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D.I.Kazakov

FINITENESS OF NONLINEAR SIGMA-MODELS ON RICCI-FLAT MANIFOLDS

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1. Recently two-dimensional nonlinear sigma-models have been the subject of renewed interest, and the ultraviolet behaviour of these theories has been studied [1-3]. This is due to their relevance in superstring theories and, in particular, in the compactification of extra dimensions [4]. It has been argued that the superstring consistency requires the σ-model to be conformally invariant. In a curved space one must require that generalized β-functions of the σ-model vanish, that is, the model is finite. The corresponding equations are interpreted as the equations of motion of the massless fields of superstring [5,6]. The graviton equation then defines the metric d possible choice of a compact six-dimensional manifold.

At the one-loop level it is known that in both supersymmetc and nonsupersymmetric cases the generalized β-function the graviton field is proportional to the Ricci tensor of the ssociated manifold. Therefore the theory is consistent if the tter is Ricci-flat. In higher loops the metric of the consisnt theory acquires corrections proportional to $\alpha' \sim 1/T$, where T is a string tension. Thus, the phenomenologically favored mpactification on the Ricci-flat Kähler manifold with $SU(3)$ olonomy (the Calabi-Yau manifold) becomes only approximate and y be considered as a starting point.

In the present note we propose an alternative approach to e finiteness of nonlinear σ-models. We show that the theory n be made finite and conformally invariant by a proper choice the background bare metric in the form $g^B_{ij}(\varphi) = \sum_{k\geq 0} g^{(k)}_{ij}(\varphi)\varepsilon^k$, where ε is the parameter of dimensional regularization. The finiteness is achieved by the mechanism proposed in ref.[7].

The classical metric $g^{(0)}_{ij}$ corresponds to $\varepsilon = 0$ and must be Ricci-flat.

2. We briefly review the renormalization properties of nonlinear σ-models. The counterterms are the local expressions of dimension two and result in the renormalization of the bare metric

$$g^B_{ij} = g^R_{ij} + \sum_{n=1}^{\infty} \frac{1}{\varepsilon^n} T^{(n)}_{ij}(g^R). \tag{1}$$

The generalized β-function is determined by the coefficient of a simple pole [3]

$$\beta_{ij}(g^R) = \left(\lambda \frac{\partial}{\partial\lambda} - 1\right) T^{(1)}_{ij}(\lambda g^R)\Big|_{\lambda=1}. \tag{2}$$

If one reconstructs the α' dependence of the counterterms

$$T^{(n)}_{ij}(g^R) = \sum_{L\geq n} (\alpha')^L\, T^{(L,n)}_{ij}(g^R), \tag{3}$$

then the β-function appears to be

$$\beta_{ij}(g^R) = \frac{\partial}{\partial\alpha'} T^{(1)}_{ij}(g^R). \tag{4}$$

In the one-loop order $T^{(1)}_{ij} \sim R_{ij}$, where R_{ij} is the Ricci tensor of the associated manifold. In higher loops the contributions to the β-function are proportional to the Riemann curvature tensor and do not vanish when restricted to the Ricci-flat manifolds. They modify the metric which can be found iteratively in the form

$$g^R_{ij} = g^{(0)}_{ij} + \alpha' g^{(1)}_{ij} + (\alpha')^2 g^{(2)}_{ij} + \cdots, \tag{5}$$

with $g^{(0)}_{ij}$ beeing Ricci-flat. Just the appearance of the correction to the metric $\sim(\alpha')^3$ in supersymmetric σ-models has driven to the conclusion that the Calabi–Yau compactification of superstring theory should be modified [8].

It is sometimes said [6] that conformal invariance, i.e. vanishing of β_{ij} is not the same as finiteness, i.e. vanishing of all $T_{ij}^{(n)}$. This is because the condition $\partial/\partial\alpha'\, T_{ij}^{(1)} = 0$ does not lead to $T_{ij}^{(1)} = 0$ and further on to $T_{ij}^{(n)} = 0$. The discrepancy comes from inaccurate formulation of the renormalization procedure in $2-2\varepsilon$ dimensions. The problem is solved by introducing the ε -dependence into eq. (5) in the following way [9,7]

$$g_{ij}^{R} = \sum_{n\geq 0} (\alpha')^n g_{ij}^{(n)}(\varepsilon), \quad g_{ij}^{(n)}(\varepsilon) = \sum_{k\geq 0} g_{ij}^{(n,k)}\, \varepsilon^k . \qquad (6)$$

This leads to a simultaneous vanishing of the β -function and the counterterms.

3. An alternative way to construct a finite σ -model is to reformulate eq. (6) for the bare metric. In full analogy with finite supersymmetric gauge theories [7] it appears to be more simple. The following theorem holds:

<u>Theorem</u> A two-dimensional nonlinear σ -model can be made finite by choosing the bare metric to be

$$g_{ij}^{B} = \sum_{k\geq 0} g_{ij}^{(0,k)}\, \varepsilon^k , \qquad (7)$$

where $g_{ij}^{(0,0)}$ is Ricci-flat.

<u>The proof</u>

Consider a singular regularized (but not renormalized) expression. (We do not care about the infrared singularities,

supposing that they are removed somehow). It has the form:

$$\sum_{n\geq 1} (\alpha')^n \left[\frac{\tilde{T}_{ij}^{(n,n)}}{\varepsilon^n} + \frac{\tilde{T}_{ij}^{(n,n-1)}}{\varepsilon^{n-1}} + \dots + \frac{\tilde{T}_{ij}^{(n,1)}}{\varepsilon} \right] . \qquad (8)$$

Coefficients $\tilde{T}_{ij}^{(n,k)}$ are uniquely connected with MS-counterterms of eq. (3). According to the generalized pole equations [1] higher-order poles are not independent but governed by a simple pole $\tilde{T}_{ij}^{(n,1)}$.

We are looking for finiteness, i.e. vanishing of all poles. Consider them order by order in α'.

<u>1 loop</u>

$$\frac{1}{\varepsilon}: \quad \tilde{T}_{ij}^{(1,1)} = -T_{ij}^{(1,1)} \sim R_{ij} \quad \Rightarrow \quad R_{ij}(g^{(0,0)}) = 0 .$$

<u>2 loops</u>

$\frac{1}{\varepsilon^2}: \quad \tilde{T}_{ij}^{(2,2)}(g^{(0,0)}) = 0$ because it is totally determined by $T_{ij}^{(1,1)}$. Otherwise, there would be a nonlocal divergence which cannot be removed by local counterterms. This is forbidden by the general structure of R-operation, as far as one-loop counterterms are absent.

$\frac{1}{\varepsilon}$: With account of eq. (7) we get

$$\tilde{T}_{ij}^{(2,1)}(g^{(0,0)}) + \frac{d}{d\eta}\tilde{T}_{ij}^{(2,2)}(g^{(0,0)} + \eta g^{(0,1)})\Big|_{\eta=0} = 0 . \qquad (9)$$

Hence even if $\tilde{T}_{ij}^{(2,1)}(g^{(0,0)}) \neq 0$ one can achieve finiteness properly choosing $g_{ij}^{(0,1)}$ provided $\tilde{T}_{ij}^{(2,2)}$ has a first order zero at $g_{ij} = g_{ij}^{(0,0)}$.

<u>n loops</u>

All higher-order poles have to vanish when $g_{ij}^{B} = \sum_{k=0}^{n-1} g_{ij}^{(0,k)}\varepsilon^k$ for the same reason as above. It is a consequence of the statement that all the infinities can be removed by local counter-terms within R-operation and of the absence of lower order counterterms. The appearance of ε -dependence in eq. (6) does not change the situation and results only in a modification of the renormalization scheme. In case of a finite theory eqs. (1),(6) correspond to a finite renormalization.

For a simple pole we have

$$\tilde{T}_{ij}^{(n,1)}(g^{(0,0)}) + \dots + \frac{d}{d\eta}\tilde{T}_{ij}^{(n,n)}(g^{(0,0)} + \eta\, g^{(0,n)})\Big|_{\eta=0} = 0. \qquad (10)$$

Thus again if $\tilde{T}_{ij}^{(n,n)}$ possesses a simple zero at $g_{ij} = g_{ij}^{(0,0)}$ one can achieve finiteness. To show this, we note that

$$\tilde{T}_{ij}^{(n,n)} = (-)^n\, T_{ij}^{(n,n)}$$

and $T_{ij}^{(n,n)}$ is governed by $T_{ij}^{(1,1)}$ and can be calculated using the pole equations. The result is

$$T_{ij}^{(n,n)} \sim (-)^n (\nabla^2)^{n-1} R_{ij} + \text{terms with a higher-order zero.}$$

Note that for the variation of the metric $g = g^{(0)} + \delta g$ we have the following variation of the Ricci tensor

$$\delta R_{ij} = \frac{1}{2}\left(\nabla^2 \delta g_{ij} + \nabla_i \nabla_j \delta g_m{}^m - \nabla_j \nabla^m \delta g_{im} - \nabla_i \nabla^m \delta g_{jm}\right), \qquad (11)$$

$$\delta R = \nabla^2 \delta g_i{}^i - \nabla^i \nabla^j \delta g_{ij} .$$

This completes our proof.

4. As examples illustrating our theorem we consider $N = 0$ and N=2 nonlinear supersymmetric sigma-models.

N=0 NLSM

$$S = \frac{1}{4\pi\alpha'}\int d^2 z\, g_{ij}(\varphi)\,\partial_\mu \varphi^i \partial_\mu \varphi^j, \quad i,j = 1,2,\dots,26 \qquad (12)$$

The counterterms are [6]

$$\begin{aligned} T_{ij}^{(1,1)} &= -R_{ij}, \\ T_{ij}^{(2,2)} &= -\tfrac{1}{4}\left[\nabla^2 R_{ij} - 2R_{ik}R_j{}^k + 2R_{ai\ell j}R^{a\ell}\right], \\ T_{ij}^{(2,1)} &= -\tfrac{1}{4} R_{iabc}R_j{}^{abc}, \\ \beta_{ij} &= -R_{ij} - \tfrac{\alpha'}{2} R_{iabc}R_j{}^{abc}. \end{aligned} \qquad (13)$$

Substituting eq. (6) into eqs. (1), (13) and requiring the vanising of all poles and the β -function, we get

$$\begin{aligned} & R_{ij}(g^{(0,0)}) = 0, \\ & \delta R_{ij}(g^{(1,0)}) + \tfrac{1}{2} R_{iabc}(g^{(0,0)}) R_j{}^{abc}(g^{(0,0)}) = 0, \\ & \nabla^2 \delta R_{ij}(g^{(0,1)}) + 2R_{ia\ell j}(g^{(0,0)})\,\delta R^{a\ell}(g^{(0,1)}) = R_{iabc}(g^{(0,0)}) R_j{}^{abc}(g^{(0,0)}). \end{aligned} \qquad (14)$$

The regularized expressions (8) in this case are

$$\begin{aligned} \tilde{T}_{ij}^{(1,1)} &= R_{ij}, \\ \tilde{T}_{ij}^{(2,2)} &= -\tfrac{1}{4}\left[\nabla^2 R_{ij} - 2R_{ik}R_j{}^k + 2R_{ai\ell j}R^{a\ell}\right], \\ \tilde{T}_{ij}^{(2,1)} &= \tfrac{1}{4} R_{iabc}R_j{}^{abc} \dots \end{aligned} \qquad (15)$$

Substituting into eq. (15) eq. (7) with $g_{ij}^{(0,0)}$ and $g_{ij}^{(0,1)}$ defined as in eq. (14) we confirm ourselves that all the poles identically vanish.

N=2 NLSM

$$S = \frac{1}{4\pi\alpha'}\int d^2 z\, d^2\theta\, d^2\bar{\theta}\, K(\Phi^\mu, \bar{\Phi}^{\bar{\nu}}). \qquad (16)$$

Here $K(\Phi,\bar{\Phi})$ is a Kähler potential and $g_{\mu\bar{\nu}} = \dfrac{\partial^2 K}{\partial\Phi^\mu \partial\bar{\Phi}^{\bar{\nu}}}$ is a Kähler metric.

The counterterms of N=2 NLSM are calculated up to five loops [8,10]. The β -function is

$$\beta_{\mu\bar\nu} = -R_{\mu\bar\nu} - \frac{2\zeta(3)}{3}(\alpha')^3\partial_\mu\partial_{\bar\nu}\Delta K - \frac{\zeta(4)}{4}(\alpha')^4\partial_\mu\partial_{\bar\nu}\nabla^2\Delta K, \quad (17)$$

where

$$\Delta K = R^{\mu}{}_{\nu}{}^{\tau}{}_{\lambda}\left(R^{\rho}{}_{\mu}{}^{\sigma}{}_{\tau}R^{\nu}{}_{\rho}{}^{\lambda}{}_{\sigma} + R^{\lambda}{}_{\tau}{}^{\rho}{}_{\sigma}R^{\sigma}{}_{\rho}{}^{\nu}{}_{\mu}\right).$$

It does not vanish on Ricci-flat manifolds thus leading to the corrections to the renormalized metric $\sim(\alpha')^3$ and $(\alpha')^4$. Note, however, that $(\alpha')^4$ term is renormalization-scheme-dependent.

The regularized expressions are

$$\begin{aligned}
\tilde K^{(1,1)} &= \mathrm{tr}\ln g_{\mu\bar\nu}, \\
\tilde K^{(2,2)} &= \tfrac{1}{2}R, \\
\tilde K^{(3,3)} &= \tfrac{1}{3!}\left[\nabla^2 R - R^{\mu\bar\nu}R_{\mu\bar\nu}\right], \\
\tilde K^{(4,4)} &= \tfrac{1}{4!}\left[(\nabla^2)^2R + 2R^{\rho\bar\rho}R_{\rho}{}^{\sigma}R_{\sigma\bar\rho} - \nabla^2(R^{\rho\bar\rho}R_{\rho\bar\rho}) - 3R^{\rho\bar\rho}\nabla_\rho\nabla_{\bar\rho}R\right], \\
\tilde K^{(4,1)} &= \tfrac{\zeta(3)}{4!}\,4\,\Delta K, \\
\tilde K^{(5,5)} &= \tfrac{1}{5!}\left[(\nabla^2)^3R + \dots\right], \quad \tilde K^{(5,2)} = \tfrac{\zeta(3)}{5!}\left[4\nabla^2\Delta K + \dots\right], \\
\tilde K^{(5,1)} &= \tfrac{\zeta(4)}{5!}\left[6\nabla^2\Delta K + \dots\right],
\end{aligned} \quad (18)$$

where the dots mean the terms that vanish on Ricci-flat manifolds.

Substituting eq. (7) into eq. (18) and requiring the vanishing of all poles we get corrections to the metric. Up to five loops the results are

$$g^B_{\mu\bar\nu} = g^{(0)}_{\mu\bar\nu} + \left[4\zeta(3)\varepsilon^3 + 6\zeta(4)\varepsilon^4\right]g^{(1)}_{\mu\bar\nu}, \quad (19)$$

where

$$\begin{aligned}
&R_{\mu\bar\nu}(g^{(0)}) = 0, \\
&\nabla^2\delta R(g^{(1)}) + \Delta K(g^{(0)}) = 0.
\end{aligned} \quad (20)$$

Note that the ε^3 correction automatically cancels the $1/\varepsilon^2$ term in eq. (18) according to the general theorem.

In n loops

$$\tilde K^{(n,n)} = \frac{1}{n!}\left[(\nabla^2)^{n-2}R + \dots\right]$$

and eq. (7) becomes

$$K^B = K^{(0)} + \dots + \varepsilon^n K^{(n)}, \quad (21)$$

where $K^{(n)}$ obeys the equation

$$(\nabla^2)^{n-2}\delta R(g^{(n)}) = F^{(n)}\Big|_{K^B = K^{(0)} + \dots + \varepsilon^{n-1}K^{(n-1)}} \quad (22)$$

Hence N=2 NLSM can be made finite in all orders in α' choosing the bare Kähler potential as in eq. (21) with corrections proportional to the powers of ε found from eq. (22).

5. We conclude that nonlinear sigma-models can be made finite by a proper choice of the bare metric. The physical meaning appears to be attached only to the classical metric which is Ricci-flat. This means that despite the appearance of nonzero β-functions in the standard approach one can construct a conformally invariant σ-model with a Ricci-flat metric. Thus, the superstring compactification on Calabi-Yau manifolds becomes possible.

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Казаков Д.И. E2-87-16
Конечность нелинейных сигма-моделей
на Риччи-плоских многообразиях

Показано, что нелинейные сигма-модели могут быть сделаны конечными в $2-2\epsilon$ измерениях, если затравочная метрика выбрана в виде $g_{ij}(\phi) = \sum_{k\geq 0} g^{(k)}_{ij} \epsilon^k$, причем, $g^{(o)}_{ij}$ – Риччи-плоская. Это означает, что несмотря на появление ненулевых β-функций в стандартном подходе, можно построить конформно-инвариантные сигма-модели с Риччи-плоской метрикой. Следовательно компактификация суперструны на многообразиях Калаби-Яо становится возможной.

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Kazakov D.I. E2-87-16
Finiteness of Nonlinear Sigma-Models
on Ricci-Flat Manifolds

It is shown that nonlinear sigma-models can be made finite in $2-2\epsilon$ dimensions if the background bare metric is chosen to be $g_{ij}(\phi) = \sum_{k\geq 0} g^{(k)}_{ij} \epsilon^k$, where $g^{(o)}_{ij}$ is Ricci-flat. This means that despite the appearance of non-zero β-functions in the standard approach, we can construct conformally invariant sigma-models with Ricci-flat metric. Thus the superstring compactification on Calabi-Yau manifolds becomes possible.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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