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SECONDARY GAUGE CONDITIONS IN FIELD THEORY

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I. INTRODUCTION

The problem of the choice of conditions fixing the gauge seems from the first glance to be quite a technical problem connected with the choice of the most optimal methods of calculation of a physical process but when passing to the non-Abelian theory to be connected with other problems of the principal importance (Gribov's ambiguities, the proof of the S-matrix unitarity and so on). In this connection, the problems arising in the class of noncovariant ("axial") gauges $n^\mu A_\mu = 0$ widely used in QCD should be mentioned, an intensive discussion of which has emerged again/1/ from the recently found disagreement of the results of calculation of the gauge-invariant object, Wilson loop performed in this class of gauges, with the results obtained in the Coulomb and Feynman gauges/2/.

On the other hand it is known that the problem of quantization of constrained systems is fightly connected with the definition of the asymptotic behaviour of quantised fields. Thus, it has been shown in/3/ that the standard procedure of gauge-field quantization by the functional integral method contradicts the physical boundary conditions of vanishing of fields faster than $1/|x|$ infinity.

The mentioned difficulties lead to the necessity to modify the standard gauge-field quantisation. The present paper is devoted to the solution of this problem. Here, we shall restrict our consideration to the level of the classical theory of Yang-Mills fields. Firstly, we shall consider, using QED as an example, the connection between the choice of gauge conditions with the physical boundary condition that the field should obey. The result will be formulated in the form of a "Criterion of Unique Attainability of the gauge condition", that will be formulated in the form of a theorem. Then, a general theorem would be proved claiming that the imposing on electromagnetic field of a gauge condition that satisfies the "Criterion of Unique Attainability", leads, with equations of motion, to that the same potentials obey the secondary gauge condition appearing here as

an analog of the secondary constraint* and having the form of the Lorentz gauge condition,

$$\partial^\mu A_\mu(x)=0 \text{-secondary gauge condition} \qquad (1).$$

An analogous consideration will be performed also in a non-Abelian case. We shall introduce a new class of path-dependent generalized non-Abelian fields that under a particular choice of a path do coincide with the ordinary Yang-Mills fields. It will be shown that all the relations of QED (i.e. the equations of motion, the formulae of the connection of the field potential with the strength tensor, etc.) take place also for these non-Abelian fields with the only substitution of the ordinary derivatives in these relations by the Mandelstam path-derivatives. Correspondingly, the secondary gauge condition for these fields (and as a particular case also for the ordinary Yang-Mills fields in the corresponding gauges) has the form like (1) with the substitution of the ordinary derivatives by Mandelstam path-derivatives.

2. THE SECONDARY CONSTRAINT IN THE FORM OF THE LORENTZ CONDITION

Let us consider the case of free electrodynamics for which the action**

$$S = -\frac{1}{4}\int d^4x\, F^{\mu\nu}(x)F_{\mu\nu}(x), \qquad (2)$$

is invariant under the gauge transformations

$$A_\mu(x) \rightarrow A^\lambda_\mu(x) = A_\mu(x) + \partial_\mu\lambda(x). \qquad (3)$$

The gauge parameter λ in a general case is a functional of the field A, taken in an arbitrary gauge

$$\lambda = \Lambda(A,x). \qquad (4)$$

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*The secondary constraints follow from the primary ones with the use of equations of motion /4-6/.

**In what follows we shall be interested in developing a perturbation theory, i.e. we shall work in the interaction representation in which the free fields are just quantized.

Choosing the concrete form of functional (4) we obtain in this case in the left-hand side of (3) the field that obeys the gauge condition (GC) corresponding to this choice

$$A_\mu^{\varphi}(x) = A_\mu^{any}(x) + \partial_\mu \Lambda^{\varphi}(A^{any}, x). \qquad (5)$$

The symbol φ signifies that the field A_μ^{φ} obeys the GC*

$$\varphi(A, x) = 0. \qquad (6)$$

All of the most widely used in the theory GC can be presented in the form

$$\hat{\varphi}^{\mu} A_\mu^{\varphi}(x) = 0, \qquad (7)$$

where different forms of the operator $\hat{\varphi}^{\mu}$ correspond to the following GC:

∂^{μ} -Lorentz GC: $\quad \partial^{\mu} A_\mu^{(L)}(x) = 0, \qquad (8a)$

n^{μ} -the class of the non-covariant gauges $\quad n^{\mu} A_\mu^{(n)}(x) = 0,$ ** $\qquad (8b)$

$(x-\xi)^{\mu}$ -Fock GC/6/: $\quad (x-\xi)^{\mu} A_\mu^{(F)}(x) = 0.$ *** $\qquad (8c)$

Thus, the functional Λ^{φ} in (5) plays the role of a projector on a definite gauge.

In what follows we shall essentially use the boundary conditions (BC) imposed on gauge fields. They have the form/7/:

$$\lim_{|x|\to\infty} |x| A_\mu(x) = 0, \quad |x| = \sqrt{x_0^2 - \vec{x}^2}. \qquad (9a)$$

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*As it is mentioned before, we shall call this GC as primary ones. The meaning of this terminology will be clear from what follows.

**Here n is an arbitrary vector independent of x. The special choices of the vector n in (8b) correspond to such widely used gauges as, for instance, $n = (1,0,0,0)$ -Hamiltonian (temporal) gauge $A_0 = 0$, $n = (0,0,10)$ -axial $A_3 = 0$ gauge; $n^2 = 0$ -light-like gauge.

***Here ξ is an arbitrary, one and the same for all x, fixed point of the space-time. This gauge was rediscovered later by Schwinger/13/ and a number of authors.

Let us note that these BC are the only possible in the perturbation theory*. It is easy to see from relation (3) that the BC (9a) can be rewritten for the parameter of the gauge transformation λ in the following way

$$\lim_{|x|\to\infty} \lambda(x) = 0, \quad |x| = \sqrt{x_0^2 - \vec{x}^2}. \qquad (9b)$$

Thus, in the theory only such projectors $\Lambda(A, x)$ are admissible that do not contradict the boundary conditions (9a). It is not difficult to check that the projectors on the GC(8)

$$\Lambda^{(L)}(A, x) = -\int d^4y\, \partial_x^{\nu} \mathcal{D}^s(x-y) A_\nu(y); \qquad (10a)$$

$$\left(\mathcal{D}^s(x) \equiv \frac{1}{(2\pi)^4} \int d^4p\, V.P.(1/p^2)\right)$$

$$\Lambda^{(n)}(A, x) = -\int_{-\infty}^{0} d\alpha\, n^{\nu} A_\nu(x + \alpha n); \qquad (10b)$$

$$\Lambda^{(F)}(A, x) = -\int_0^1 d\alpha\, (x-\xi)^{\nu} A_\nu(\xi + \alpha(x-\xi)) \qquad (10c)$$

do satisfy condition (9b).

The role of the BC(9) in gauge-field quantization is very important. The existence of the BC allows us to draw a conclusion about the projector (4) to be the only admissible in the chosen gauge (6), i.e. that the equation $\varphi(A^{\Lambda}, x) = 0$ at fixed values of x and A has a unique solution for the functional Λ^{φ}.

Let us emphasize that the requirement of the uniqueness of this solution is one of the most important requirements imposed on the GC admissible in field quantization/4,5,8/. It is just this requirement that can be interpreted as one-to-one correspondence between the choice of the GC φ and the projector $\Lambda^{\varphi}(A, x)$ corresponding to it: $\Lambda^{\varphi}(A, x) \longleftrightarrow \varphi(A, x)$, what allows the application of the Fadeev-Popov procedure/8/ of the introduction of unity into the naive integral over trajectories.

The above-said will be formulated as the following criterion for the choice of the gauge condition (GC) - $\varphi(A, x) = 0$.

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*Only with this choice of BC it becomes possible to combine the requirement of finitness of action (2) with the possibility of integrating by parts, which is necessary for the construction of perturbation theory.

"Criterion of unique attainability of the gauge condition":

"The primary GC $\varphi(A;x)=0$ can be admissible in electrodynamics if there exists under GC(9) a unique functional $\Lambda^{\varphi}(A,x)$ with which these GC can be attained by the gauge transformation (5)".

It is obvious that the fulfilment of the criterion of unique attainability for GC is equivalent to the absence for it of the remaining gauge arbitrariness when the GC (9) are valid.

For the gauge conditions of the form of (7) and (8) the fulfilment of the criterion of unique attainability means that under the GC(9) there exists a unique functional $\Lambda^{\varphi}(A,x)$ that satisfies the equation

$$\hat{\varphi}^{\mu}\partial_{\mu}\Lambda^{\varphi}(A,x)=-\hat{\varphi}^{\mu}A_{\mu}(x), \qquad (11)$$

where A_{μ} is the electromagnetic field taken in an arbitrary gauge.

For illustration, let us consider an example of the primary GC $A_0=0$ (temporal or Hamiltonian gauge). We want to show that this GC does obey the criterion of unique attainability. Firstly, the projector onto this GC that satisfies equation (11) and the GC(9b) exists and has form (10b) with the choice of $n=(1,0,0,0)$: $\Lambda^{H}(A,x)=-\int_{-\infty}^{0}d\alpha\, A_0(x_0+\alpha,\vec{x})$*. Secondly, it is easy to prove the absence of the gauge arbitrariness under the BC (9b). Really, let us assume for the moment that besides the field $A^{H}_{\mu}(A,x)=A_{\mu}(x)+\partial_{\mu}\Lambda^{H}(A,\vec{x})$ there exists another field $A'^{H}_{\mu}=A^{H}_{\mu}+\partial_{\mu}\lambda$ must satisfy the equation $\partial/\partial x^{0}\lambda(x)=0$. Hence $\lambda(x)$ does not depend on time, i.e. $\lambda(x)=C(\vec{x})$. On the other hand, the BC (9a) for spatial components of the field A'^{H}_{μ} mean:

$$\lim_{x^0\to\infty}A_i(x)+\lim_{x^0\to\infty}\partial_i\int_{-\infty}^{0}d\alpha\,A_0(x_0+\alpha,\vec{x})+\lim_{x^0\to\infty}\partial_i\lambda(x)=0$$

The first two terms here turn out to be equal to zero due to (9a). Therefore, $\lim_{x^0\to\infty}\lambda(x)=0$, and as a result, $C(\vec{x})=0$.

Let us formulate the following main theorem.

Theorem 1.

"The gauge condition $\varphi(A,x)$ is uniquely attainable if first, there exists for it a projector $\Lambda^{\varphi}(A,x)$ that satisfies the boundary conditions (9), and second, this projector satisfies the relations

$$\partial_{\mu}\Lambda^{\varphi}(A+\partial\lambda,x)-\partial_{\mu}\Lambda^{\varphi}(A,x)=-\partial_{\mu}\lambda(x), \qquad (12)$$

where $A_{\mu}(x)$ is a field taken in an arbitrary gauge, $\lambda(x)$ is the parameter of the gauge transformations (3)

$$\Lambda^{\varphi}(A^{\varphi},x)=0, \qquad (13)$$

where A^{φ}_{μ} is the field defined with an accuracy up to the remaining gauge arbitrariness and satisfying the GC (6)".

Let us prove this theorem.

1. First, we shall prove the fact that the existence and uniqueness of the projector onto the GC leads to the fulfilment of relations (12) and (13) of theorem 1 for this projector. For this aim we shall show that the opposite statement is wrong. Let us suppose that the criterion of the unique attainability takes place for GC (7). It means that, first, there exists the projector $\Lambda^{\varphi}(A,x)$ that satisfies equation (11) for each fixed gauge of the field A:

$$\begin{aligned}\partial^{\mu}\hat{\varphi}_{\mu}\Lambda^{\varphi}(A^{\varphi_1},x)+\hat{\varphi}^{\mu}A^{\varphi_1}_{\mu}(x)&=\\ =\partial^{\mu}\hat{\varphi}_{\mu}\Lambda^{\varphi}(A^{\varphi_2},x)+\hat{\varphi}^{\mu}A^{\varphi_2}_{\mu}(x)&=0,\end{aligned} \qquad (14)$$

where A^{φ_1} and A^{φ_2} are fields taken in two different gauges φ_1 and φ_2, respectively, and, second, there does not exist the second functional $\Lambda'^{\varphi}\neq\Lambda^{\varphi}$ with the same property. Let us suppose for a moment that relation (12) of theorem 1 does not hold. It is easy to see that this supposition is equivalent to the inequality

$$A^{(1)\varphi}_{\mu}\neq A^{(2)\varphi}_{\mu}, \qquad (15)$$

where the fields $A^{(1)\varphi}_{\mu}=A^{\varphi_1}_{\mu}+\partial_{\mu}\Lambda^{\varphi}(A^{\varphi_1})$ and $A^{(2)\varphi}_{\mu}=A^{\varphi_2}_{\mu}+\partial_{\mu}\Lambda^{\varphi}(A^{\varphi_2})$ are obtained by projecting the fields given in different gauges φ_1 and φ_2, respectively. On the other hand, due to (14) both fields $A^{(1)\varphi}_{\mu}$ and $A^{(2)\varphi}_{\mu}$ do obey the GC (7): $\hat{\varphi}^{\mu}A^{(1)\varphi}_{\mu}=\hat{\varphi}^{\mu}A^{(2)\varphi}_{\mu}=0$. Further, due to the fact that the electromagnetic fields can differ only in the gradient transformation (4), we, with account of inequality (15), easily get $A^{(2)\varphi}_{\mu}=A^{(1)\varphi}_{\mu}+\partial_{\mu}\lambda^{\varphi}$, where $\partial^{\mu}\hat{\varphi}_{\mu}\lambda^{\varphi}=0$ and

*It is easy to check by the substitution $\tau=x_0+\alpha$ that the functional Λ^{H} does satisfy equation (11) with the choice of $\hat{\varphi}^{\mu}$ in form (8b), where $n=(1,0,0,0)$: $\partial/\partial x^{0}\Lambda^{H}(A,x)=-A_0(x)$.

$\lambda^{\varphi}(x) \neq$ const. Herefrom it becomes obvious that the functional $\Lambda'^{\varphi} = \Lambda^{\varphi} + \lambda^{\varphi}$ as well as the functional Λ^{φ} do obey relation (4), and $\Lambda'^{\varphi} \neq \Lambda^{\varphi}$ *. Thus, we have arrived at the contradiction with the requirement of the uniqueness of the projector and have shown at the same time that the fulfilment of relation (12) serves as the necessary condition for fulfilment of the criterion of the unique attainability.

Let us suppose now that relation (13) of theorem 1 does not hold. It means that there exists the field A'^{φ}_{μ} that does satisfy GC (7) and for which $\Lambda^{\varphi}(A'^{\varphi}, x) \neq 0$. It is obvious that the field A'^{φ}_{μ} could not be equal to the field $A^{\varphi}_{\mu} = A_{\mu} + \partial_{\mu}\Lambda^{\varphi}(A, x)$, because $\partial^{\mu}\Lambda^{\varphi}(A, x) = 0$ (the field should concide with itself). Therefore, $A'^{\varphi}_{\mu} = A^{\varphi}_{\mu} + \partial_{\mu}\lambda^{\varphi}$, where $\partial^{\mu}\varphi_{\mu}\lambda^{\varphi} = 0$ and we arrive at the same contradiction with the attainability of the GC: $\Lambda'^{\varphi} \neq \Lambda^{\varphi}$ but do satisfy (14).

So, we have proved that the fulfilment of relations (12) and (13) of theorem 1 serves as the necessary condition of the absence of the remaining gauge arbitrariness,i.e. of the uniqueness of the projector.

2. We shall prove now that, inversely, the absence of the gauge arbitrariness is the necessary condition for fulfilment of relations (12) and (13) of theorem 1.

Let us suppose that apart from the field $A^{\varphi}_{\mu} = A_{\mu} + \partial_{\mu}\Lambda^{\varphi}(A, x)$ (where the projector Λ^{φ} satisfies relations (12) and (13)) there exists another, different from A^{φ}_{μ}, field A'^{φ}_{μ} also satisfying GC (7) $A'^{\varphi}_{\mu} = A^{\varphi}_{\mu} + \partial_{\mu}\lambda^{\varphi}$, where $\partial^{\mu}\hat{\varphi}_{\mu}\lambda^{\varphi} = 0$ and $\lambda^{\varphi} \neq$ const. Then, due to (12) we shall have: $A'^{\varphi}_{\mu}(x) = A'^{\varphi}_{\mu}(x) + \partial_{\mu}\Lambda(A'^{\varphi}, x) + \partial_{\mu}\lambda^{\varphi}$. Taking into account relation (13) we finally obtain the equality $A'^{\varphi}_{\mu}(x) = A'^{\varphi}_{\mu}(x) + \partial_{\mu}\lambda^{\varphi}(x)$ and therefore, $\lambda^{\varphi}(x) =$ const. So, we see that the supposition about the difference of A^{φ}_{μ} and A'^{φ}_{μ} leads to the contradiction of the type const. $\neq$ const., and therefore, it was wrong.

At the first glance it may seem that this proof does not use BC (9). But it is not so. We have used relation (13) that seems to be obvious from the explicit form of projectors (10). But the fulfilment of (13) implies the fulfilment of BC. Thus, for example, the projector onto the class of noncovariant gauges (8b) has in the momentum

*The nontrivial inequality $\Lambda'^{\varphi}(A, x) - \Lambda^{\varphi}(A, x) \neq$ const is implied. The difference of the parameters of gauge transformation (4) up to a constant is equivalent to their coincidence because the field does not change in both cases.

representation the form $\Lambda^{(n)}(A, P) = -(nA)/(nP)$ (it is easy to verify that the field $A^{(n)}_{\mu}(P) = A_{\mu}(P) + iP_{\mu}\Lambda^{(n)}(A, P)$ obeys GC (1)). The requirement $\Lambda^{(n)}(A^{(n)}, x) = 0$, if $A^{(n)}$ is an arbitrary field satisfying (8b) (it is the requirement of relation (13) of theorem (1)), is equivalent to the requirement $n^{\nu}[P_{\nu}\lambda(P)]/(nP) = 0$, where $(nP)\lambda(P) = 0$. It is obvious that if the way of going around the pole $(nP)^{-1}$ is not defined, then a nontrivial solution of the last equation $\lambda(P) = \delta(nP)\tilde{\lambda}(P)$ is possible. With this solution requirement (13) is not valid and so the possibility of the nontrivial remaining gauge arbitrariness is left: $A'^{(n)}_{\mu}(P) = A^{(n)}_{\mu}(P) + iP_{\mu}[\delta(n^{\nu}P_{\nu})\lambda(P)]$. But the existence of the Fourier-transform of the projector $\Lambda^{(n)}(A, P)$ requires the prescription of going around the pole, for example, $(nP)^{-1} \to [(nP) + i\varepsilon]^{-1}$.

In this case relation (13) is valid for all solutions of the equation $(nP)\lambda(P) = 0$. The requirement of the existence of the Fourier-transform of the projector is just an account of the BC. Really, it is not difficult to verify that projector (10b) do satisfies BC (9b) with the condition of fulfilment of BC (9a) for the fields, taken in all other gauges different from (8b).

It is easy to check that relations (12) and (13) take place for all of the projectors of form (10) and therefore the corresponding to them GC (8) are uniquely attainable.

Let us mention that after the gauge transformation (3) with the functional (4) that depends on the electromagnetic field A_{μ} in an arbitrary gauge, we can consider the field A^{φ}_{μ} as the functional of these primary fields $A: A^{\varphi}_{\mu} = A^{\varphi}_{\mu}[A]$. The relation (12) leads to the gauge-invariance of this functional under the gauge transformations over the primary fields $A \; : \; A^{\varphi}_{\mu}[A + \partial\lambda, x] = A^{\varphi}_{\mu}[A, x]$. This looks like a paradox: the electromagnetic field in a fixed gauge seems to be a gauge invariant itself. But there is no any paradox in reality. This property of the "gauge invariance" means only that the fields , given in different gauges by (3) and (4), are projected onto one and the same field A^{φ}_{μ} , satisfying the GC (6). For example, it is not difficult to check that the functional relation

$$A^{(L)}_{\mu}(P) = A_{\mu}(P) - P_{\mu}\frac{(PA)}{P^2} \left(\Lambda^{(L)}(A, P) = -i\frac{(PA)}{P^2}\right)$$

is the Fourier transform of the projector (10a)) defining the field in the Lorentz gauge $P^{\mu}A^{(L)}_{\mu}(P) = 0$ is invariant under the gauge transformations: $A'_{\mu}(P) = A_{\mu}(P) + iP_{\mu}\lambda(P)$.

Now we shall show that the unique attainability of GC has a very important consequence, to be formulated as a theorem.

T h e o r e m 2.

"The electromagnetic field (with action (2)) that satisfies the criterion of unique attainability with the condition that the functional $\Lambda^{\varphi}(A,x)$ does satisfy the relation

$$\Box \Lambda(A,x) = \Lambda(\Box A, x) + f(\partial^{\mu} A_{\mu}, x), \qquad (16)$$

where $f(0,x) = 0$, and on which the primary gauge condition (6) is imposed by the gauge transformation (5), satisfies, on the equations of motion, the condition $\partial^{\mu} A_{\mu}(x) = 0$ that appears as the secondary gauge condition".

P r o o f :

Due to theorem 1, the unique attainability of the GC $\varphi(A,x)=0$ means that relations (12) and (13) take place for the projector $\Lambda^{\varphi}(A,x)$. Let us take in (12) $\lambda = \Lambda^{(L)}(A,x)$, where $\Lambda^{(L)}(A,x)$ is the projector (10a) on the primary Lorentz GC. Then relation (12) can be rewritten in the form

$$\partial^{\mu} \Lambda^{\varphi}(A^{(L)}, x) - \partial^{\mu} \Lambda^{\varphi}(A,x) = -\partial^{\mu} \Lambda^{(L)}(A,x).$$

Here the field A_{μ} is defined in an arbitrary gauge, therefore we can take it as given in gauge (6), i.e. we shall put in the last formula $A_{\mu} = A_{\mu}^{\varphi}$. Then due to (13) we have

$$\partial^{\mu} \Lambda^{\varphi}(A^{(L)}, x) = -\partial^{\mu} \Lambda^{(L)}(A^{\varphi}, x). \qquad (17)$$

Taking the 4-divergence of both sides of this equality with account of the fact that $\partial^{\mu} A_{\mu}^{(L)} = \partial^{\mu} A_{\mu} + \Box \Lambda^{(L)}(A,x)$, we obtain the relation

$$\partial^{\mu} A_{\mu}^{\varphi} = -\Box \Lambda^{\varphi}(A^{(L)}, x).$$

So with the use of this relation and (16) and the Maxwell equations $\Box A_{\mu}^{(L)} = 0$ we get that

$$\partial^{\mu} A_{\mu}^{\varphi}(x) = 0. \qquad (18)$$

Thus the theorem is proved. It is important to note that condition (18) holds for the simultanious account of the gauge condition imposed onto the field and the equations of motion (Maxwell equations in QED)*. Here one can see a complete analogy with the division of the constraints into the primary and secondary ones (see, for instance/3/). Following this analogy we shall call condition (18) the secondary GC**.

3. THE SECONDARY GAUGE CONDITION IN NON-ABELIAN THEORY

Let us consider the non-Abelian Yang-Mills theory with the Lagrangian $\mathcal{L}(x) = -\frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x)$ that is invariant under the gauge transformations

$$A_{\mu} \to A_{\mu}^{\omega} = \omega A_{\mu} \omega^{-1} + \frac{1}{g} \partial_{\mu} \omega \omega^{-1} \qquad (19a)$$

$$F_{\mu\nu} \to F_{\mu\nu}^{\omega} = \omega F_{\mu\nu} \omega^{-1}. \qquad (19b)$$

In what follows we shall need some formulae obtained in/11,12/. Thus, in the paper/12/, where the Mandelstam formulation of QED without potentials was generalized to the non-Abelian case, the gauge-invariant (under transformations (19)) field strength tensor

$$F_{\mu\nu}(x/C) = U^{+}(A, x/C) F_{\mu\nu}(x) \cdot U(A, x/C) \qquad (20)$$

was introduced.

Here $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + ig[A_{\mu}, A_{\nu}]$ is the usual gauge-invariant tensor that transforms according to (19b), and the operator $U(A, x/C)$ has the form

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*Relation (18) was obtained in a particular case of gauges (8b) and (8c) in/9/ on the basis of the so-called inversion formulae that express the fields in gauges (8b) and (8c) through the tension tensor /9/. By the same method it was shown then/10/ that in the non-Abelian case an analog of (18) has the form $\tilde{\partial}^{\mu} A_{\mu}^{a}(x) = 0$, where $\tilde{\partial}^{\mu}$ is the Mandelstam path derivative. Now we see that relation (17) for gauges (8b) and (8c) appears as a particular case of theorem 2.

**Let us note that the secondary GC takes the form of the Lorentz condition only in the case of a free electromagnetic field. In the case of interaction of the electromagnetic and spinor fields, for example, in the Fock gauge instead of (18), we shall have the following condition $\partial^{\mu} A_{\mu}^{(F)}(x/\xi) = \int_0^1 d\alpha\, \alpha (x-\xi)^{\nu} J_{\nu}(\xi + \alpha(x-\xi))$, where J^{ν} is a spinor current. But here we shall be interested in the perturbation theory in the framework of which free fields are quantized, thus we shall use the secondary GC in form (18).

$$\mathcal{U}(A,x/C)=\exp[ig\mathcal{P}\int_{-\infty}^{x}d\eta^{\nu}A_{\nu}(\eta)]. \tag{21}$$

The integration in the exponent is performed along the unclosed path C of an arbitrary form that goes from $-\infty$ up to x. The symbol $\mathcal{P}$ means the ordering along the path. In /12/ it has been shown that the tensor (20) obeys the next equation of motion

$$\tilde{\partial}^{\mu}\mathcal{F}_{\mu\nu}(x/C)=0 \tag{22}$$

and the equality

$$\tilde{\partial}_{\rho}\mathcal{F}_{\mu\nu}+\tilde{\partial}_{\mu}\mathcal{F}_{\nu\rho}+\tilde{\partial}_{\nu}\mathcal{F}_{\rho\mu}=0. \tag{23}$$

Here $\tilde{\partial}_{\mu}$ is the Mandelstam path derivative /11,12/:

$$\tilde{\partial}_{\mu}\mathcal{U}(A,x/C)=\lim_{\Delta x^{\mu}\to 0}[\mathcal{U}(A,x+\Delta x/C')-\mathcal{U}(A,x/C)]/\Delta x^{\mu}, \tag{24}$$

where two paths C and C' differ from each other only by Δx^{μ}.

We would like to note that in the present article, in contrast with papers /11,12/ when authors have refused from the use of the potentials and have chosen instead of them the strength tensor as the main object for quantization, we develop an alternative approach. In our formalizm the field potentials would appear as the main object for quantization and the strength tensor as the auxiliary object.

For simplicity we shall restrict our consideration to two particular cases of two primary gauge conditions (8b) and (8c) imposed on the field A_{μ}.

Let us consider firstly the case when the primary gauge condition (8b) is imposed on A_{μ} with the help of the gauge transformation (19a): $n^{\mu}A_{\mu}^{(n)}(x)=0$. It is obvious that the field $A_{\mu}^{(n)}$ can equivalently be rewritten in the form

$$A_{\mu}^{(n)}(x)=A_{\mu}^{(n)}(x)-\partial_{\mu}\int_{-\infty}^{0}d\alpha\, n^{\nu}A_{\nu}^{(n)}(x+\alpha n)+g\int_{-\infty}^{0}d\alpha\, n^{\nu}[A_{\mu}^{(n)},A_{\nu}^{(n)}].$$

From here we get that

$$A_{\mu}^{(n)}(x)=-n^{\nu}\int_{-\infty}^{0}d\alpha\, F_{\mu\nu}(x+\alpha n), \tag{25}$$

where $F_{\mu\nu}^{(n)}=\partial_{\mu}A_{\nu}^{(n)}-\partial_{\nu}A_{\mu}^{(n)}+g[A_{\mu}^{(n)},A_{\nu}^{(n)}]$ is the strength tensor of the Yang-Mills field in the gauge (8b). Further, with the help of the equality $n^{\mu}A_{\mu}^{(n)}=0$ we can rewrite the relation (25) in the next way

$$A_{\mu}^{(n)}(x)=-n_{\nu}\int_{-\infty}^{0}d\alpha\, \exp[-ig\bar{\mathcal{P}}\int_{-\infty}^{0}d\beta\, n^{\nu}A_{\nu}(z+\beta n)]\cdot F_{\mu\nu}^{(n)}(z(\alpha))\exp[ig\mathcal{P}\int_{-\infty}^{0}d\beta\, n^{\nu}A_{\nu}^{(n)}(z+\beta n)]. \tag{26}$$

Here, in accordance with the notation (21)

$$\mathcal{U}^{+}(A^{(n)},z/C^{(n)})F_{\mu\nu}^{(n)}(z)\mathcal{U}(A^{(n)},z/C^{(n)}). \tag{27}$$

The integration is performed along the unclosed path $C^{(n)}$ of the form

$$C^{(n)}:\ \eta(\beta)=z+\beta n,\ -\infty<\beta\leqslant 0;\quad z(\alpha)=x+\alpha n \tag{28}$$

and $\mathcal{P}$ is the symbol of the ordering ($\bar{\mathcal{P}}$ is antiordering) along the parameter β. The expression (27) is nothing more but the gauge-invariant strength tensor (20) with the particular choice of the path $C=C^{(n)}$. Using the gauge invariance of the expression (20) we easily get

$$\mathcal{U}^{+}(A^{(n)}/C^{(n)})\cdot F_{\mu\nu}^{(n)}(z)\mathcal{U}(A^{(n)}/C^{(n)})=$$
$$=\mathcal{F}_{\mu\nu}^{(n)}(z/C^{(n)})=\mathcal{F}_{\mu\nu}(z/C^{(n)})=\mathcal{U}^{-1}(A,z/C^{(n)})\cdot$$
$$\cdot F_{\mu\nu}(z)\mathcal{U}(A,z/C^{(n)}),$$

where A is the field in an arbitrary gauge.

So, we have obtained that the Yang-Mills field obeying the primary gauge condition (8b) can be expressed through the gauge-invariant strength tensor (22) with the help of the next formula (the "inversion formula")

$$A_\mu^{(n)}(x/C^{(n)}) = -n^\nu \int_{-\infty}^{0} d\alpha\, \mathcal{F}_{\mu\nu}(z(\alpha)/C^{(n)}). \qquad (29)$$

It is easy to see that there exists a possibility to generalize formula (29) by refusing from the strict fixation $C = C^{(n)}$ and keeping the possibility of arbitrariness in the choice of the path C. Let us introduce the new fields

$$B_\mu^{(n)}(x/C) = -n^\nu \int_{-\infty}^{0} d\alpha\, \mathcal{F}_{\mu\nu}(z(\alpha)/C), \qquad (30)$$

where C is a path of an arbitrary form. It is clear that due to the antisymmetry of the tensor $\mathcal{F}_{\mu\nu}$ those new gauge-dependent fields $B_\mu^{(n)}$ as well as the fields $A_\mu^{(n)}$ obey the primary gauge condition (8b). But in the general case the fields (30) with an arbitrary choice of the path do not coincide with the ordinary Yang-Mills fields and only with the choice of the path $C = C^{(n)}$ in the form of a straight line, parallel to the 4-vector n^μ, we would obtain instead of (30) an ordinary field $A_\mu^{(n)}$ in an auxiliary gauge.

Let us consider the properties of the fields $B_\mu^{(n)}$. With the help of equality (23) it can be shown from (30) that the fields $B_\mu^{(n)}$ introduced by this formula, can be related with the tensor (20) also by

$$\mathcal{F}_{\mu\nu}(x/C) = \tilde{\partial}_\mu B_\nu^{(n)}(x/C) - \tilde{\partial}_\nu B_\mu^{(n)}(x/C) \qquad (31)$$

that inverts formula (30). The relation (31) was previously given in paper/12/ but without explicit definition of the form of the fields that can fulfil the relation (31).

So, we see that the connection of the gauge-invariant tensor $\mathcal{F}_{\mu\nu}(x/C)$ with the fields $B_\mu(x/C)$ introduced here is analogous to the well-known relation that holds in the Abelian case up to the substitution of ordinary derivatives by the Mandelstam path derivatives.

It is easy to find with the help of the inversion formula (30) that on the equations of motion (22) the next relation takes place.

$$\tilde{\partial}^\mu B_\mu^{(n)}(x/C) = 0, \qquad (32)$$

that appears here as the generalization of the secondary gauge condition (18) for the non-Abelian case and has the sense of the seconary constraint as well.

The formulae (31) and (32) are valid for any choice of the path and in particular for the choice $C = C^{(n)}$. Thus, they are also valid for the ordinary Yang-Mills fields taken in the gauge (8b). As in the previous case we shall obtain in an analogous way firstly the relation

$$A_\mu^{(F)}(x) = -\int_0^1 d\alpha\, \alpha (x-\xi)^\nu F_{\mu\nu}^{(F)}(\xi + \alpha(x-\xi)), \qquad (33)$$

where is

$$F_{\mu\nu}^{(F)} = \partial_\mu A_\nu^{(F)} - \partial_\nu A_\mu^{(F)} + g[A_\mu^{(F)}, A_\nu^{(F)}]$$

is the strength tensor in the Fock gauge, and secondly, the expression

$$A_\mu^{(F)}(x) = -\int_0^1 d\alpha\, \alpha (x-\xi)^\nu \mathcal{F}_{\mu\nu}(\xi + \alpha(x-\xi)/C_\xi^{(F)}), \qquad (34)$$

where $\mathcal{F}_{\mu\nu}(z/C_\xi)$ is the gauge-invariant strength tensor

$$\mathcal{F}_{\mu\nu}(z/C_\xi) = \exp[-ig\bar{\mathcal{P}} \int_\xi^z d\eta^\nu A_\nu(\eta)] F_{\mu\nu}(z) \cdot \exp[ig\mathcal{P}\int_\xi^z d\eta^\nu A_\nu(\eta)], \qquad (35)$$

where the particular choice of the path in the form of the straight line that connects the points ξ and z is chosen

$$C_\xi = C_\xi^{(F)}: \quad \eta = \xi + \beta(z-\xi), \quad 0 \leq \beta \leq 1. \qquad (36)$$

The difference between the tensors (35) and (20) consists in the fact that the integration in the exponents (35) is performed not along the infinitely long path but along the path C_ξ of the finite length and arbitrary form. Due to this, the tensor (35) transforms when the local gauge transformations (12) of the fields $A_\mu(z)$ are performed; but those transformations have the global form

$$\mathcal{F}_{\mu\nu}(z/C_\xi) \longmapsto \omega(\xi)\mathcal{F}_{\mu\nu}(z/C_\xi)\omega^{-1}(\xi). \qquad (37)$$

That is why we shall consider the tensor (35) as a gauge-invariant one only up to this reservation.

By analogy with the previous case let us introduce new fields :

$$B_\mu(x/C_\xi) = -\int_0^1 d\alpha\, \alpha\, (x-\xi)^\nu \mathcal{F}_{\mu\nu}(\xi+\alpha(x-\xi)/C_\xi), \quad (38)$$

where $\mathcal{F}_{\mu\nu}(z/C_\xi)$ is the tensor (35) with the arbitrary path C_ξ that connects the points ξ and z. As in the case of the gauge-conditions (8b), these new fields that obey the gauge condition (8c) they do coincide with the ordinary Yang-Mills fields in the Fock gauges only at the particular choice of the path $C_\xi = C_\xi^{(F)}$ (36).

With the help of integration by parts and making use of the antisymmetry of the tensor $\mathcal{F}_{\mu\nu}(z/C_\xi)$ and of the equation of motion (22) that is also valid for the tensor (35) we easily find

$$\mathcal{F}_{\mu\nu}(x/C_\xi) = \tilde{\partial}_\mu B_\nu(x/C_\xi) - \tilde{\partial}_\nu B_\mu(x/C_\xi) \quad (39)$$

and

$$\tilde{\partial}^\mu B_\mu(x/C_\xi) = 0. \quad (40)$$

The relation (40) appears here as a secondary gauge condition. It is not difficult to see that formulae (39) and (40) are valid also for the ordinary Yang-Mills fields $A_\mu^{(F)}$ obeying the primary gauge condition (86) as a particular case corresponding to the choice $C_\xi = C_\xi^{(F)}$ in (38)-(40).

In the conclusion of this section let us remind that in our approach in the non-Abelian case there appears just the same formalism as in QED, i.e. formulae (29) and (33) do express the fields in the fixed gauges (8b) and (8c) through a gauge invariant (up to the reservation mentioned before for the Fock gauge) strength tensor. Thus, those fields defined in the gauges (8b) and (8c) by formulae (29) and (33) through the strength tensor being considered as the functionals of the primary fields A in arbitrary

*It is obvious that in the Abelian case the tensors (20) and (35) would transform into the tensor $f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the relation (38) would transform into a well-known inversion formula $A_\mu^{(F)}(x) = -\int_0^1 d\alpha\, \alpha\, (x-\xi)^\nu f_{\mu\nu}(\xi+\alpha(x-\xi))$ and (30) would transform into relation $A_\mu^{(n)} = -n^\nu \int_{-\infty}^0 d\alpha\, f_{\mu\nu}(x+\alpha n)$ that has previously been derived in /10/.

gauges are gauge-invariant quantities in the non-Abelian as well as in Abelian case. (Here, it should be mentioned that, as it is seen from (33) and (36), the field in the Fock gauge, in contrast with the field defined in the gauge (8b), transforms in a global way $A_\mu^{(F)}(A^\omega; x) = \omega(\xi) A_\mu^{(F)} \omega^{-1}(\xi)$ under the transformations (19a) over the primary fields . Due to the physical boundary conditions $\omega(\infty) = +1$ this residual gauge arbitrariness disappear only in the limit $\xi \to \infty$. Let us note that in this limit the translation invariance of the field $A_\mu^{(F)}(x/C = C_\xi)$ restores; so, this limit can be interpreted as a physical one).

The property of the gauge invariance of the fields in the fixed gauges is nothing more but the manifestation of the property of the unique attainability of the gauge conditions (8b) and (8c) that for this gauges take place for the Abelian as well as for non-Abelian cases.

In view of this property let us especially stress the next important feature. If we would like to repeat for the field $A_\mu^{(L)}$ (defined in the gauge (8a)) the same procedure that was used for the derivation of the inversion formulae (29) and (33) for the fields in the gauges (8b) and (8c), then due to the fact that $\partial_x^\mu [A_\mu^{(L)}(x), A_\nu^{(L)}(x)] \neq 0$ we would not come to the inversion formula. Thus, we see that in the non-Abelian case there does not exist formula expressing the field in the Lorentz gauge (8a) through the gauge-invariant strength tensor. So, the gauge condition (8a) in the non-Abelian case is not a unique attainable condition in contrast with the gauge conditions (8b) and (8c). This fact is directly connected with the fact of existence of Gribov's ambiguities that, as it is well known, exist for the gauge (8a) and from which the gauges (8b) and (8c) are free.

CONCLUSION

Thus, it is shown in the present paper that in gauge theories the fields (Abelian and non-Abelian ones) after imposing on them gauge condition do satisfy the secondary gauge condition. This condition has the sense of the secondary constraint because it fulfils the equations of motion.

In the following paper we shall whow how this condition can be included into the system of constraints at the quantization of fields (partly the results were presented in /13/) and to what physical consrquences it can lead.

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Вторичные калибровочные условия в теории поля

Показано, что после наложения на калибровочные поля одного калибровочного условия они подчиняются также вторичному калибровочному условию, имеющему в КЭД форму условия Лоренца. В неабелевом случае введен новый класс калибровочно-инвариантных и контурно-зависимых полей. Показано, что для них имеют силу все соотношения КЭД, т.е. уравнения движения, формулы связи с тензором напряженности и вторичное калибровочное условие и т.д., с заменой обычных производных на производные Мандельстама.

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Secondary Gauge Conditions in Field Theory

It is shown that upon imposing one gauge condition on the gauge fields the latter turn out to obey the secondary gauge condition that in QED is in form a Lorentz gauge condition. A new class of gauge-invariant and part-dependent non-Abelian fields is introduced. It is shown that all of the QED relations, i.e. the equations of motion, the formulae of the connection with the strength tensor, the secondary gauge condition etc., take place for these new fields with the only substitution of the ordinary derivatives by the Mandelstam ones.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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