



**ОБЪЕДИНЕННЫЙ  
ИНСТИТУТ  
ЯДЕРНЫХ  
ИССЛЕДОВАНИЙ  
ДУБНА**

N 56

**E2-87-279**

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**QUANTIZATION  
OF AXIAL ELECTRODYNAMICS**

Submitted to the International Conference  
on High Energy Physics, Sweden, 1987.

**1987**

## 1. INTRODUCTION

The absence of anomalies in gauge theories represents one of the fruitful principles for constructing physical theories <sup>/1,2,3/</sup>.

At the same time there is an opinion <sup>/4/</sup> that the gauge theories with anomalies can be considered as physical ones. However, up to now despite numerous attempts <sup>/5,6,7/</sup> there is no consistent quantization of the theories with anomalies.

In the present paper, we shall study the possibility of Hamiltonian quantization <sup>/9-11/</sup> of such gauge theories with anomalies.

## 2. THE FORMULATION OF THE THEORY

Let us consider the classical theory of massless free fermions in a four-dimensional space-time.

$$S = \int d^4x \mathcal{L}_0(\mathbf{x}), \quad \mathcal{L}_0(\mathbf{x}) = \bar{\Psi} i \hat{\partial} \Psi; \quad (\hat{\partial} = \gamma_\mu \partial_\mu; \quad \bar{\Psi} = \Psi^+ \gamma_0). \quad (1)$$

We demand that the theory (1) should be invariant with respect to the axial gauge transformation

$$\Psi(\mathbf{x})^\beta = e^{i\gamma_5 \beta(\mathbf{x})} \Psi(\mathbf{x}). \quad (2)$$

According to the classical principle of the local gauge invariance the invariance of the theory (1) is achieved by introducing the fermion interaction with an axial-vector field

$$S = \int d^4x \bar{\Psi} i \hat{D} \Psi \equiv \int d^4x (\bar{\Psi} i \hat{\partial} \Psi + J_\mu^5 A_\mu), \quad (3)$$

$$i \hat{D} = \gamma_\mu (i \partial_\mu + \gamma_5 A_\mu), \quad J_\mu^5 = \bar{\Psi} \gamma_\mu \gamma_5 \Psi,$$

whose transformations

$$A_\mu \rightarrow A_\mu^\beta(\mathbf{x}) = A_\mu(\mathbf{x}) + \partial_\mu \beta(\mathbf{x}), \quad (4)$$

compensate the transformations (2).

However, if the fermion fields are quantum and satisfy the commutation relation

$$\{\Psi_\alpha^+(\vec{x}, t), \Psi_\beta(\vec{y}, t)\} = \delta_{\alpha\beta} \delta^3(\vec{x} - \vec{y}),$$

so that their axial transformations are made by the generator

$$\Psi^\beta(x) = U \Psi(x) U^{-1}, \quad (5)$$

$$U = \exp\{iQ_5(\beta)\}, \quad Q_5(\beta) = \int d^3x J_0^5(x) \beta(x),$$

then the "classical" principle of the local gauge invariance is broken.

What is the physical cause of this breaking?

The quantum fermions differ from classical ones by the Dirac Sea (continuum) that arises from the requirement for the quantum Hamiltonian being positive. In the external (classical) axial-vector field (3) the Dirac sea is rearranged so that the current commutators become anomalous<sup>12-16</sup>

$$i[J_0^5(\vec{x}, t), J_0^5(\vec{y}, t)] = -\frac{1}{6\pi^2} B_k(y) \frac{\partial}{\partial x_k} \delta^3(\vec{x} - \vec{y}),$$

$$i[J_1^5(\vec{x}, t), J_0^5(\vec{y}, t)] = -\frac{1}{6\pi^2} \epsilon_{ikl} A_k(y) \frac{\partial}{\partial x_l} \delta^3(\vec{x} - \vec{y}), \quad (6)$$

$$B_k(y) = \epsilon_{kij} \partial_i A_j(y), \quad A_k(y) = \frac{\partial}{\partial t} A_k(y).$$

Solving by these commutators the Heisenberg equation for

$$\frac{\partial}{\partial t} J_0^5(x) = i[H, J_0^5(x)], \quad (7)$$

where H is the Hamiltonian of the theory (3)

$$H = \int d^3x [\bar{\Psi} i \gamma_i \partial_i \Psi - J_\mu^5 A_\mu],$$

we get the anomalous divergence of the axial current

$$\partial_\mu J_\mu^5(x) = \frac{1}{48\pi^2} F_{\mu\nu} \tilde{F}_{\mu\nu}, \quad (8)$$

$$\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}.$$

Formula (8) is consistent with the calculation of the anomalous triangle diagrams<sup>12-16</sup>.

From eq. (8) we see that the action (3) for the quantum fermions is noninvariant under the axial gauge transformation (5) and acquires the extra term

$$\Delta S = \int d^4x \beta(x) \partial_\mu J_\mu^5(x) \equiv \int d^4x \beta(x) \frac{1}{48\pi^2} F_{\mu\nu} \tilde{F}_{\mu\nu}. \quad (9)$$

### 3. QUANTIZATION OF AXIAL-VECTOR FIELD

We consider the interaction of massless quantum fermions with an external axial-vector field

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu}^2 + \bar{\Psi} \gamma_\mu (i\partial_\mu + A_\mu \gamma_5) \Psi. \quad (10)$$

As it is pointed above, this lagrangian is not invariant with respect to the transformations (4), (5), (see eq. (9)).

We can restore the symmetry of the theory (3) by introducing into the Lagrangian an extra term whose transformation compensates the anomalous reaction (9) of the initial action (3).

For example, we choose the following extra term<sup>11</sup>

$$\mathcal{L}_{\text{mod.}}(x) = \mathcal{L}(x) - \Lambda \mathcal{L}(x), \quad \Lambda \mathcal{L}(x) = -\frac{1}{\square} (\partial_\mu A_\mu) \frac{F_{\alpha\beta} \tilde{F}_{\alpha\beta}}{48\pi^2}, \quad (11)$$

or a classical equivalent

$$\Delta \mathcal{L}(x) = \partial_\mu \chi (\partial_\mu \phi + A_\mu) + \phi \frac{F_{\alpha\beta} \tilde{F}_{\alpha\beta}}{48\pi^2}, \quad (12)$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \chi)} = 0 \quad \Rightarrow \quad \phi = -\frac{1}{\square} \partial_\mu A_\mu.$$

For quantization of the axial-vector field we choose transverse variables which are fixed by transformations

$$\begin{aligned}
\mathbf{A}_\mu^T &= \mathbf{A}_\mu + \partial_\mu \theta^T(\mathbf{A}) \\
\Psi^T &= e^{i\theta^T(\mathbf{A})} \Psi \\
\phi^T &= \phi - \theta^T(\mathbf{A}) \\
\chi^T &= \chi
\end{aligned}
\tag{13}$$

$$\theta^T(\mathbf{A}) = -\frac{1}{\partial_k^2} (\partial_i \mathbf{A}_i).$$

The transverse physical fields (13) are nonlocal (gauge invariant) functionals of the initial fields  $\mathbf{A}, \Psi, \phi$  <sup>/11/</sup>.

The transverse variables are convenient for the Hamiltonian quantization and are only variables consistent with the classical equation <sup>/17/</sup> for the time component of the field ( $\mathbf{A}_0$ ).

Due to the nonlocality (13) the gauge of the variables is not fixed and follows the time-axis rotation in the course of relativistic transformations <sup>/9,11/</sup>.

Upon passing to the transverse variables we have only one nondynamic variable  $\mathbf{A}_0^T$  and the constraint equation

$$\frac{\delta S}{\delta \mathbf{A}_0^T} = 0 \quad \Rightarrow \quad \partial_k^2 \mathbf{A}_0^T = [-\mathbf{J}_0^{5T} + \partial_0 \chi^T + \frac{\partial_i \phi^T \mathbf{B}_i^T}{6\pi^2}].
\tag{14}$$

The Hamiltonian of the theory (11),(12) has the form

$$\begin{aligned}
H &= \int d^3x T_{00}, \\
T_{00} &= \dot{\mathbf{A}}_k^T \mathbf{E}_k^T + \dot{\phi}^T \pi_\phi^T + \dot{\chi}^T \pi_\chi^T + \dot{\Psi}^T \pi_\Psi^T - \mathcal{L}_{\text{mod.}}^T = \\
&= \frac{1}{2} [\dot{\mathbf{A}}_k^{T2} + (\partial_k \mathbf{A}_0^T)^2 + \mathbf{B}_k^{T2}] + \pi_\phi^T \pi_\chi^T - \mathbf{A}_i^T \mathbf{J}_i^{5T} + \\
&+ \dot{\Psi}^T i \gamma_i \partial_i \Psi^T + \partial_i \chi^T \partial_i \phi^T,
\end{aligned}
\tag{15}$$

where the canonical conjugate momenta  $\mathbf{E}_k^T, \pi_\phi^T, \pi_\chi^T, \pi_\Psi^T$  are given by the following formulae

$$\mathbf{E}_k^T = \dot{\mathbf{A}}_k^T + \pi_{ki} \left( \frac{\phi^T \mathbf{B}_i^T}{6\pi^2} \right), \quad (\pi_{ki} = \delta_{ki} - \delta_k \frac{1}{\partial^2} \partial_i),
\tag{16}$$

$$\pi_\chi^T = \partial_0 \phi^T + \mathbf{A}_0^T, \quad \pi_\phi^T = \partial_0 \chi^T, \quad \pi_\Psi^T = i\Psi^T,
\tag{17}$$

$$\mathbf{A}_0^T = \frac{1}{\partial_k^2} (-\mathbf{J}_0^{5T} + \pi_\phi^T + \frac{\partial_i \phi^T \mathbf{B}_i^T}{6\pi^2}).
\tag{18}$$

For the boson operators we choose the usual commutation relations

$$\begin{aligned}
i[\mathbf{E}_k^T(\vec{x}, t), \mathbf{A}_j^T(\vec{y}, t)] &= (\delta_{kj} - \partial_k \frac{1}{\partial^2} \partial_j) \delta^3(\vec{x} - \vec{y}), \\
i[\pi_\phi^T(\vec{x}, t), \phi^T(\vec{y}, t)] &= i[\pi_\chi^T(\vec{x}, t), \chi^T(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y}), \\
[\mathbf{J}_0^{5T}(\vec{x}, t), \mathbf{A}_i^T(\vec{y}, t)] &= 0.
\end{aligned}
\tag{19}$$

Then we check the divergence of the axial current with the help of the Heisenberg equation

$$\frac{\partial}{\partial t} \mathbf{J}_0^{T5}(\mathbf{x}) = i[H, \mathbf{J}_0^{T5}(\mathbf{x})].
\tag{20}$$

We get an expression of the type (8) with a new term

$$\partial_\mu \mathbf{J}_\mu^{T5}(\mathbf{x}) = \frac{1}{48\pi^2} F_{\alpha\beta} \tilde{F}_{\alpha\beta} + \int d^3y = [\frac{1}{2} \dot{\mathbf{A}}_k^{T2}(y), \mathbf{J}_0^{5T}(\mathbf{x})],
\tag{21}$$

that appears owing to the axial field quantization. This term can easily be calculated

$$\begin{aligned}
\int d^3y [\frac{1}{2} \dot{\mathbf{A}}_k^{T2}(y), \mathbf{J}_0^{T5}(\mathbf{x})] &= \int d^3y \frac{1}{2} \{ \dot{\mathbf{A}}_k^T(y) [\dot{\mathbf{A}}_k^T(y), \mathbf{J}_0^{T5}(\mathbf{x})] + \\
&+ [\dot{\mathbf{A}}_k^T(y), \mathbf{J}_0^{T5}(\mathbf{x})] \dot{\mathbf{A}}_k^T(y) \} = -\frac{1}{12\pi^2} \frac{\partial}{\partial t} (\mathbf{A}_k^T \mathbf{B}_k^T).
\end{aligned}
\tag{22}$$

Thus, instead of eq. (8) we get

$$\frac{\partial}{\partial t} [\mathbf{J}_0^{T5}(\mathbf{x}) - \frac{1}{12\pi^2} (\mathbf{A}_k^T(\mathbf{x}) \mathbf{B}_k^T(\mathbf{x}))] = \partial_i \mathbf{J}_i^{T5}(\mathbf{x}) + \frac{1}{48\pi^2} F_{\alpha\beta} \tilde{F}_{\alpha\beta}.
\tag{23}$$

The usual choice of the commutators (19) leads also to nonzero Jacobi brackets

$$\mathcal{J}(\mathbf{A}_k^T(\mathbf{x}), H, \mathbf{J}_0^{T5}(y)) = \{[\mathbf{A}_k^T(\mathbf{x}), [H, \mathbf{J}_0^{T5}(y)]] +$$

$$+ [J_0^{T5}(y)[A_k^T(x), H]] + [H[J_0^{T5}(y), A_k^T(x)]] = \\ = \frac{i}{6\pi^2} B_k(x) \delta^3(\vec{x} - \vec{y}).$$

Violation the Jacobi-identities for the very currents is noted in refs <sup>6,7</sup>. Thus, the ordinary quantum theory with axial anomalies is not consistent for the quantum gauge fields. In any case the application of the ordinary part integral is problematic <sup>5</sup>.

## CONCLUSION

We have attempted to construct the Hamiltonian operator formalism for the Abelian axial gauge theory. It is show that the ordinary commutation relation for the gauge fields breaks the Jacobi identity for operators of the Hamiltonian, the time component of the current and the very field. These operators do not satisfy the associativity law with respect to their multiplications and cannot be represented by linear operators in the Hilbert space.

Acknowledgements. The authors would like to thank Prof. B.M.Barbashov, A.V.Efremov, G.V.Efimov for useful comments.

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Received by Publishing Department  
on April 22, 1987.

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E2-87-279

Квантование аксиальной электродинамики

В работе изучается возможность построения гамильтонова квантования для аксиальной электродинамики с аномалиями в 4-мерном пространстве. Показано, что при каноническом квантовании этой теории нарушается тождество Якоби для операторов гамильтониана, тока и самого поля, и квантовая теория самосогласована лишь в случае нулевого магнитного поля.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1987

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E2-87-279

Quantization of Axial Electrodynamics

The possibility of constructing the Hamiltonian quantization for axial electrodynamics with anomalies in a four-dimensional space is studied. It is shown that in this theory the Jacobi identity for operators of the Hamiltonian, of a time component of the current and of the very field is broken. The usual quantum theory is consistent only for a zero magnetic field.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1987