\,g^{\mu\nu}$$

then for (3), (9) when $\frac{\alpha}{r} \to \infty$ we get

$$\mathcal{U}^{00} = -\frac{c^4}{16\pi G}\,\frac{2\alpha^2}{r^4}\left(7 + 20\frac{\alpha}{r} - 2\mathcal{K}\frac{\alpha}{r} + \dots\right)$$

so that this energy density expression depends on $\mathcal{K}$ too.

This result leads to conclusion coinciding completely with the long-standing one by Schrödinger [4] and Bauer [10] about the dependence of the energy-momentum pseudotensor on the choice of any coordinate system. Now it is not in favour of the hypothesis by Fock and other authors (e.g.,[11]) about the privilegy of harmonic coordinates and the possibility of removing some known difficulties of general relativity by using them.

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Асанов Р.А. E2-87-612
Метрика Шварцшильда и условия Де Дондера

Рассматриваются необычные математические формы метрики Шварцшильда в гармонических координатах. Показано, что выражение для плотности энергии для этих форм обладает обычными для общей теории относительности свойствами.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Asanov R.A. E2-87-612
Schwarzschild Metric and De Donder Condition

Non-ordinary mathematical form of the Schwarzschild metric in harmonic coordinates is considered. The ordinary property of energy density for it is demonstrated.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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