сообщения
объединенного
института
ядерных
исследований
дубна

E2-87-599

J.Dittrich, P.Exner

A NON-RELATIVISTIC MODEL OF TWO-PARTICLE DECAY.

Relation to the scattering theory, spectral concentration, and bound states

1987

1. Introduction

This is the fourth part of the paper devoted to a detailed analysis of a Lee-type model of two-particle decay. Let us recollect briefly the contents of the previous parts /1-3/, hereafter referred to as I-III. The model itself has been described in I, where we have proven also its Galilean invarince *). After separating the centre-of-mass motion, the analysis simplifies to the perturbation problem of a simple eigenvalue embedded into a non-simple continuous spectrum. Its solution depends crucially on the properties of the reduced resolvent. In II, we show that for small enough values of the coupling constant g, the reduced resolvent has a simple second-sheet pole whose position depends analytically on g.

Finally, in III we study the pole approximation in which the analytically continued reduced resolvent is replaced by the pole term alone. We show that under some mild regularity requirements on the function v that characterizes the interaction, the deviations of the true decay law from the exponential one resulting from the approximation is of order g^4. This gives us further a possibility of establishing the rigorous validity of Fermi golden rule for the present model.

Here we shall be concerned with three other problems. In the next section, we shall discuss relations of the model under consideration to the scattering theory. We establish existence and completeness of the wave operators for the situation when the two light particles that have played the role of decay products scatter elastically. Then we express the R-matrix and show that for a sufficiently weak coupling, the system has just one resonance. Its position as a position of the R-matrix pole is the same as that of the pole considered in Theorem II.3.6 and further on.

---

*) In the preprint version, division of the material between the first two parts is different - cf. a comment on this point in the introduction to III.

It is much more difficult to prove that the wave operators are also asymptotically complete, i.e., that $\sigma_{sing}(H_g) = \emptyset$. This will be done in Section 3 under some additional assumptions concerning smoothness of the function v . As a technical tool, we compute here the full resolvent of the Hamiltonian H_g . Section 4 deals with the problem of spectral concentration : we show that the spectral projections $E_{H_g}(\Delta_g)$ on a family of intervals Δ_g that shrink around E tend to the projection E_u , provided the shrinking is slower than quadratic in g . In this way, the decaying system remembers the embedded eigenvalue which "dissolved" once the perturbation was turned on. Finally, the existence of bound states is discussed in Section 5 .

The next part will deal with the quntum-kinematical aspect of the decay under consideration, and with the relations of an appropriate kinematical model to the dynamical one discussed in I-IV .

2. Connection with the scattering theory

Consider the scattering process between the two light particles of masses m_1 and m_2 which have played the role of decay products in the previous parts of the paper. In view of (II.2.4), the non-trivial part of the scattering problem concerns the centre-of-mass coordinates only. In the following, therefore, we shall use the "relative" quantities without mentioning this fact explicitly.

The wave operators for the pair (H_g, H_0) are defined by

$$\Omega_{\pm}(H_g, H_0) = \operatorname*{s-lim}_{t\to\pm\infty} e^{iH_g t} e^{-iH_0 t} P_{ac}(H_0) \quad , \tag{2.1}$$

where $P_{ac}(H_0)$ is the projection referring to the absolutely continuous subspace of H_0 . We have the following

Theorem 2.1 : The wave operators (2.1) exist and are complete, i.e.,

$$\operatorname{Ran} \Omega_+(H_g, H_0) = \operatorname{Ran} \Omega_-(H_g, H_0) = \operatorname{Ran} P_{ac}(H_g) \quad . \tag{2.2}$$

Furthermore, assume (a)-(c) and (e) (cf.Section III.2). If the function $\hat{v}_1$ has a piecewise continuous derivative, then $\Omega_{\pm}(H_g, H_0)$ are also asymptotically complete, i.e., $\sigma_{sing}(H_g) = \emptyset$.

Since the interaction Hamiltonian gV is of a finite rank, the existence and completeness follow from Kato-Rosenblum theorem (cf. Ref.4, theorem XI.8). It is much more difficult to establish the asymptotic completeness ; we postpone it to Section 3 .

The wave operators (2.1) determine the S-matrix of our problem by

$$S = \Omega_+^* \Omega_- \quad . \tag{2.3}$$

In order to derive an explicit expression for S , we employ the following Lippmann-Schwinger-type result (Ref.5, Proposition 6.11) : suppose that the wave operators exist and are complete, $H_g = H_0 + gV$ is self-adjoint and $D(V) \supset D(H_0)$, then

$$S = P_{ac}(H_0) + \operatorname*{s-lim}_{\eta\to 0+} \operatorname*{s-lim}_{\delta\to 0+} \int_{\mathbb{R}} \left[R(\lambda - i\eta, H_0) - R(\lambda + i\eta, H_0)\right] \times$$
$$\times \left[gV - g^2 V R(\lambda + i\delta, H_g) V\right] P_{ac}(H_0)\, dE_\lambda^{(0)} \quad , \tag{2.4}$$

where, as in Sec.III.5, the decomposition of unity $\{E_\lambda^{(0)}\}$ refers to the free Hamiltonian H_0 . The assumptions are obviously valid in our case, so (2.4) holds. For brevity, we shall work in the followig with the R-matrix defined by

$$R = S - P_{ac}(H_0) \quad . \tag{2.5}$$

Our goal is now to express matrix elements of this operator. It is clear that both S and R commute with $P_{ac}(H_0)$; in fact, S is a partial isometry of the subspace $P_{ac}(H_0) = \mathcal{H}_d$ into itself (Ref.5, Proposition 4.6). Hence we shall consider vectors of the form $\Psi = \begin{pmatrix} 0 \\ \psi_d \end{pmatrix}$, $\Phi = \begin{pmatrix} 0 \\ \varphi_d \end{pmatrix}$ only, restricting our attention to those with $\hat{\psi}_d, \hat{\varphi}_d \in C_0^\infty(\mathbb{R}^3)$.

Let us first express the integral on the rhs of (2.4). We can write it as

$$C \equiv \int_0^\infty B(\lambda) P_{ac}(H_0)\, dE_\lambda^{(0)} \quad , \tag{2.6a}$$

where $B(\lambda)$ denotes the product of the square brackets, because the support of $dE_\lambda^{(0)}$ is $\mathbb{R}^+$. Moreover, in the corresponding expression of $(\Psi, C\Phi)$ the integral is actually taken over a finite interval $\Delta = [0, \lambda_0]$, since the supports of $\hat{\psi}_d, \hat{\varphi}_d$ are by assumption contained in a ball of radius $\sqrt{2m\lambda_0}$ for some λ_0 . The integral is defined (cf,Ref.5, Section 6.1) as a strong limit of the corresponding Riemannian sums, so we have

$$(\Psi, C\Phi) = \lim_{\delta_n \to 0} \sum_j \left(B(\mu_j^n)^* \Psi, E_0(\Delta_j^n) \Phi\right) \quad , \tag{2.6b}$$

where λ_j^n are the points of some partition of Δ, $\mu_j^n \in \Delta_j^n = (\lambda_j^n, \lambda_{j+1}^n]$ and $\delta_n = \max_j \delta_j^n$, where δ_j^n is the length of Δ_j^n. The vectors appearing in the last expression can be written down explicitly in the momentum representation,

$$(E_0(\Delta_j^n)\Phi)_d(\vec p) = \hat\varphi_d(\vec p)\chi_{\Delta_j^n}\left(\frac{\vec p^{\,2}}{2m}\right) \tag{2.7a}$$

and

$$(B(\lambda)^*\Psi)_d(\vec p) = -g^2\hat v(\vec p)\,\overline{r_u(\lambda+i\delta,H_g)}\int_{\mathbb{R}^3}\overline{\hat v(\vec k)}\frac{2i\eta}{\left(\frac{\vec k^2}{2m}-\lambda\right)^2+\eta^2}\hat\psi_d(\vec k)\,d\vec k\,, \tag{2.7b}$$

where the term linear in g is absent in view of the condition (II.3.3). Substituting from here into (2.6b), we get

$$(\Psi,C\Phi) = \lim_{\delta_n\to 0}\sum_j C(\mu_j^n,\lambda_j^n,\delta_j^n)\delta_j^n\,, \tag{2.8a}$$

where

$$C(\mu,\lambda,\varepsilon) = \varepsilon^{-1}\int_{M(\lambda,\varepsilon)}\overline{(B(\mu)^*\Psi)_d(\vec p)}\,\hat\varphi_d(\vec p)\,d\vec p \tag{2.8b}$$

and $M(\lambda,\varepsilon) = B_3(\sqrt{2m(\lambda+\varepsilon)})\setminus B_3(\sqrt{2m\lambda})$. The sum on the rhs of (2.8a) is just the integral of the step function F_n that assumes the value

$$F_n(\lambda) = C(\mu_j^n,\lambda_j^n,\delta_j^n)$$

for $\lambda\in\Delta_j^n$. So we have

$$(\Psi,C\Phi) = \lim_{n\to\infty}\int F_n(\lambda)\,d\lambda\,, \tag{2.8c}$$

provided $\lim_{n\to\infty}\delta_n = 0$.

Suppose now that the function $\hat v_1$ fulfils the assumption (d). Since (2.7b) is a continuous function of λ and $\vec p$, one checks easily that

$$\lim_{n\to\infty}F_n(\lambda) = m\sqrt{2m\lambda}\int_{S_2}\overline{(B(\lambda)^*\Psi)_d(\sqrt{2m\lambda}\,\vec n)}\,\hat\varphi_d(\sqrt{2m\lambda}\,\vec n)\,d\Omega_{\vec n}\,, \tag{2.9a}$$

where S_2 is the unit sphere and $\lambda\in\Delta$. Further (2.7b) and (2.8b) gives the estimate

$$|C(\mu,\lambda,\varepsilon)| \le \frac{2g^2}{\varepsilon\eta\delta}(|\hat v|,|\hat\psi_d|)\int_{M(\lambda,\varepsilon)}|\overline{\hat v(\vec p)}\,\hat\varphi_d(\vec p)|\,d\vec p\,.$$

In view of the stated assumption (recall that $\hat\varphi_d\in C_0^\infty(\mathbb{R}^3)$), modulus of the integrand in the last integral is bounded by a positive K within $B_3(\sqrt{2m\lambda_0})$. Then the integral may be estimated by $K\int_{M(\lambda,\varepsilon)}d\vec p \le 4\pi m\sqrt{2m\lambda_0}\,\varepsilon$ for $0\le\lambda<\lambda+\varepsilon\le\lambda_0$, and we have the following ε-independent bound

$$|C(\mu,\lambda,\varepsilon)| \le \frac{8\pi mK\sqrt{2m\lambda_0}}{\eta\delta}\,g^2(|\hat v|,|\hat\psi_d|)\,. \tag{2.9b}$$

The relations (2.8c) and (2.9) together with the dominated-convergence theorem then show that

$$(\Psi,C\Phi) = \int_0^\infty d\lambda\, m\sqrt{2m\lambda}\int_{S_2}\overline{(B(\lambda)^*\Psi)_d(\sqrt{2m\lambda}\,\vec n)}\,\hat\varphi_d(\sqrt{2m\lambda}\,\vec n)\,d\Omega_{\vec n} =$$

$$= \int_{\mathbb{R}^3}\overline{\left(B\left(\frac{\vec p^{\,2}}{2m}\right)^*\Psi\right)_d(\vec p)}\,\hat\varphi_d(\vec p)\,d\vec p\,.$$

Combining now the relations (2.4), (2.5) and (2.7b), we get

$$(\Psi,R\Phi) = \tag{2.10}$$

$$= g^2\lim_{\eta\to 0+}\lim_{\delta\to 0+}\int_{\mathbb{R}^3}d\vec p\int_{\mathbb{R}^3}d\vec k\,\overline{\hat v(\vec p)}\hat\varphi_d(\vec p)\,r_u\left(\frac{p^2}{2m}+i\delta,H_g\right)\frac{2i\eta}{\left(\frac{\vec k^2}{2m}-\frac{\vec p^{\,2}}{2m}\right)^2+\eta^2}\,\hat v(\vec k)\overline{\hat\psi_d(\vec k)}\,.$$

Assume now that for each $\lambda>0$, there is a finite limit $\lim_{\substack{z\to\lambda\\ \mathrm{Im}\,z>0}} r_u(z,H_g) = r_u(\lambda+i0,H_g)$ and that the function $(\lambda,\delta)\mapsto r_u(\lambda+i\delta,H_g)$ is bounded in $(0,\lambda_0)\times(0,\varepsilon_0)$ for some ε_0. This is true, e.g., under the assumptions (a)-(e) for a sufficiently small coupling constant $g\neq 0$. Since $\hat\varphi_d,\hat\psi_d$ have compact supports, the dominant-convergence theorem implies

$$(\Psi,R\Phi) =$$

$$= g^2\lim_{\eta\to 0+}\int_{\mathbb{R}^3}d\vec p\int_{\mathbb{R}^3}d\vec k\,\overline{\hat v(\vec p)}\hat\varphi_d(\vec p)\,r_u\left(\frac{p^2}{2m}+i0,H_g\right)\frac{2i\eta}{\left(\frac{\vec k^2}{2m}-\frac{\vec p^{\,2}}{2m}\right)^2+\eta^2}\,\hat v(\vec k)\overline{\hat\psi_d(\vec k)}\,.$$

Next we must interchange the limit $\eta\to 0+$ with the integrals. Consider first the second integral : we can write it as

$$\lim_{\eta\to 0+}\int_0^\infty dk\,\frac{2i\eta k^2}{\left(\frac{k^2}{2m}-\frac{\vec p^{\,2}}{2m}\right)^2+\eta^2}\,\hat v_1(k)\int_{S_2}\overline{\hat\psi_d(k\vec n)}\,d\Omega_{\vec n}\,.$$

Since $\hat{\psi}_d \in C_0^\infty(\mathbb{R}^3)$, the last integral is a bounded continuous function of k; we denote it as $I_\psi(k)$. Introducing the variable $y = (k^2 - \vec{p}^2)/2m\eta$, the last expression can be rewritten as

$$2im \lim_{\eta \to 0+} \int_{-\frac{\vec{p}^2}{2m}}^{\infty} \frac{dy}{1+y^2} \sqrt{\vec{p}^2 + 2m\eta y}\; \hat{v}_1(\sqrt{\vec{p}^2+2m\eta y})\; I_\psi(\sqrt{\vec{p}^2+2m\eta y}) \quad . \qquad (2.11)$$

However, $\hat{v}_1$ is bounded by the assumption (d), and the limit can be therefore calculated by the dominated-convergence theorem to be $2\pi imp\hat{v}_1(p)I_\psi(p)$. Using once more the fact that the integral in (2.11) can be estimated independently of η in an interval $(0,\eta_0)$, we are able to justify application of the dominated-convergence theorem to the interchange of the limit with the first integral. Together we get

$$(\Psi, R\Phi) = 2\pi i m g^2 \int_{\mathbb{R}^3} d\vec{p}\; \hat{\varphi}_d(\vec{p})\, |\hat{v}_1(p)|^2\, p\, r_u\!\left(\frac{p^2}{2m} + i0\,, H_g\right) \int_{S_2} \bar{\hat{\psi}}_d(p\vec{n})\, d\Omega_{\vec{n}} =$$

$$= 2\pi i m g^2 \int_{\mathbb{R}^3} d\vec{p}\; \bar{\hat{\psi}}_d(\vec{p})\, |v_1(p)|^2\, p\, r_u\!\left(\frac{p^2}{2m} + i0\,, H_g\right) \int_{S_2} \varphi_d(p\vec{n}')\, d\Omega_{\vec{n}'} \;,$$

where the second equality follows from Fubini theorem. Since $C_0^\infty(\mathbb{R}^3)$ is dense in $\mathcal{H}_d$, we get

$$(R\Phi)_d(\vec{p}) = 2\pi i m g^2 |\hat{v}_1(p)|^2\, p\, r_u\!\left(\frac{p^2}{2m} + i0,\; H_g\right) \int_{S_2} \hat{\varphi}_d(p\vec{n}')\, d\Omega_{\vec{n}'} \;. \qquad (2.12a)$$

The operator R is bounded so by standard density arguments the validity of the last relation may be extended to each $\Phi \in \mathcal{H}$.

Let us conclude the above discussion.

<u>Theorem 2.2</u> : Assume (a)-(e), then the relation (2.12a) holds. In other words, R is a decomposable operator on $\mathcal{H}_d = L^2(\mathbb{R}^+; L^2(S_2))$ and its component $R(\lambda)$ for a given $\lambda \in \mathbb{R}^+$ is the Hilbert-Schmidt operator with the kernel

$$R(\lambda; \vec{n}, \vec{n}') = 2\pi i m g^2 |\hat{v}_1(\sqrt{2m\lambda})|^2 \sqrt{2m\lambda}\; r_u(\lambda + i0\,,\, H_g) \quad . \qquad (2.12b)$$

Now we are able to write down the amplitude of the elastic scattering with the initial momentum $\vec{p}' = p\vec{n}'$ and final momentum $\vec{p} = p\vec{n}$. It equals (cf.Ref.5, eq.(7.48)) $-2\pi i\,(2m\lambda)^{-1/2} R(\lambda; \vec{n}, \vec{n}')$, i.e.,

$$f(\lambda; \vec{p}' \to \vec{p}) = 4\pi^2 m g^2 |\hat{v}_1(\sqrt{2m\lambda})|^2\, r_u(\lambda + i0\,,\, H_g) \quad . \qquad (2.13)$$

Furthermore, the assumptions of Theorem 2.2 ensure that both $\lambda \mapsto |v_1(\sqrt{2m\lambda})|^2$ and $\lambda \mapsto r_u(\lambda + i0, H_g)$ can be continued analytically from the upper halfplane to the region Ω that contains $(0,\infty)$ (as for the first of them, recall that we choose the cut of the square root conventionally along the negative real axis). For small enough g, the continued function $\lambda \mapsto r_u(\lambda+i0,H_g)$ has, according to Theorem II.3.6, a simple pole whose position depends analytically on g. The same is then true for the scattering amplitude. In the commonly accepted terminology, this fact can by expressed as

<u>Proposition 2.3</u> : Under the assumptions (a)-(e), there are a positive g_0 and a complex region $\Omega_1 \supset (0,\infty)$ such that for $0 < |g| < g_0$, the above described scattering system has just one resonance in Ω_1 whose position is given by the relations (II.3.16).

In order to illustrate typical features of the resonance scattering, let us calculate the cross section and the phase shift. The differential cross section is equal to the squared modulus of (2.13). In order to get the squared modulus of $r_u(\lambda+i0,H_g)$, one must use the relations (II.3.11) and (II.3.15). A short calculation then gives

$$\frac{d\sigma}{d\Omega}(\lambda; \vec{p}' \to \vec{p}) = \frac{16\pi^4 m^2 g^4 |\hat{v}_1(\sqrt{2m\lambda})|^4}{[\lambda - E - 4\pi g^2 I(\lambda, v)]^2 + 32\pi^4 m^3 g^4 \lambda |\hat{v}_1(\sqrt{2m\lambda})|^4} \quad . \qquad (2.14)$$

It is clear that the scattering is <u>isotropic</u> as a consequence of the fact that v is rotationally invariant. In that case, the total cross section does not depend on the chosen initial direction and equals $4\pi(d\sigma/d\Omega)$. It means that both of them have the same shape as functions of the energy. For σ_{tot}, e.g., we have

$$\sigma_{tot}(\lambda) = \frac{\pi}{2m\lambda} \frac{\Gamma(g,\lambda)^2}{(\lambda - E(g,\lambda))^2 + \frac{1}{4}\Gamma(g,\lambda)^2} \quad , \qquad (2.15a)$$

where

$$E(g,\lambda) = E + 4\pi g^2 I(\lambda, v) \quad , \qquad (2.15b)$$

$$\Gamma(g,\lambda) = 8\pi^2 m g^2 |\hat{v}_1(\sqrt{2m\lambda})|^2 \sqrt{2m\lambda} \quad . \qquad (2.15c)$$

If the coupling constant g is sufficiently small and the function $\hat{v}_1$ is slowly varying around E, then we obtain an approximative expression to (2.15a) replacing $E(g,\lambda)$ by $E(g,E)$ and $\Gamma(g,\lambda)$ by $\Gamma(g,E) = \Gamma_F(g)$. The cross-section formula acquires the familiar Breit-Wigner shape with the peak situated at $E(g,E)$, and with the width given again by the Fermi-rule expression (III.5.2). At the same time, it is clear that the cross section may have accidental peaks connected with the shape of the function $\hat{v}_1$.

Let us turn now to the phase shift. According to (2.5) and Theorem 2.2, S is a decomposable operator on $\mathcal{H}_d$ and its component for a given λ is $S(\lambda) = I + R(\lambda)$. Only the s-wave part is non-trivial according to (2.12a). The s-wave phase shift $\delta_0(\lambda)$ is determined by the relation

$$S_0(\lambda) = e^{2i\delta_0(\lambda)} I = (1 + 4\pi R(\lambda;\vec{n},\vec{n}'))I \quad ; \qquad (2.16a)$$

the higher phase shifts $\delta_l(\lambda) = 0$, $l = 1,2,\ldots$. Using now the abbreviations (2.15b,c), one finds after a short calculation that

$$S_0(\lambda) = \frac{-\lambda + E(g,\lambda) + \frac{1}{2}\Gamma(g,\lambda)}{-\lambda + E(g,\lambda) - \frac{1}{2}\Gamma(g,\lambda)} \quad , \qquad (2.16b)$$

and therefore

$$\delta_0(\lambda) = \operatorname{arctg} \frac{\frac{1}{2}\Gamma(g,\lambda)}{E(g,\lambda) - \lambda} \quad (\mathrm{mod}\ \pi) \quad . \qquad (2.17)$$

For a weak coupling and a slowly varying $\hat{v}_1$, one has an approximative expression of this function around $E(g,E)$, namely

$$\delta_0(\lambda) \approx \begin{cases} \operatorname{arctg} \dfrac{\frac{1}{2}\Gamma_F(g)}{E(g,E) - \lambda} & \ldots \quad \lambda < E(g,E) \\ \dfrac{\pi}{2} & \ldots \quad \lambda = E(g,E) \\ \pi + \operatorname{arctg} \dfrac{\frac{1}{2}\Gamma_F(g)}{E(g,E) - \lambda} & \ldots \quad \lambda > E(g,E) \end{cases} \quad , \qquad (2.18)$$

again modulo π . It shows that the phase shift has a sheer increase which changes its value on about π . This is another characteristic feature of a resonance.

The last approximative formula holds in the resonant region only. The asymptotic behaviour for large and small λ can be also easily obtained under some additional, not very restrictive assumptions. From Lemmas II.3.4 and III.2.2, we know that $I(\cdot,v)$ is bounded,

$$|I(\lambda,v)| \le C_2 \quad ,$$

under the assumtions (a),(b) and (d). Therefore eq.(2.17) gives

$$\delta_0(\lambda) \approx -4\pi^2 m g^2 |\hat{v}_1(\sqrt{2m\lambda})|^2 \sqrt{\frac{2m}{\lambda}} \quad (\mathrm{mod}\ \pi) \qquad (2.19a)$$

as $\lambda \to \infty$, the asymptotic region being specified by the requirement $\lambda \gg E + 4\pi g^2 C_2$. In particular, one has

$$\lim_{\lambda\to\infty} \delta_0(\lambda) = 0 \quad (\mathrm{mod}\ \pi) \quad . \qquad (2.19b)$$

To find the behaviour for $\lambda \to 0+$, we need to know the limit of $I(\lambda,v)$ first.

<u>Lemma 2.4</u> : Assume (a) and (d), then

$$I(0+,v) \equiv \lim_{\lambda\to 0+} I(\lambda,v) = -2m \int_0^\infty |\hat{v}_1(p)|^2 dp \quad . \qquad (2.20)$$

<u>Proof</u> : To a given $\lambda > 0$, we choose a positive A so that $A^2 > 4m\lambda$. Then $I(\lambda,v)$ exists according to Lemma II.3.3 and

$$I(\lambda,v) = \mathcal{P}\int_0^A \frac{|\hat{v}_1(p)|^2 p^2}{\lambda - \frac{p^2}{2m}} dp + \int_A^\infty \frac{|\hat{v}_1(p)|^2 p^2}{\lambda - \frac{p^2}{2m}} dp \quad .$$

By the dominated-convergence theorem,

$$\lim_{\lambda\to 0+} \int_A^\infty \frac{|\hat{v}_1(p)|^2 p^2}{\lambda - \frac{p^2}{2m}} dp = -2m \int_A^\infty |v_1(p)|^2 dp$$

(note that $\hat{v}_1 \in L^2(\mathbb{R}^+)$ according to the assumptions (a) and (d)). A simple calculation gives

$$\mathcal{P}\int_0^A \frac{|\hat{v}_1(p)|^2 p^2}{\lambda - \frac{p^2}{2m}} dp = \int_0^A \frac{v_2(p) - v_2(\sqrt{2m\lambda})}{\lambda - \frac{p^2}{2m}} p\, dp + m\, v_2(\sqrt{2m\lambda}) \ln \frac{\lambda}{\frac{A^2}{2m} - \lambda} \quad . \qquad (2.21)$$

Since $|v_2(\sqrt{2m\lambda})| \le C_1\sqrt{2m\lambda}$ by the assumption (d), the second term tends to zero as $\lambda \to 0+$. With the help of the finite-difference formula, one checks that the integrand in the first term on the rhs of (2.21) is bounded uniformly with respect to λ . So the integral exists in the sense of Lebesgue and the dominated-convergence theorem yields

$$\lim_{\lambda\to 0+} \mathcal{P}\int_0^A \frac{|\hat{v}_1(p)|^2 p^2}{\lambda - \frac{p^2}{2m}} dp = -2m \int_0^A |\hat{v}_1(p)|^2 dp \quad .$$

Summing the two contributions, we arrive at (2.20). ∎

Let us now denote

$$g_{cr} = E^{1/2}\left[8\pi m \int_0^\infty |\hat{v}_1(p)|^2\, dp\right]^{-1/2} \quad . \tag{2.22}$$

Proposition 2.5 : Assume (a)-(e) and $|g| \neq g_{cr}$. Then

$$\lim_{\lambda\to 0+} \delta_0(\lambda) = 0 \quad (\mathrm{mod}\ \pi) \quad . \tag{2.23a}$$

Furthermore, if $|\hat{v}_1|^2 \in C([0,\infty))$ and $|\hat{v}_1(0)| \neq 0$, then

$$\delta_0(\lambda) \approx \frac{4\pi^2 m g^2 |\hat{v}_1(0)|^2 \sqrt{2m\lambda}}{E\left[1-(g/g_{cr})^2\right]} \quad (\mathrm{mod}\ \pi) \tag{2.23b}$$

as $\lambda \to 0+$.

Proof : The relations (2.23) follow from (2.17) and Lemma 2.4. ■

For $|g| = g_{cr}$, we are able to deduce the analogous conclusion under an additional assumption on v . We need an estimate on $I(\lambda,v)$ first.

Lemma 2.6 : Assume (a),(d) and $\frac{d}{dp}|\hat{v}_1(p)|^2$ bounded in $(0,\infty)$. Then there is a positive c such that

$$|I(\lambda,v) - I(0+,v)| < c \ln\frac{\lambda}{m} \tag{2.24}$$

for $\lambda > 0$.

Proof : Let us choose $A > 0$ and denote $v_4(p) \equiv |v_1(p)|^2$. Since $I(.,v)$ is bounded, it is sufficient to consider $\lambda \in (0, A^2/4m)$. By Lemma 2.4 ,

$$I(\lambda,v) - I(0+,v) = 2m\lambda\, \mathcal{P}\int_0^\infty \frac{v_4(p)\,dp}{\lambda - \frac{p^2}{2m}} \quad . \tag{2.25}$$

We estimate

$$\left| \int_A^\infty \frac{v_4(p)}{\lambda - \frac{p^2}{2m}}\, dp \right| \leq \frac{4m}{A^2} \int_A^\infty v_4(p)\, dp < \infty \quad .$$

Furthermore,

$$\mathcal{P}\int_0^A \frac{v_4(p)}{\lambda - \frac{p^2}{2m}}\, dp = v_4(\sqrt{2m\lambda})\sqrt{\frac{m}{2\lambda}}\, \ln\frac{A+\sqrt{2m\lambda}}{A-\sqrt{2m\lambda}} - 2m\int_0^A \frac{v_4'(\xi(\lambda,p))}{p+\sqrt{2m\lambda}}\, dp \quad ,$$

where $\xi(\lambda,p)$ is a number between $\sqrt{2m\lambda}$ and p . Since

$$0 < \sqrt{\frac{m}{2\lambda}}\, \ln\frac{A+\sqrt{2m\lambda}}{A-\sqrt{2m\lambda}} = \sqrt{\frac{m}{2\lambda}}\, \ln\left(1 + \frac{2\sqrt{2m\lambda}}{A-\sqrt{2m\lambda}}\right) \leq \frac{2(2+\sqrt{2})\, m}{A}$$

and

$$0 < \int_0^A \frac{dp}{p+\sqrt{2m\lambda}} = \ln\left(1+\frac{A}{\sqrt{2m\lambda}}\right) \leq \ln\left(\sqrt{2}\,\frac{A}{m}\right) - \frac{1}{2}\ln\frac{\lambda}{m} \quad ,$$

the inequality (2.24) follows with a suitable c . ■

Proposition 2.7 : Assume (a)-(e). Let the function $|\hat{v}_1|^2$ be continuous in $[0,\infty)$ and continuously differentiable in $(0,\infty)$ with $|\hat{v}_1(0)| \neq 0$, and $|g| = g_{cr}$. Then

$$\lim_{\lambda\to 0+} \delta_0(\lambda) = \frac{\pi}{2} \quad (\mathrm{mod}\ \pi) \quad . \tag{2.25}$$

Proof : The eqs.(2.16b) and (2.22) give

$$S_0(\lambda) = \frac{-\lambda + O(\lambda|\ln\frac{\lambda}{m}|) + 4\pi^2 i m g^2 |\hat{v}_1(\sqrt{2m\lambda})|^2 \sqrt{2m\lambda}}{-\lambda + O(\lambda|\ln\frac{\lambda}{m}|) - 4\pi^2 i m g^2 |\hat{v}_1(\sqrt{2m\lambda})|^2 \sqrt{2m\lambda}}$$

so, according to Lemma 2.6, $S_0(\lambda)$ tends to -1 as $\lambda \to 0+$ and (2.25) follows. ■

3. Asymptotic completeness

Now we would like to complete the proof of Theorem 2.1 . In order to do that, we need a more complete information about the full resolvent $(H_g - z)^{-1}$ of the Hamiltonian. The obtained formulae will be useful also in the next section.

Proposition 3.1 : Let $\hat{\psi}_d \in \mathcal{H}_d$, and assume that z belongs to the resolvent sets of H_g and H_0 , then

$$E_d(\hat{H}_g - z)^{-1} E_d \hat{\psi}_d = (\hat{H}_0 - z)^{-1}\hat{\psi}_d + g^2 r_u(z, H_g)(\hat{\varphi}_{2z}, \hat{\psi}_d)\hat{\varphi}_{1z} \quad , \tag{3.1a}$$

where the vectors $\hat{\varphi}_{jz}$ are defined by

$$\hat{\varphi}_{1z} = (\hat{H}_0 - z)^{-1}\hat{v} \quad , \quad \hat{\varphi}_{2z} = (H_0 - \bar{z})^{-1}\hat{v} \quad . \tag{3.1b}$$

In the last relation, $(H_0 - z)^{-1}$ stands as a shorthand for multiplication by $(p^2/2m - z)^{-1}$ – cf.(II.2.9).

Proof : The first part of the argument is similar to the proof of Proposition II.3.1, with the roles of $\mathcal{H}_u, \mathcal{H}_d$ reversed. We start from the second resolvent identity which yields the relations.

$$E_d(\hat{H}_g-z)^{-1}E_d = E_d(\hat{H}_0-z)^{-1}E_d - gE_d(\hat{H}_g-z)^{-1}E_u\hat{V}E_d(\hat{H}_0-z)^{-1}E_d -$$
$$-gE_d(\hat{H}_g-z)^{-1}E_d\hat{V}E_d(\hat{H}_0-z)^{-1}E_d \quad ,$$
$$E_d(\hat{H}_g-z)^{-1}E_u = -gE_d(\hat{H}_g-z)^{-1}E_u\hat{V}E_u(\hat{H}_0-z)^{-1}E_u -$$
$$-gE_d(\hat{H}_g-z)^{-1}E_d\hat{V}E_u(\hat{H}_0-z)^{-1}E_u \quad ,$$

where we have used commutatitvity of E_u, E_d with $\hat{H}_0$, orthogonality of these projections and the relation $E_u+E_d = I$. The last term on the rhs of the first equality is zero in view of the Friedrichs condition ; the same is true for the first rhs term of the second equality, because $E_u V E_u = 0$. Now one has to substitute from the second relation to the first one, and to multiply the obtained operator identity by $\hat{H}_0-z$ from the right. It gives

$$E_d(\hat{H}_g-z)^{-1}E_d\{E_d(\hat{H}_0-z)E_d - g^2E_d\hat{V}E_u(\hat{H}_0-z)^{-1}E_u\hat{V}E_d\} = E_d \quad . \tag{3.2a}$$

This equation is solved by

$$E_d(\hat{H}_g-z)^{-1}E_d = \{E_d[\hat{H}_0 - z - g^2\hat{V}E_u(\hat{H}_0-z)^{-1}E_u\hat{V}]E_d\}^{-1} \quad , \tag{3.2b}$$

where the inverse refers, of course, to $\mathcal{B}(\mathcal{H}_d)$.

Comparing to Proposition II.3.1, the situation is now more difficult, because we are looking for inverse of an infinite-dimensional operator. Fortunately, the problem is solvable. The curly-bracket operator in (3.1), which we denote by A_z for a moment, acts as

$$(A_z\hat{\psi}_d)(\vec{p}) = \left(\frac{\vec{p}^2}{2m}-z\right)\hat{\psi}_d(\vec{p}) - g^2\,\frac{\hat{v}(\vec{p})(\hat{v},\hat{\psi}_d)}{E-z} \quad . \tag{3.3}$$

It differs therefore from the projection of $\hat{H}_0-z$ to $\mathcal{H}_d$ by a rank-one operator. Then one has to employ the corollary of the second-resolvent identity which is known as Krein formula /6/ ; according to it, the difference of the corresponding resolvents is again of rank one. It suggests the following guess,

$$(E_d(\hat{H}_g-z)^{-1}E_d\hat{\psi}_d)(\vec{p}) = \left(\frac{\vec{p}^2}{2m}-z\right)^{-1}\hat{\psi}_d(\vec{p}) + \alpha(z,g)(\hat{\varphi}_{2z},\hat{\psi}_d)\hat{\varphi}_{1z}(\vec{p}) \quad , \tag{3.4}$$

where the vectors φ_{jz} are given by (3.1b) (recall that $\hat{v}\in L^2(\mathbb{R}^3)$) and $\alpha(z,g)$ is an unknown complex number. Then we express $(E_d(\hat{H}_g-z)^{-1}E_dA_z\hat{\psi}_d)(\vec{p})$ using the relations (3.3) and (3.4), the expression

$$\hat{\varphi}_{1z}(\vec{p}) = \frac{\hat{v}(\vec{p})}{\frac{\vec{p}^2}{2m}-z} \; .$$

and the analogous one for $\hat{\varphi}_{2z}$ (with z replaced by $\bar{z}$). In view of (3.2b), it must be equal to $\hat{\psi}_d(\vec{p})$. This requirement leads to a condition which yields

$$\alpha(z,g) = \frac{g^2}{E-z-g^2(\hat{\varphi}_{2z},\hat{v})} \quad . \tag{3.5a}$$

The same argument applied to $(A_zE_d(\hat{H}_g-z)^{-1}E_d\hat{\psi}_d)(\vec{p})$ gives

$$\alpha(z,g) = \frac{g^2}{E-z-g^2(\hat{v},\hat{\varphi}_{1z})} \quad . \tag{3.5b}$$

The conditions (3.5) are, however, consistent because $(\hat{\varphi}_{2z},\hat{v}) = (\hat{v},\hat{\varphi}_{1z}) = -G(z)$. Combining the relations (3.5) with (II.3.4), we get $\alpha(z,g) = g^2r_u(z,H_g)$, i.e., the desired result. ■

Remark 3.2 : With the help of the relations used in the proof, one can easily write down also the "non-diagonal blocks" of the resolvent. For any $\hat{\psi}_d\in\mathcal{H}_d$, we have

$$E_u(\hat{H}_g-z)^{-1}E_d\hat{\psi}_d = -gr_u(z,H_g)(\hat{\varphi}_{2z},\hat{\psi}_d) \quad . \tag{3.6a}$$

The result is, of course, a complex number, i.e., an element of $\mathcal{H}_u$ (recall that we work with $\mathcal{H}^{rel} = \mathbb{C}\oplus L^2(\mathbb{R}^3)$). On the other hand, for any $\psi_u\in\mathcal{H}_u$ we have

$$E_d(\hat{H}_g-z)^{-1}E_u\psi_u = -g\psi_ur_u(z,H_g)\hat{\varphi}_{1z} \quad . \tag{3.6b}$$

The relations (3.6) together with (3.1) and (II.3.4) describe the resolvent $(\hat{H}_g-z)^{-1}$ completely ; after a short calculation, we obtain for any $\hat{\Psi} = \begin{pmatrix}\psi_u\\ \hat{\psi}_d\end{pmatrix}$ the expression

$$(\hat{\Psi},(\hat{H}_g-z)^{-1}\hat{\Psi}) = (\hat{\psi}_d,(\hat{H}_0-z)^{-1}\hat{\psi}_d) + $$
$$+ r_u(z,H_g)\,[\psi_u - g(\hat{\varphi}_{2z},\hat{\psi}_d)][\bar{\psi}_u - g(\hat{\psi}_d,\hat{\varphi}_{1z})] \quad . \tag{3.7}$$

Now we are ready to deal with the problem left from the preceding section :

Proof of the asymptotic completeness - Theorem 2.1 : First of all, we must estimate $\mathrm{Im}(\hat{\Psi},(\hat{H}_g-z)^{-1}\hat{\Psi})$ for $\hat{\Psi}=\begin{pmatrix}\psi_u\\ \hat{\psi}_d\end{pmatrix}$ of a dense set in $\mathcal{H}$. We take $\hat{\psi}_d\in C_0^\infty(\mathbb{R}^3)$. Further we choose a finite interval $[a,b]\subset\subset\Omega_1\cap(0,\infty)$ (cf.Theorem II.3.6) and consider $z=\lambda+i\varepsilon$ with $\lambda\in[a,b]$ and $\varepsilon>0$. For the free resolvent, we have

$$\mathrm{Im}\,(\hat{\psi}_d,(\hat{H}_0-z)^{-1}\hat{\psi}_d)=\int_{\mathbb{R}^3}\frac{\varepsilon|\hat{\psi}_d(\vec{p})|^2}{\left(\frac{\vec{p}^2}{2m}-\lambda\right)^2+\varepsilon^2}\,d\vec{p}\quad. \tag{3.8}$$

If we denote $\alpha=\lambda/\varepsilon$, the rhs of (3.8) may be estimated as follows

$$(2m)^{3/2}\sqrt{\varepsilon}\int_{\mathbb{R}^3}\frac{|\hat{\psi}_d(\sqrt{2m\varepsilon}\,\vec{k})|^2}{(\vec{k}^2-\alpha)^2+1}\,d\vec{k}\leq 4\pi(2m)^{3/2}\sqrt{\varepsilon}\sup_{\vec{p}\in\mathbb{R}^3}|\hat{\psi}_d(\vec{p})|^2\int_0^\infty\frac{k^2\,dk}{(k^2-\alpha)^2+1}.$$

In the last integral, we substitute $u=k^2-\alpha$ and estimate it as

$$\int_{-\alpha}^{\infty}\frac{\sqrt{u+\alpha}}{2(u^2+1)}\,du\leq\int_{-\infty}^{\infty}\frac{\sqrt{|u|}+\sqrt{\alpha}}{2(u^2+1)}\,du=\frac{\pi}{\sqrt{2}}+\frac{\pi}{2}\sqrt{\frac{\lambda}{\varepsilon}}\quad.$$

Together we get

$$0\leq\mathrm{Im}\,(\hat{\psi}_d,(\hat{H}_0-z)^{-1}\hat{\psi}_d)\leq 8\pi^2m^{3/2}\sup_{\vec{p}\in\mathbb{R}^3}|\psi_d(\vec{p})|^2\left(\sqrt{\varepsilon}+\sqrt{\frac{\lambda}{2}}\right) \tag{3.9}$$

so the imaginary part (3.8) is a bounded function of z in $[a,b]\times[0,\varepsilon_0]$ for any $\varepsilon_0>0$. We shall set $\varepsilon_0=1$ in the following. The relations (3.7) and (3.9) then give

$$|\mathrm{Im}\,(\hat{\Psi},(\hat{H}_g-z)^{-1}\hat{\Psi})|\leq 8\pi^2m^{3/2}\left(1+\sqrt{\frac{b}{2}}\right)\sup_{\vec{p}\in\mathbb{R}^3}|\hat{\psi}_d(\vec{p})|^2+$$

$$+|r_u(z,H_g)|\left[|\psi_u|+|g|\left|\int_{\mathbb{R}^3}\frac{\overline{\hat{v}(\vec{p})}\hat{\psi}_d(\vec{p})}{\frac{\vec{p}^2}{2m}-z}\,d\vec{p}\right|\right]\left[|\psi_u|+|g|\left|\int_{\mathbb{R}^3}\frac{\hat{v}(\vec{p})\overline{\hat{\psi}_d(\vec{p})}}{\frac{\vec{p}^2}{2m}-z}\,d\vec{p}\right|\right].$$

Under the assumptions (a)-(c), the function $r_u(.,H_g)$ is continuous for a sufficiently small g, and therefore bounded in $[a,b]\times[0,1]$. Suppose that the function $\hat{v}_1$ has a bounded derivative in $[a,b]$. Since $\hat{\psi}_d\in C_0^\infty(\mathbb{R}^d)$, the same argument as in the proofs of Lemmas II.3.3 and II.3.4 shows that the function defined on $[a,b]\times(0,1]$ by

$$z\mapsto g\int_{\mathbb{R}^3}\frac{\overline{\hat{v}(\vec{p})}\hat{\psi}_d(\vec{p})}{\frac{\vec{p}^2}{2m}-z}\,d\vec{p}$$

and extended continuously to $\mathrm{Im}\,z=0$ is continuous within $[a,b]\times[0,1]$,

and therefore bounded ; the same is true for the other function that appears in the last estimate. Collecting these results, we see that

$$\sup_{0<\varepsilon<1}\int_a^b|\mathrm{Im}\,(\hat{\Psi},(\hat{H}_g-\lambda-i\varepsilon)\hat{\Psi})|^q\,d\lambda<\infty \tag{3.10}$$

for any $q>1$. The well-known criterion (cf.Ref.I.27, Theorem XIII.19) then implies $E_{\hat{H}_g}((a,b))\hat{\Psi}\subset\mathcal{H}_{ac}(\hat{H}_g)$. Since $C_0^\infty(\mathbb{R}^3)$ is dense in $L^2(\mathbb{R}^3)$, it follows that $\mathrm{Ran}\,E_{\hat{H}_g}((a,b))\subset\mathcal{H}_{ac}(\hat{H}_g)$.

Now we would like to extend this result to the whole $\mathbb{R}^+$. According to the assumption (e) and Lemma II.3.4, $r_u(.,H_g)$ may be continued analytically accross $\mathbb{R}^+$, with a possible exception of the points $\lambda\in\mathbb{R}^+$ in which $\lambda=E+g^2G_\Omega(\lambda)$. These points correspond to eigenvalues of $\hat{H}_g$. However, the analyticity assumptions (b) and (e) imply that such points are isolated with no accumulation points except infinity. If none of them is contained in $[a,b]$, the above described procedure may be carried out.

We have required $\hat{v}_1$ to have a continuous derivative within $[a,b]$. By assumption, $\hat{v}_1'$ is piecewise continuous, i.e., its discontinuity points are isolated and have no accumulation points except infinity. Together we have a closed subset $M=\{\lambda_j\}\subset\mathbb{R}^+$ such that in its points either $r_u(.,H_g)$ is not bounded or $\hat{v}_1'$ is discontinuous. Let λ_j,λ_{j+1} be any two neighbouring points of M. Then $E_{\hat{H}_g}((a,b))\subset\subset\mathcal{H}_{ac}(\hat{H}_g)$ for any $\lambda_j<a<b<\lambda_{j+1}$, i.e., $E_{\hat{H}_g}((\lambda_j,\lambda_{j+1}))\subset\mathcal{H}_{ac}(\hat{H}_g)$. It means that $E_{\hat{H}_g}(\mathbb{R}^+\setminus M)\subset\mathcal{H}_{ac}(\hat{H}_g)$, and therefore $\sigma_{sing}(\hat{H}_g)$ is void. ∎

Remark 3.3 : (a) The result holds without any limitation on g. If $|g|$ is small enough, then Theorem II.3.6 ensures that $r_u(.,H_g)$ has no real-axis poles around E. Such a pole can appear in the strong-coupling case, however, it is always an isolated point which does not contribute to $\sigma_{sing}(H_g)$.
(b) In some cases, H_g has no eigenvalues - cf.Lemma **III.2.4**. The eigenvalue problem will be further discussed in Section 5.
(c) The smoothness requirement on $\hat{v}_1$ does not follow from the assumptions III.2.1, since the latter concern the modulus of $\hat{v}_1$ only.

4. Spectral concentration

We have seen that the embedded eigenvalue E corresponding to the initial particle disappears once the interaction is turned on. Nevertheless, the system "remembers" the dissolved eigenvalue : if the coupling is sufficiently weak, the states whose energy support does not contain a small interval around E are nearly orthogonal to the original eigenstate ; the weaker is the coupling, the smaller is this interval. This effect is known as spectral concentration ; we are going to formulate it now for the model under consideration.

Theorem 4.1 : Let α, β be positive numbers, $\beta<2$, and denote

$$\Delta_g = (E-\alpha|g|^\beta, E+\alpha|g|^\beta) \quad . \tag{4.1a}$$

Suppose the assumptions (a)-(c) are valid and $\hat{v}_1$ is continuously differentiable in some neighbourhood of E , then

$$\operatorname*{s-lim}_{g\to 0} E_{\hat{H}_g}(\Delta_g) = E_u \quad . \tag{4.1b}$$

Proof : In view of Theorem II.3.6, we can choose a positive g_0 such that the functions $r_u(\cdot,H_g)$ and $\hat{v}_1'$ are continuous in $\bar{\Delta}_g$ for all g such that $0<|g|<g_0$. The proof presented in the previous section then shows that $\operatorname{Ran} E_{\hat{H}_g}(\bar{\Delta}_g) \subset \mathcal{H}_{ac}(\hat{H}_g)$. Then we have

$$(\hat{\Psi},E_{\hat{H}_g}(\Delta_g)\hat{\Psi}) = (\hat{\Psi},E_{\hat{H}_g}(\bar{\Delta}_g)\hat{\Psi}) = \lim_{\varepsilon\to 0+} \frac{1}{\pi}\int_{\Delta_g} \operatorname{Im}(\hat{\Psi},(H_g-\lambda-i\varepsilon)^{-1}\hat{\Psi})\,d\lambda \ , \tag{4.2}$$

where the second equality follows from the Stone formula - cf. Ref.I.27, Theorem VII.13 . Now we choose $\hat{\psi}_d \in C_0^\infty(\mathbb{R}^3)$ and calculate the limit of (4.2) as $g\to 0$. In view of (3.7), we have

$$\lim_{g\to 0} (\hat{\Psi},E_{\hat{H}_g}(\Delta_g)\hat{\Psi}) = \frac{1}{\pi}\lim_{g\to 0}\lim_{\varepsilon\to 0+}\int_{\Delta_g}\operatorname{Im}\Big\{\int_{\mathbb{R}^3}\frac{i\varepsilon|\hat{\psi}_d(\vec{p})|^2 d\vec{p}}{\left(\frac{\vec{p}^2}{2m}-\lambda\right)^2+\varepsilon^2} +$$

$$+ r_u(\lambda+i\varepsilon,H_g)\left[\psi_u - g\int_{\mathbb{R}^3}\frac{\overline{\hat{v}(\vec{p})}\hat{\psi}_d(\vec{p})}{\frac{\vec{p}^2}{2m}-\lambda-i\varepsilon}\,d\vec{p}\right]\times \tag{4.3}$$

$$\left[\bar{\psi}_u - g\int_{\mathbb{R}^3}\frac{\overline{\hat{\psi}_d(\vec{p})}\hat{v}(\vec{p})}{\frac{\vec{p}^2}{2m}-\lambda-i\varepsilon}\,d\vec{p}\right]\Big\}\,d\lambda \quad .$$

The estimate (3.9) gives

$$0 \le \lim_{g\to 0}\lim_{\varepsilon\to 0+}\int_{\Delta_g}\int_{\mathbb{R}^3}\frac{\varepsilon|\hat{\psi}_d(\vec{p})|^2 d\vec{p}}{\left(\frac{\vec{p}^2}{2m}-\lambda\right)^2+\varepsilon^2}\,d\lambda \le$$

$$\le \lim_{g\to 0}\lim_{\varepsilon\to 0+} 8\pi^2 m^{3/2}\sup_{\vec{p}\in\mathbb{R}^3}|\hat{\psi}_d(\vec{p})|^2\left(\sqrt{\varepsilon}+\sqrt{\frac{E+\alpha|g|^\beta}{2}}\right)\cdot 2\alpha|g|^\beta =$$

$$= \lim_{g\to 0} 16\alpha\pi^2 m^{3/2}\sup_{\vec{p}\in\mathbb{R}^3}|\psi_d(\vec{p})|^2\sqrt{\tfrac{1}{2}(E+\alpha|g|^\beta)}\,|g|^\beta = 0 \quad ,$$

i.e., the contribution of the first term is zero. We shall use once more the proof of the preceding section : it shows that the second term in the curly bracket in (4.3) is bounded. Then we may interchange the limit $\varepsilon\to 0+$ with the integral obtaining in this way

$$\lim_{g\to 0}(\hat{\Psi},E_{\hat{H}_g}(\Delta_g)\hat{\Psi}) =$$

$$= \frac{1}{\pi}\lim_{g\to 0}\int_{\Delta_g}\operatorname{Im} r_u^{\Omega}(\lambda,H_g)\left[\psi_u - g\lim_{\varepsilon\to 0+}\int_{\mathbb{R}^3}\frac{\overline{\hat{v}(\vec{p})}\hat{\psi}_d(\vec{p})}{\frac{\vec{p}^2}{2m}-\lambda-i\varepsilon}\,d\vec{p}\right]\times \tag{4.4}$$

$$\times\left[\bar{\psi}_u - g\lim_{\varepsilon\to 0+}\int_{\mathbb{R}^3}\frac{\overline{\hat{\psi}_d(\vec{p})}\hat{v}(\vec{p})}{\frac{\vec{p}^2}{2m}-\lambda-i\varepsilon}\,d\vec{p}\right]$$

provided the limits in the square brackets exist. It can be verified, however, in the same way as in the preceding section ; since $\hat{v}_1'$ and ψ_d' are continuous, the limits

$$\lim_{\varepsilon\to 0+}\int_{\mathbb{R}^3}\frac{\overline{\hat{v}(\vec{p})}\hat{\psi}_d(\vec{p})}{\frac{\vec{p}^2}{2m}-\lambda\mp i\varepsilon}\,d\vec{p}$$

exist. Moreover, they are continuous and bounded functions of λ . The same is then true for the product of the two square brackets in (4.4), which we denote for a moment as $\mathcal{G}(\lambda,g)$. Next one has to express $r_u^{\Omega}(\lambda,H_g)$ from (II.3.1) and (II.3.15), and calculate the limit

$$\lim_{g\to 0}\int_{\Delta_g}\operatorname{Im} r_u^{\Omega}(\lambda,H_g)\mathcal{G}(\lambda,g)\,d\lambda =$$

$$= \lim_{g\to 0}\int_{E-\alpha|g|^\beta}^{E+\alpha|g|^\beta}\frac{4\pi^2 g^2 m|\hat{v}_1(\sqrt{2m\lambda})|^2\sqrt{2m\lambda}}{[E-\lambda+4\pi g^2 I(\lambda,v)]^2+32\pi^4 g^4 m^3\lambda|\hat{v}_1(\sqrt{2m\lambda})|^4}\,\mathcal{G}(\lambda,g)\,d\lambda \ .$$

One substitutes $x=g^{-2}(\lambda-E)$ and uses the dominated-convergence theorem, then the limit equals

$$4\pi^2 m|\hat{v}_1(\sqrt{2mE})|^2\sqrt{2mE}\,\mathcal{G}(E,0)\int_{-\infty}^{\infty}\frac{dx}{[x-4\pi I(E,v)]^2+32\pi^4 m^3 E|\hat{v}_1(\sqrt{2mE})|^4} =$$

$$= \pi\mathcal{G}(E,0) \quad .$$

Since $\mathcal{G}(E,0) = |\psi_u|^2$, the relation (4.4) implies

$$\lim_{g\to 0} (\hat{\Psi}, E_{\hat{H}_g}(\Delta_g)\hat{\Psi}) = |\psi_u|^2 = (\hat{\Psi}, E_u\hat{\Psi}) \qquad (4.5)$$

for all $\hat{\Psi} = \begin{pmatrix}\psi_u \\ \hat{\psi}_d\end{pmatrix}$ with $\hat{\psi}_d \in C_0^\infty(\mathbb{R}^3)$. Using the fact that this set is dense in $L^2(\mathbb{R}^3)$ together with the polarization identity, we get

$$\text{w-}\lim_{g\to 0} E_{\hat{H}_g}(\Delta_g) = E_u \quad .$$

At the same time, (4.5) together with the density argument give

$$\lim_{g\to 0} \| E_{\hat{H}_g}(\Delta_g)\hat{\Psi}\|^2 = \| E_u\hat{\Psi}\|^2$$

for all $\hat{\Psi} \in \mathcal{H}$. The last two relations imply

$$\text{s-}\lim_{g\to 0} E_{\hat{H}_g}(\Delta_g) = E_u$$

which is the p-representation form of (4.1b). Hence the theorem is proved. ∎

5. Bound states

Let us turn now to the problem of the existence of bound states, i.e., eigenstates of the Hamiltonian H_g .

Proposition 5.1 : A bound state $\Psi = \begin{pmatrix}\alpha \\ \psi\end{pmatrix}$ with energy $\mathcal{E}$ exists iff $\hat{\psi} \in L^2(\mathbb{R}^3)$,

$$\hat{\psi}(\vec{p}) = \frac{g\alpha \hat{v}_1(p)}{\mathcal{E} - \frac{p^2}{2m}} \qquad (5.1)$$

and

$$\mathcal{E} = E + 4\pi g^2 \int_0^\infty \frac{|\hat{v}_1(p)|^2}{\mathcal{E} - \frac{p^2}{2m}}\, p^2 dp \qquad (5.2)$$

(the last integral exists because $\hat{\psi} \in L^2(\mathbb{R}^3)$).

Proof : The assertion follows immediately from the Schrödinger equation $\hat{H}_g\hat{\Psi} = \mathcal{E}\hat{\Psi}$. ∎

Proposition 5.2 : Assume $g \neq 0$. Let $|\hat{v}_1| \in C(0,\infty)$ and $\mathcal{E} > 0$ is an eigenvalue of H_g , then

$$\hat{v}_1(\sqrt{2m\mathcal{E}}) = 0 \quad . \qquad (5.3)$$

Moreover, if $|\hat{v}_1|$ belongs to $C^1(0,\infty)$, then a positive $\mathcal{E}$ is an eigenvalue of H_g iff the relations (5.2) and (5.3) hold.

Proof : One has only to use Proposition 5.1, and to realize that for $|\hat{v}_1| \in C^1(0,\infty)$, the function (5.1) is square-integrable iff (5.3) holds. ∎

Proposition 5.3 : Let $g \neq 0$, then $\mathcal{E} = 0$ is eigenvalue of H_g iff the function $p \mapsto p^{-2}\hat{v}_1(p) \in L^2(\mathbb{R}^+, p^2dp)$ and

$$E = 8\pi g^2 m \int_0^\infty |\hat{v}_1(p)|^2 dp \quad .$$

Proof is an immediate application of Proposition 5.1 . ∎

Proposition 5.4 : (a) There is at most one bound state with energy $\mathcal{E} < 0$.
(b) Let $\hat{v}_1 \in L^2(\mathbb{R}^+)$, then a bound state with energy $\mathcal{E} < 0$ exists iff

$$E < 8\pi g^2 m \int_0^\infty |\hat{v}_1(p)|^2 dp \quad . \qquad (5.4)$$

Proof : If $\mathcal{E} < 0$, then the function (5.1) is in $L^2(\mathbb{R}^3)$. The lhs of (5.2) is increasing, while the rhs is non-increasing with respect to $\mathcal{E}$ in $(-\infty, 0)$ so the equation (5.2) has at most one solution. Furthermore, both sides of (5.2) are continuous functions of $\mathcal{E}$ in $(-\infty, 0)$,

$$\lim_{\mathcal{E}\to-\infty} \left[E + 4\pi g^2 \int_0^\infty \frac{|\hat{v}_1(p)|^2}{\mathcal{E} - \frac{p^2}{2m}}\, p^2 dp\right] = E > 0 \quad ,$$

and for $\hat{v}_1 \in L^2(\mathbb{R}^+)$,

$$\lim_{\mathcal{E}\to 0-} \left[E + 4\pi g^2 \int_0^\infty \frac{|\hat{v}_1(p)|^2}{\mathcal{E} - \frac{p^2}{2m}}\, p^2 dp\right] = E - 8\pi m g^2 \int_0^\infty |v_1(p)|^2 dp \quad .$$

A solution $\mathcal{E} < 0$ to (5.2) clearly exists iff the last limit is negative. ∎

Corollary 5.5 : Let $\hat{v}_1 \in L^2(\mathbb{R}^+, p^2dp)$ be a function with non-zero values whose modulus is continuous in $(0,\infty)$ and $g \neq 0$. Then there is a bound state (just one, and with a negative energy) iff

$$g^2 > g_{cr}^2 \equiv \frac{E}{8\pi m \int_0^\infty |\hat{v}_1(p)|^2 dp} \quad . \qquad (5.5)$$

Proof : According to Propositions 5.2 and 5.3, there is no non-negative eigenvalue of H_g under our assumptions. Since $|\hat{v}_1|$ is continuous, and therefore bounded in a (right) vicinity of zero, $\hat{v}_1 \in L^2(\mathbb{R}^+)$. Then a bound state exists iff $g^2 > g_{cr}^2$ due to Proposition 5.4b. ∎

Remark 5.6 : We have seen in Proposition 2.5 that the s-wave phase shift changes its behaviour at $\lambda \to 0+$ for $|g| = g_{cr}$. Since the bound state emerges at the same value of g , the validity of a Levinson-type theorem is indicated – in this connection cf. Ref.7 .

References

1 J.Dittrich, P.Exner, Czech.J.Phys.B, 1987, v.37, pp.503-515.
2 J.Dittrich, P.Exner, preprint JINR E2-86-709, Dubna 1986.
3 J.Dittrich, P.Exner, preprint JINR E2-86-750, Dubna 1986
4 M.Reed, B.Simon : Methods of Modern Mathematical Physics, III.Scattering Theory, Academic Press, New York 1979.
5 W.O.Amrein, J.M.Jauch, K.B.Sinha : Scattering Theory in Quantum Mechanics, W.A.Benjamin, Reading, Mass., 1979.
6 N.I.Akhiezer, I.M.Glazman : The Theory of Linear Operators in Hilbert Space (in Russian), Nauka, Moscow 1966 ; § 106 .
7 M.Wollenberg, Math.Nachr., 1977, v.78, pp.223-230.

We do not reproduce the full list of references from the preceding parts of the paper. Here we have referred to:

I.27 M.Reed, B.Simon : Methods of Modern Mathematical Physics, I.Functional Analysis, IV.Analysis of Operators, Acadaemic Press, New York 1972, 1978.

Received by Publishing Department
on July 30, 1987.

Диттрих Я., Экснер П. E2-87-599

Нерелятивистская модель двухчастичного распада. Связь с теорией рассеяния, спектральная концентрация и связанные состояния

В настоящей работе, которая представляет собой четвертую часть серии, посвященной анализу простой модели двухчастичного распада типа модели Ли, обсуждаются три проблемы. Первая из них касается связи модели с теорией рассеяния. Выведена асимптотическая полнота для упругого рассеяния двух легких частиц. Доказано, что в случае достаточно слабой связи, система обладает в точности одним резонансом, положение которого совпадает с положением полюса, определяющего главный вклад в закон распада. Вторая проблема касается спектральной концентрации; доказано, что она имеет место для семейств отрезков вокруг E, стягивающихся медленнее чем квадратично по отношению к g. Наконец, обсуждаются необходимые и достаточные условия существования связанных состояний.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1987

Dittrich J., Exner P. E2-87-599

A Non-Relativistic Model of Two-Particle Decay. Relation to the scattering theory, spectral concentration, and bound states

The present paper, which represents the fourth part of the series devoted to analysis of a simple Lee-type model of two-particle decay, deals with three problems. The first one concerns relation of the model to the scattering theory. We prove asymptotic completeness for the elastic scattering of the two light particles and show that for a sufficiently weak coupling, this system has just one resonance whose position is the same as that of the pole which yields the main contribution to the decay law. The second problem concerns spectral concentration; we prove its occurrence for families of intervals around E that shrink slower than quadratically in g. Finally, necessary and sufficient conditions for existence of bound states are discussed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1987