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THE THEORIES WITH HIGHER DERIVATIVES AND GAUGE-TRANSFORMATION CONSTRUCTION

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1. In describing the elementary particle dynamics in the framework of field theory, singular or degenerated Lagrangians [1-4] are mainly used. Usually, the singularity of a Lagrangian is caused by the invariance of the action with respect to the transformations of field functions which depend on an arbitrary function of the coordinates and time. Such transformations determined in the phase space of the coordinates and velocities (tangent bundle) are often called the gauge transformations; and the corresponding theories, gauge theories.

The general description, as on a classical and on a quantum level, of the systems with singular Lagrangians was proposed by Dirac [1] on the basis of the extended Hamilton formalism. In the framework of the functional integral the quantization method in phase space of the models with singular Lagrangians was given in papers [5,6], the detailed presentation of those approaches can be found in [3,4,7,8].

But the problem of finding the gaige transformations for a given Lagrangian is studied insufficiently. A consistent scheme which might give a possibility of determining the gauge transformations, assuming the Lagrangian to be known, was not constructed yet. In paper [9], on the basis of the second Noether theorem in the framework of the Lagrangian formalism, infinitesimal gauge transformations in the tangent bundle are constructed by the iteration method. As mentioned by the authors, the method proposed by them is not extended to Lagrangians with higher derivatives.

But the gauge degrees of freedom more naturally arise when using the generalized Hamilton formalism in phase space [1,2]. Therefore in the given paper, we first find the changes of dynamical variables which do not change the physical state of a system, in phase space, and then construct corresponding gauge transformations in the tangetnt bundle. The method proposed by us, is easily generalized to Lagrangians with higher derivations.

We mark that for a definite class of Lagrangians, not containing higher derivatives, the question of construction of the gauge trasnformation generators was considered in papers [11,12].

The paper is organized as follows: in the next section, main formulae of the extended Hamilton formalism for Lagrangians without higher derivatives are given and infinitesimal transformations of dynamical variables preserving the action invariance are constructed. In the 3rd section the method is generalized to Lagrangians with higher derivatives. In section 4 examples are analysed.

2. For simplicity first let us consider a system with a finite number of degrees of freedom; assume, the system is described by Lagragian $\mathfrak{L}(q, \dot{q})$, where $q = (q_1, q_2, ..., q_n)$ are generalized coordinates and $\dot{q}_i = \frac{d}{dt} q_i$ are corresponding velocities. Also we assume that there exist transformations of the coordinates which depend on arbitrary functions of the time and their derivatives preserving the action S invariance

$$S \equiv \int dt \mathfrak{L}(q, \dot{q}) = \int dt \mathfrak{L}(q + \delta q, \dot{q} + \delta \dot{q}), \qquad (2.1)$$

$$\delta q_i = a^k_{ij} \overset{(k)}{\lambda}_j (t); \quad \delta \dot{q}_i \equiv \frac{d}{dt} \delta q_i, \quad j = 1, ..., m. \qquad (2.2)$$

Here $\overset{(k)}{\lambda}(t)$ denotes a k-th order derivative of an arbitrary function $\lambda(t)$ and coefficients a^k_{ij} are, in general, functions of dynamical variables and time. Over the repeated indices in (2) and throughout the paper we assume summation.

From (2.1) and (2.2) it follows that for the symmetric matrix (Hessian)

$$W_{ij}(q, \dot{q}) = \frac{\partial^2 \mathfrak{L}(q, \dot{q})}{\partial \dot{q}_i \partial \dot{q}_j}, \quad i, j = 1, 2, ..., m \qquad (2.3)$$

there exists m eigenvectors $\eta^s_i(q, \dot{q})$ with a zero eigenvalue:

$$\eta^s_i(q, \dot{q}) W_{ij}(q, \dot{q}) = W_{ij}(q, \dot{q}) \eta^s_j(q, \dot{q}) = 0, \quad s = 1, ..., m.$$

Define in a standard way the canonical momentum:

$$p_i(q, \dot{q}) = \frac{\partial \mathfrak{L}}{\partial \dot{q}_i}. \qquad (2.4)$$

As the rank of matrix $W_{ij}(q, \dot{q})$ is less than n in that case not all momenta are independent and in the thoery there arise, in the Dirac terminology /1/, m primary constraints

$$\phi^1_k(q, p) \approx 0 \quad k = 1, ..., m, \qquad (2.5)$$

where $\approx$ denotes a weak equality. Note that (2.5) is fulfilled identically for q and p.

We also assume that

$$\text{Rang} \left|\left| \frac{\partial \phi^1_k(q, p)}{\partial p_i} \right|\right| = m.$$

This condition rules out the possibility of appearance of unimportant constraints in the theory.

In what follows, we shall be interested in such transformations of the dynamical variables which do not change the physical state of the system, therefore we assume that there are no second-order constraints. In other words, for any constraint, which arises in the theory, its Poisson bracket with all other constraints and the canonical Hamiltonian

$$H_c = p_i \dot{q}_i - \mathfrak{L}(q, \dot{q}) \qquad (2.6)$$

equals in a weak sense zero.

In the framework of the extended Hamilton formalism the equation of motion for an arbitrary dynamical variable has the from /1-4/

$$\dot{g} = \{g, H^T\}, \qquad (2.7)$$

$$H^T = H_c + v_k \phi^1_k. \qquad (2.8)$$

Here H^T is the total Hamiltonian, v_k are arbitrary multipliers, and ϕ^1_k are primary constraints of the first-class, whereas the Poisson brackets are defined by

$$\{f, g\} = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p^i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p^i}.$$

From (2.7) and (2.8), taking into account the arbitrariness of coefficients v_k, one may obtain the variance Δg of the dynamical variable g, which is not connected with the change of the phasical state /1-8/. It is given by the formula:

$$\Delta g = \epsilon^1_a \{g, \phi_a\},$$

where $\epsilon^1_a = \delta t (v_a - v'_a)$.

The requirement for primary constraints being stationary in time may give new limitations on dynamical variables q and p. Those limitations are called secondary constraints /1/.

Denote then by ϕ_k^2. If a secondary constraint arises in the theory it should be required that it will also remain stationary in time. This process will continue unless the requirement of stationarity will turn into identity. Secondary constraints of first order also may generate infinitesimal transformations of Hamilton variables not connected with the change of the physical state, but this statement is in general incorrect [3,10]. The group properties for transformation (2.9) in a general case are fulfilled only in a weak sense [3], which makes it difficult to construct the generators of gauge transformations.

Let us rewrite (2.9) in the following form:

$$\Delta g = -\tilde{\Phi}(\epsilon_\alpha^1 \phi_\alpha^1) g, \tag{2.9}$$

where operator $\tilde{\Phi}(\epsilon_\alpha^1 \phi_\alpha^1)$ is given by the formula

$$\tilde{\Phi}(\epsilon_\alpha^1 \phi_\alpha^1) = \frac{\partial(\epsilon_\alpha^1 \phi_\alpha^1)}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial(\epsilon_\alpha^1 \phi_\alpha^1)}{\partial p_i} \frac{\partial}{\partial q_i}. \tag{2.10}$$

In analogy with (10) we construct another operator

$$\Phi(\epsilon_\alpha^{m_\alpha} \phi_\alpha^{m_\alpha}) = \frac{\partial(\epsilon_\alpha^{m_\alpha} \phi_\alpha^{m_\alpha})}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial(\epsilon_\alpha^{m_\alpha} \phi_\alpha^{m_\alpha})}{\partial p_i} \frac{\partial}{\partial q_i}. \tag{2.10'}$$

Here $\alpha = 1,\dots,m$ and $m_\alpha = 1,\dots,M_\alpha$, where M_α is a maximal number of secondary constraints obtained from the requirement of stationarity of ϕ_α^1. The difference between (2.10) and (2.10') consists in that the latter includes both the primary and secondary constraints. For arbitrary values of $\epsilon_\alpha^{m_\alpha}$ the operator (2.10') does not keep the action (2.1) invariant, in the general case.

Let us proceed in the following way: Assume that

$$\begin{aligned} q'(t) &= [1 + \Phi(\epsilon_\alpha^{m_\alpha} \phi_\alpha^{m_\alpha})] q(t), \\ p'(t) &= [1 + \Phi(\epsilon_\alpha^{m_\alpha} \phi_\alpha^{m_\alpha})] p(t) \end{aligned} \tag{2.11}$$

further, require that under such transformations the action remains invariant:

$$\delta S = \int dt\, \delta \mathcal{L} = \int dt\, \delta(\dot{q}_i p_i - H^T) \tag{2.12}$$

and find the limitations on the coefficients $\epsilon_\alpha^{m_\alpha}$, which in general are functions of q,p and t.

From the assumption that all constraints are first-class constraints, there follow the equations:

$$\begin{aligned} \{\phi_k^i, \phi_j^\ell\} &= f_{kjn}^{i\ell m} \phi_n^m, \\ \{H_c, \phi_k^i\} &= g_{kn}^{ij} \phi_n^j. \end{aligned} \tag{2.13}$$

Here the coefficeints f and g may be functions of q and p. Inserting (2.11) and (2.10') into (2.12) we find:

$$\begin{aligned} \delta S &= \int dt [\dot{q}\delta p - \dot{p}\delta q + \frac{d}{dt}(p\delta q) - \delta H_c - \delta v_\alpha \phi_\alpha^1 - v_\alpha \delta\phi_\alpha^1] = \\ &= \int dt [\dot{\epsilon}_\alpha^{m_\alpha} \phi_\alpha^{m_\alpha} - \epsilon_\alpha^{m_\alpha}\{H_c, \phi_\alpha^{m_\alpha}\} - \phi_\alpha^1 \delta v_\alpha - v_\alpha \epsilon_\alpha^{m_\alpha}\{\phi_\alpha^1, \phi_\alpha^{m_\alpha}\} + \\ &+ \frac{d}{dt}(p_i \epsilon_\alpha^{m_\alpha} \frac{\partial \phi_\alpha^{m_\alpha}}{\partial p_i} - \epsilon_\alpha^{m_\alpha} \phi_\alpha^{m_\alpha})]. \end{aligned} \tag{2.14}$$

Up to this step our consideration was of a general character. Now we make one suggestion, namely, we require that the Poisson bracket of primary constraints with the first order constraints be equal to a linear combination of the primary constraints *

$$\{\phi_{\alpha'}^1, \phi_\alpha^{m_\alpha}\} = f_{\alpha\alpha'\beta}^{1 m_\alpha' 1} \phi_\beta^1. \tag{2.15}$$

The requirement $\delta S = 0$ means that the sum of the coefficients in front of the primary and secondary constraints separately turns into zero in a strong sence. Collecting the coefficients of secondary constraints and taking into account (2.15), from (2.14) we get

$$\dot{\epsilon}_\alpha^{m_\alpha} - \epsilon_{\alpha'}^\beta g_{\alpha'\alpha}^{\beta m_\alpha} = 0 \qquad m_\alpha > 1. \tag{2.16}$$

From this equation it is seen that because of the presence of $g_{\alpha'\alpha}^{\beta m_\alpha}$ in it, in the general case $\epsilon_\alpha^{m_\alpha}$ is also a function of

*As is easily seen, the requirement is a sufficient condition for constructing gauge transformations in the tangent bundle.

q and p. The relation (2.16) gives sufficient limitations on the function $\epsilon_a^{m_a}$ in order that the operators (2.10) give such changes of coordinates and momenta, at which the physical state of the system is not changed. For each value of a in (2.16) we choose a maximum value $M_a = \max\{m_a\}$ and consider $\epsilon_a^{m_a}$ as an arbitrary function of time $\lambda(t)$. Then all other $\epsilon_a^{m_a}$ will depend on $\lambda(t)$, q and p. The form of this dependence is determined by formula (2.16).

In the phase space of coordinates and velocities we find*

$$\delta q = \Phi(\epsilon_a^{m_a} \phi_a^{m_a}) q, \tag{2.17}$$

$$\delta \dot{q} = \frac{d}{dt} \delta q \tag{2.18}$$

(into (2.17) and (2.18) the determination (2.5) is inserted).

3. In this section, we consider a physical system described by Lagrangians with higher derivatives. For simplicity we restrict ourselves to the case when the Lagrangian consists only of second-order derivatives

$$\mathcal{L}(x, \dot{x}, \ddot{x}), \quad \dot{x} = \frac{dx(t)}{dt}, \qquad x = (x_1, \dots, x_n). \tag{3.1}$$

Canonical variables for such Lagrangians are determined as follows:

$$\begin{aligned} &q_{1i} = x_i, \qquad q_{2i} = \dot{x}_i, \\ &p_{1i} = \frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \ddot{x}_i}, \quad p_{2i} = \frac{\partial \mathcal{L}}{\partial \ddot{x}_i}. \end{aligned} \tag{3.2}$$

The Lagrangian (3.1) is called singular if canonical variables satisfy the relations [13]

$$\phi_k^1(q_1, q_2, p_1, p_2) = 0, \quad k = 1, \dots, m \tag{3.3}$$

*Note that the momentum-transformation rule formula (2.11) in the phase space may not coincide with the momentum $p(q, \dot{q})$ transformation rule in the tangent bundle obtained by (2.17) and (2.18).

or, which is the same, rank $\|\lambda_{ij}\| = n - m$, where the matrix λ_{ij} is determined by

$$\lambda_{ij} = \frac{\partial^2 \mathcal{L}}{\partial \ddot{x}_i \, \partial \ddot{x}_j}. \tag{3.4}$$

The canonical Hamiltonian of the theory is constructed by the Ostrogradsky [14] method

$$H_0 = p_1 \dot{x} + p_2 \ddot{x} - \mathcal{L}(x, \dot{x}, \ddot{x}), \tag{3.5}$$

which will be a function only of canonical variables. Poisson brackets are determined in the standard way:

$$\{f, g\} = \frac{\partial f}{\partial q_{ik}} \frac{\partial g}{\partial p_{ik}} - \frac{\partial f}{\partial p_{ik}} \frac{\partial g}{\partial q_{ik}}. \tag{3.6}$$

Then the equation of motion for dynamical variables will take a form completely similar to (2.7) and (2.8) with the canonical Hamiltonian (3.5) and primary constraints (3.3).

As before, secondary constraints are obtained by the Dirac iteration method. As far as we demand that all constraints are of the first order, the relations (2.13) hold valid.

The action for Lagrangians with second derivatives is written in the form:

$$S = \int dt [p_1 \dot{x} + p_2 \ddot{x} - H^T]. \tag{3.7}$$

Then following the considerations analogous to section 3 for $\epsilon_a^{m_a}$ coefficients entering into the definition of the operator $\Phi(\epsilon_a^{m_a} \phi_a^{m_a})$ (2.10') we again obtain relation (2.16).

Thus, our method of the construction of infinitesimal gauge transformations can be applied to the Lagrangians depending on coordinates and velocities, as well as to the Lagrangians with higher derivatives.

4. As a first example, we consider the Yang-Mills Lagrangian

$$\begin{aligned} S &= -\frac{1}{4} \int d^4x \, F^a_{\mu\nu} F_a^{\mu\nu}, \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g C^a_{\beta\gamma} A^\beta_\mu A^\gamma_\nu. \end{aligned} \tag{4.1}$$

We construct canonical momenta and find primary constraints

$$\pi^a_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}^a_\mu} = - F^a_{0\mu}, \qquad (4.2)$$

$$\phi^{1a}_1 = \pi^{0a} \approx 0 .$$

The canonical Hamiltonian has the form:

$$H_c = \int d^3\vec{x} \left(\frac{1}{2}\pi^a_i \pi^a_i - A^a_0 \partial_i \pi^a_i + \frac{1}{4} F^a_{ij} F^a_{ij} + g C^a_{\beta\gamma} A_0 A^a_i \pi^a_i \right). \qquad (4.3)$$

For the calculation of coefficients $g^{\beta m_a}_{a' a}$ we have the following Poisson brackets

$$\{\phi^{1a}_1, H_c\} = \partial_i \pi^a_i - g C^\gamma_{a\beta} A^\beta_i \pi^i_\gamma = \phi^{2a}_1 \approx 0 ,$$
$$\{\phi^{2a}_1, H^T\} = g C^\gamma_{a\beta} A^\beta_0 \phi^{2a}_1 . \qquad (4.4)$$

The constraints ϕ^{1a}_1 and $\phi^{2\beta}_1$ are constraints of the first-class.

Formula (2.11) in field theory is generalized in the standard way

$$\delta A_\sigma(\vec{x}', t) = \int d^3\vec{x}\, \epsilon^{m_a}_a(\vec{x}, t) \left[\frac{\delta \phi^{m_a}_a}{\delta A_\mu(\vec{x}, t)} \frac{\delta}{\delta \pi^\mu(\vec{x}, t)} - \frac{\delta \phi^{m_a}_a}{\delta \pi^\mu(\vec{x}, t)} \frac{\delta}{\delta A_\mu(\vec{x}, t)} \right] A_\sigma(\vec{x}', t) . \qquad (4.5)$$

Inserting the constraints ϕ^{1a}_1 and $\phi^{2\beta}_1$ into (4.5) we obtain

$$\delta A^a_0 = \epsilon^{1a}_1 ,$$
$$\delta A^a_i = -\partial_i \epsilon^{2a}_1 - \epsilon^{2\beta}_1 g C^a_{\beta\gamma} A^\gamma_i . \qquad (4.6)$$

From (2.16) we find

$$\dot{\epsilon}^{2a}_1 + \epsilon^{1a}_1 + g C^a_{\beta\gamma} \epsilon^{2\beta}_1 A^\gamma_0 = 0 . \qquad (4.7)$$

Parametrizing ϵ^{2a}_1 by an arbitrary function $\omega^a(\vec{x}, t)$ from (4.6) and (4.7) we finally obtain the well-known transformations

$$\delta A^a_\mu = -\partial_\mu \omega^a(\vec{x}, t) - g\omega^\beta(\vec{x}, t) C^a_{\beta\gamma} A^\gamma_\mu . \qquad (4.8)$$

As an example with higher derivatives we may choose the following model Lagrangian

$$\mathcal{L} = \left[\left(\frac{d}{dt} - yT\right)\left(\frac{d}{dt} - yT\right)\vec{x} \right]^2 ,$$

where

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad T = -i\tau_2 = \begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix} . \qquad (4.9)$$

Using the definition (3.2) we can introduce the canonical variables q and p

$$q_1 = y, \qquad \vec{q}_1 = \vec{x} ,$$
$$q_2 = \dot{y}, \qquad \vec{q}_2 = \dot{\vec{x}} . \qquad (4.10)$$

Therefore we get the following expression for canonical Hamiltonian

$$H_c = \frac{1}{4}\vec{p}^{\,2}_2 + p_1 q_2 + \vec{p}_1 \vec{q}_2 + 2q_1 \vec{p}_2 T \vec{q}_2 + q_2 \vec{p}_2 T \vec{q}_1 + q^2_1 \vec{p}_2 \vec{q}_1 . \qquad (4.11)$$

We obtain one primary constraint ϕ^1_1 and two secondary constraints ϕ^2_1 and ϕ^3_1. All of them are of first-class. For the coefficients $g^{\beta m_a}_{a' a}$ we obtain

$$g^{12}_{11} = 1, \quad g^{23}_{11} = 1$$

and all others vanish. In this case the equations (4.6) are of the form

$$\dot{\epsilon}^2_1 - \epsilon^1_1 = 0 , \qquad \dot{\epsilon}^3_1 - \epsilon^2_1 = 0 . \qquad (4.12)$$

After the parametrization of ϵ^3_1 due to the arbitrary function $\lambda(t)$ we derive

$$\delta y = \dot{\lambda}(t) ,$$
$$\delta \dot{y} = \ddot{\lambda}(t) ,$$
$$\delta \vec{x} = \lambda(t)\, T(\vec{x}) ,$$
$$\delta \dot{\vec{x}} = \dot{\lambda}(t)\, T \vec{x} + \lambda(t)\, T \dot{\vec{x}} ,$$

these transformations leave the action invariant.

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REFERENCES

1. Dirac P.A.M. "Lectures on Quantum Mechanics", Belfer Graduate School of Science, Yeshiva University, New York, 1964.
2. Sudarshan E., Mukunda N. "Classical Dynamics - a Modern Perspective", New York, Wiley, 1974, p.78-107.
3. Sundermeyer K. "Constrained Dynamics", Lecture Notes in Physics, v.169, Berlin, Springer-Verlag, 1982.
4. Nesterenko V.V., Chervyakov A.M. JINR Preprint P2-86-324, Dubna, 1986.
5. Faddeev L.D. - Theor.Math.Phys., 1969, v.1, No.1, p.3-18.
6. Fradkin E.S., Vilkoviski G.A. - Phys.Lett., 1975, v.55B, No.2, p.224.
7. Slavnov A.A., Faddeev L.D. "Gauge Fields, Introduction to Quantum Theory", Bejamin-Commings-Reading, 1980.
8. Gitman D.M., Tyutin I.V. "Canonical Quantization of Constrained Field" (in Russian). M.: Nauka, 1986.
9. Gomis J. et al. - Europhys. Lett., 1986, 2(3), p.187-194.
10. Cawlay R. - Phys.Rev.Lett., 1979, v.42, No.7, p.413-416.
11. Castellani L. - Ann. of Phys., 1982, v.143, No.2, p.357-371.
12. Di Stefano R. - Phys.Rev., 1983, D27, p.1752.
13. Nesterenko V.V. JINR Preprint E2-8729, Dubna, 1986.
14. Ostrogradsky M. - Mem de l'Acad. de St.Pet., 1850, v.1, p.385.

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