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ON A GENERALIZATION OF RENORMALIZATION GROUP EQUATIONS TO QUANTUM FIELD THEORIES OF AN ARBITRARY TYPE

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1. Traditionally, the renormalization group equations are connected with the property of multiplicative renormalizability in quantum field theory /1/. The procedure which removes the ultraviolet divergences known as the R-operation is equivalent to the introduction into the Lagrangian of some local counterterms. In renormalizable theories this results in a multiplicative renormalization of fields and couplings /1/. The renormalization procedure possesses the group structure where the continuous group parameter appears to be a normalization point λ, parameter of dimensional regularization μ, etc.

The renormalization group equations drastically restrict the renormalization arbitrariness. For instance, in any given order of perturbation theory only the lowest logarithmic divergences are independent and define the so-called β-functions. All higher divergences are uniquely determined from lower approximations. This property allows us , in particular, to perform a summation of higher logarithmic asymptotics and to predict the asymptotic behaviour of Green functions.

In the present note, we will show how the renormalization group equations can be generalized to the theories of a general type, including nonrenormalizable interactions. In spite of the absence of multiplicative renormalizability the obtained equations enable us to calculate all higher singularities (poles in the dimensional regularization) starting from the generalized β-functions like in renormalizable theories.

2. Remind first the standard procedure in renormalizable theories /2/. Further on we shall work in the framework of dimensional regularization. The R-operation in this case leads to the following expression for the "bare" coupling in terms of the renormalized one

$$g^{bare} = (\mu^2)^{\varepsilon} g Z_g = (\mu^2)^{\varepsilon}\Big[g + \sum_{n=1}^{\infty} \frac{a_n(g)}{\varepsilon^n} \Big], \quad (1)$$

where for definiteness we use the minimal subtraction scheme. Any change of the parameter $\mu \to \mu'$ is compensated by an appropriate change of the coupling $g \to g'$ so that g^{Bare} remains unchanged. Differentiating eq.(1) with respect to μ and introducing the definition

$$\mu \frac{d}{d\mu} g \Big|_{g^B} = -\varepsilon g + \beta(g) , \tag{2}$$

we get

$$0 = \varepsilon \Big[g + \sum_{n=1}^{\infty} \frac{a_n(g)}{\varepsilon^n} \Big] + \big[-\varepsilon g + \beta(g) \big] \Big[1 + \sum_{n=1}^{\infty} \frac{a_n'(g)}{\varepsilon^n} \Big] . \tag{3}$$

Equating the coefficients of powers of ε we have

$$\beta(g) = \Big(g \frac{\partial}{\partial g} - 1 \Big) a_1(g) , \tag{4}$$

$$\Big(g \frac{\partial}{\partial g} - 1 \Big) a_n(g) = \beta(g) \frac{\partial}{\partial g} a_{n-1}(g) . \tag{5}$$

Hence, knowing the coefficient of a simple pole $a_1(g)$, with the help of eq.(5) one can find all the coefficients $a_n(g)$. In particular, expanding them in a power series in g

$$a_n(g) = \sum_{k=n}^{\infty} a_{nk} \, g^{k+1} \tag{6}$$

we come to

$$\beta(g) = \sum_{k=1}^{\infty} a_{1k} \cdot k \cdot g^{k+1} ,$$

$$a_{nk} = \frac{1}{k} \sum_{\ell=n-1}^{k-1} a_{n-1\,\ell} \, (\ell+1)(k-\ell) \, a_{1\,k-\ell} ,$$

$$a_{nn} = a_{11}^{n} . \tag{7}$$

3. For the Lagrangian of a general type $\mathcal{L}$ the implementation of R -operation results in the counterterms added to the Lagrangian

$$\mathcal{L} \to \mathcal{L} + \Delta \mathcal{L} .$$

This can be described in a way like eq.(1)

$$\mathcal{L}^{Bare} = (\mu^2)^{\varepsilon} \Big[\mathcal{L} + \sum_{n=1}^{\infty} \frac{A_n(\mathcal{L})}{\varepsilon^n} \Big] , \tag{8}$$

where $A_n(\mathcal{L})$ means that the counterterms are calculated within the Lagrangian $\mathcal{L}$.

Proceeding further according to Sect.2, i.e. differentiating eq.(8) with respect to μ and introducing the definition

$$\mu \frac{d}{d\mu} \mathcal{L} \Big|_{\mathcal{L}^B} = -\varepsilon \mathcal{L} + \beta(\mathcal{L}) , \tag{9}$$

we get

$$0 = \varepsilon \Big[\mathcal{L} + \sum_{n=1}^{\infty} \frac{A_n(\mathcal{L})}{\varepsilon^n} \Big] + \big[-\varepsilon \mathcal{L} + \beta(\mathcal{L}) \big] \Big[1 + \sum_{n=1}^{\infty} \frac{\delta A_n(\mathcal{L})}{\delta \mathcal{L}} \frac{1}{\varepsilon^n} \Big] .$$

Formally, this leads to

$$\beta(\mathcal{L}) = \Big(\mathcal{L} \frac{\delta}{\delta \mathcal{L}} - 1 \Big) A_1(\mathcal{L}) , \tag{10}$$

$$\Big(\mathcal{L} \frac{\delta}{\delta \mathcal{L}} - 1 \Big) A_n(\mathcal{L}) = \beta(\mathcal{L}) \frac{\delta}{\delta \mathcal{L}} A_{n-1}(\mathcal{L}) . \tag{11}$$

The variational derivative should here be defined. For this purpose we note that in the loop expansion the counterterms $A_n(\mathcal{L})$ can be expressed like eq.(6)

$$A_n(\mathcal{L}) = \sum_{k=n}^{\infty} A_{nk}(\mathcal{L}) , \tag{12}$$

where $A_{nk}(\mathcal{L})$ are homogeneous functions of $\mathcal{L}$, i.e. the following relation holds

$$A_{nk}(\lambda \mathcal{L}) = \lambda^{k+1} A_{nk}(\mathcal{L}) . \tag{13}$$

With account of eqs. (12), (13), eqs. (10), (11) can be rewritten in the form (in case of nonlinear sigma-models this procedure has been used in refs. /3,4/):

$$\beta(\mathcal{L}) = \left(\lambda\frac{\partial}{\partial\lambda} - 1\right) A_1(\lambda\mathcal{L})\Big|_{\lambda=1} = \sum_{k=1}^{\infty} k\cdot A_{1k}(\mathcal{L}), \tag{14}$$

$$\left(\lambda\frac{\partial}{\partial\lambda} - 1\right) A_n(\lambda\mathcal{L})\Big|_{\lambda=1} = \frac{d}{d\eta} A_{n-1}(\mathcal{L} + \eta\beta(\mathcal{L}))\Big|_{\eta=0}. \tag{15}$$

For the highest poles this gives

$$n A_{nn}(\mathcal{L}) = \frac{d}{d\eta} A_{n-1\,n-1}(\mathcal{L} + \eta A_{11}(\mathcal{L}))\Big|_{\eta=0}. \tag{16}$$

It is obvious from eq.(16) that the coefficient function of the highest pole is totally defined by the one-loop approximation. However, contrary to eq.(7) A_{nn} is not merely an n-th power of A_{11} but contains also the derivatives.

There is a useful graphical interpretation of eq.(16). In the background field formalizm the one-loop counterterm for an arbitrary Lagrangian $\mathcal{L}(\varphi)$ can be obtained in a general form /5,6,7/ and is proportional to $(D = 4-2\varepsilon)$

$$A_{11} \sim \frac{\delta^2\mathcal{L}}{\delta\varphi^i\,\delta\varphi^j} \times \frac{\delta^2\mathcal{L}}{\delta\varphi^k\,\delta\varphi^l} \equiv A$$

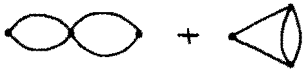
with a proper contraction of indices. Graphically, this can be represented as

$A_{11} = \mathcal{L} = \mathcal{L} =$  .

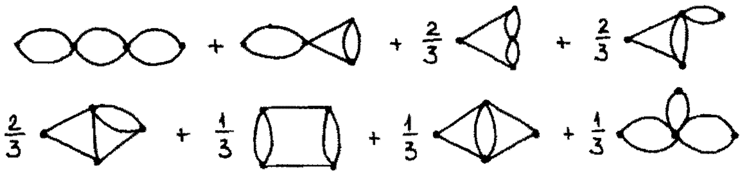
Then for A_{22} we get, according to eq.(16),

$A_{22} = \mathcal{L} = A = \mathcal{L} = \mathcal{L} = \mathcal{L} + \mathcal{L} <^{\mathcal{L}}_{\mathcal{L}}$

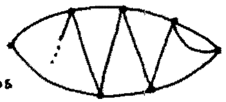
or

$A_{22} =$  .

Proceeding further, we have

$A_{33} =$ 

Clearly if $A_{11} = 0$,
i.e.  $= 0$, then all $A_{nn} = 0$. At the same time expanding eq.(16) up to n-th order we always have the diagram



n loops

which contains only one simple loop. That means that A_{nn} possesses only first order zero when $A_{11} = 0$. This fact is of great importance for the construction of finite field theories /8,9/.

4. Equations (14), (15) are the desired generalization of renormalization group equations (4), (5) to the theories of arbitrary type. In spite of a possible nonrenormalizability of a theory the counterterms happen to be connected with each other. The only independent coefficient functions are those of a simple pole. Eq.(15) appears to be very useful in various cases. In nonlinear sigma-models it was used in a refs./10,11,9/. It can be also applied to the calculation of higher counterterms in quantum gravity.

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Казаков Д.И. E2-87-132
Об одном обобщении уравнений ренормгруппы для квантово-полевых теорий произвольного вида

Дано обобщение уравнений ренормгруппы на теории с лагранжианом произвольного вида, включая неперенормируемые взаимодействия. В рамках размерной регуляризации полученные уравнения позволяют определять коэффициентные функции при старших полюсах, исходя из младшего полюса или обобщенных β-функций.

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Kazakov D.I. E2-87-132
On a Generalization of Renormalization Group Equations to Quantum Field Theories of an Arbitrary Type

A generalization of renormalization group equations to the theories with arbitrary Lagrangians including nonrenormalizable ones is presented. In the framework of dimensional regularization these equations enable us to determine the coefficient functions of higher poles starting from a simple pole or generalized β-functions.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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