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CONSTRAINTS ON EXPERIMENTAL OBSERVABLES OF THREE $(0\frac{1}{2} \to 0'\frac{1}{2}')$ REACTIONS RELATED BY UNITARY SYMMETRIES

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1. *Introduction*

It is well known that the internal symmetries of elementary particle interactions imply a large number of linear relations between the transition matrices of different reactions. The most usual reactions are going through two channels of isospin, U -spin, V -spin or full unitary spin such that the most frequent sum rules implied by unitary symmetries or quark models [1,2] are the triangular relationships. The isospin triangular inequalities for differential (polarized or unpolarized) cross-sections have been investigated by many authors[3-13]. But the isospin constraints on the differential observables of $(0\,1/2 \to 0'\,1'/2)$ reactions, which are more stronger than the usual triangular inequalities, were derived by Doncel et al. [4] and recently in ref. [5,6]. So, a remarkable equality [see eq. (4a), this paper] and the bound $4H < \lambda(\sigma)$ [see our definitions (3a,b) and (4a)] was obtained in ref.[4] while other equalities and the lower bounds on H have been proved in ref. [5,6] / see eqs. (24) and (25) from ref. [5] / . The isospin constraints have been derived [5,6] using a set of bilinear forms which can be constructed from the scattering amplitudes of two charge (or s , t , u -isospin)-channels. This form of presentation of the isospin constraints on differential observables has an advantage that the exact saturation of the bounds can be obtained in terms of the zero-trajectories of the imaginary and real parts of these bilinear form, or equivalently, in terms of $[\,n\pi, (n+1/2)\pi\,]$ phase contours. Therefore, the analysis of isospin bounds helps to locate the zeros of certain transition amplitudes [7] and to obtain strong constraints on the experimental data and amplitude analysis [6] when these bounds are exactly saturated or degenerated.

The purpose of this paper is to present a general method for derivation of all the constraints on the (differential and integrated) experimental observables of three ($0\,1/2 \to 0'\,1'/2$) reactions related by internal symmetries. So, in sect. 2, using the generalized amplitudes defined by eq. (6a) and the bilinear forms (6a,b,c,d) we prove that a general linear relation (2) alone implies the equalities (4a,b,c,d,e) and the bounds (5a,b,c,d,e) valid for any unit vector $\vec{\kappa}$ in any spin reference frame at any energies and scattering angles. All the constraints on experimental data and amplitude analyses, when the bounds are exactly saturated, as well as a hierarchy of the bounds are given in the table. In sect. 3, we have proved that the sum rule (2) alone implies that the equalities (4a) and each of the bounds listed in the table have an integrated analog. These results improve in the most general form all the constraints previously obtained /4,5,6/ .

2. *Constraints on Differential Cross-Sections and on Polarization Projections*

In order to obtain a unified treatment of all experimental consequences resulting from different triangular relationships, such as those derived from different internal symmetries [isospin invariance, SU(3) -symmetry /1/ quark models /2/], we start with the following definitions. Let T_k be the transition matrices for three ($0\,½ \to 0'\,½$) reactions written in the form:

$$T_k = f_k + i\vec{\sigma}\cdot\vec{n}\, g_k , \qquad (1)$$

where f_k and g_k are the non-spin-flip and spin-flip scattering amplitudes, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices and $\vec{n}$ is the unit vector normal to the scattering plane.

Let us assume that the transition matrices T_k satisfy the sum rule

$$\sum_{k=1}^{3} c_k T_k = 0 , \qquad (2)$$

where the c_k coefficients are real numbers [see ref./1,2/] .

Let us define

$$\lambda(x,y,z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz , \qquad (3a)$$

$$\lambda(\sigma) \equiv \lambda[\, c_1^2 \sigma_1 , c_2^2 \sigma_2 , c_3^2 \sigma_3 \,] , \qquad (3b)$$

$$\lambda(\vec{\kappa}\cdot\vec{P}\sigma) \equiv \lambda[\, c_1^2 \vec{\kappa}\cdot\vec{P}_1 \sigma_1 , c_2^2 \vec{\kappa}\cdot\vec{P}_2 \sigma_2 , c_3^2 \vec{\kappa}\cdot\vec{P}_3 \sigma_3 \,] , \qquad (3c)$$

$$\lambda_{\kappa}^{(\pm)} \equiv \lambda[\, c_1^2 (1 \pm \vec{\kappa}\cdot\vec{P}_1)\sigma_1 , c_2^2 (1 \pm \vec{\kappa}\cdot\vec{P}_2)\sigma_2 , c_3^2 (1 \pm \vec{\kappa}\cdot\vec{P}_3)\sigma_3 \,] , \qquad (3d)$$

$$H_{ij} \equiv \frac{1}{2}(1 - \vec{P}_i\cdot\vec{P}_j)\sigma_i \sigma_j , \qquad (3e)$$

where $\vec{\kappa}$ is an arbitrary unit vector, σ_k and $\vec{P}_k$ are the unpolarized differential cross-sections and the spin rotation vectors, respectively.

The sum rule (2) alone implies the following set of equalities

$$H \equiv c_1^2 c_2^2 H_{12} = c_2^2 c_3^2 H_{23} = c_3^2 c_1^2 H_{31} , \qquad (4a)$$

$$\frac{1}{4} | \lambda_{\kappa}^{(+)} - \lambda_{\kappa}^{(-)} | = [-4H - \lambda(\sigma)]^{½} [4H - \lambda(\vec{\kappa}\cdot\vec{P}\sigma)]^{½} , \qquad (4b)$$

$$|2H + \frac{1}{4}\lambda(\sigma) - \frac{1}{4}\lambda(\vec{\kappa}\cdot\vec{P}\sigma)| = [-\frac{1}{4}\lambda_{\kappa}^{(+)}]^{½} [-\frac{1}{4}\lambda_{\kappa}^{(-)}]^{½} , \qquad (4c)$$

$$|2H + \frac{1}{4}\lambda(\sigma) - \frac{1}{4}\lambda(\vec{\kappa}\cdot\vec{P}\sigma) + \frac{1}{4}\lambda_{\kappa}^{(\pm)}| = |-4H - \lambda(\sigma)|^{½} [-\frac{1}{4}\lambda_{\kappa}^{(\pm)}]^{½} , \qquad (4d)$$

$$|2H + \frac{1}{4}\lambda(\sigma) - \frac{1}{4}\lambda(\vec{\kappa}\cdot\vec{P}\sigma) - \frac{1}{4}\lambda^{(\pm)}| = [4H - \lambda(\vec{\kappa}\cdot\vec{P}\sigma)]^{½} [-\frac{1}{4}\lambda^{(\pm)}]^{½} , \qquad (4e)$$

(not all are independent) and the following set of inequalities

$$0 \leq -\frac{1}{4}\lambda_{\kappa}^{(\pm)} < \min_{(ij)} \{ c_i^2 c_j^2 (1 \pm \vec{\kappa}\cdot\vec{P}_i)(1 \pm \vec{\kappa}\cdot\vec{P}_j)\sigma_i \sigma_j \} , \tag{5a}$$

$$\max \{ -c_i^2 c_j^2 (\vec{\kappa}\cdot\vec{P}_j)(\vec{\kappa}\cdot\vec{P}_j)\sigma_i \sigma_j \} \leq \frac{1}{4}\lambda(\vec{\kappa}\cdot\vec{P}\sigma) \leq H , \tag{5b}$$

$$H \leq -\frac{1}{4}\lambda(\sigma) < \min_{(ij)} \{ c_i^2 c_j^2 \sigma_i \sigma_j \} , \tag{5c}$$

$$\Omega_{\kappa}^{(-)} \leq H \leq \Omega_{\kappa}^{(+)} , \tag{5d}$$

$$\Omega_{\kappa}^{(\pm)} = \frac{1}{3}\left(\begin{matrix}\min\\ \max\end{matrix}\right)\{[1-(\vec{\kappa}\cdot\vec{P}_i)(\vec{\kappa}\cdot\vec{P}_j) \pm [1-(\vec{\kappa}\cdot\vec{P}_i)^2]^{1/2} \times \tag{5e}$$

$$\times [1-(\vec{\kappa}\cdot\vec{P}_j)^2]^{1/2}] c_i^2 c_j^2 \sigma_i \sigma_j \} ,$$

valid for any $\vec{\kappa}$ and $\vec{P}$ in any spin reference frame at any energy and scattering angle.

There are several ways of demonstrating these results. An interesting proof, which is in particular connected with the "contours" of the relative phases of the scattering amplitudes /6/ , can be obtained using the following combinations of the scattering amplitudes:

$$F_k^{(+\kappa)} \equiv \frac{\sqrt{2}}{[1+|w|^2]^{1/2}}[f_k + w g_k], \quad F_k^{(-\kappa)} = \frac{\sqrt{2}}{[1+|w|^2]^{1/2}}[-w^* f_k + g_k], \tag{6a}$$

where w is an arbitrary complex number, and the following bilinear forms

$$M_{ij}^{(\pm\kappa)} = [F_i^{(\pm\kappa)}]^* F_j^{(\pm\kappa)} \tag{6b}$$

$$Z_{ij}^{(0)} = \frac{1}{2}[M_{ij}^{(+\kappa)} + M_{ij}^{(-\kappa)}], \quad Z_{ij}^{(\kappa)} = \frac{1}{2}[M_{ij}^{(+\kappa)} - M_{ij}^{(-\kappa)}] , \tag{6c}$$

$$Y_{ij}^{(0)} = \frac{1}{2}[F_i^{(+\kappa)} F_j^{(-\kappa)} - F_i^{(-\kappa)} F_j^{(+\kappa)}] , \tag{6d}$$

since

$$M_{kk}^{(\pm\kappa)} = (1 \pm \vec{\kappa}\vec{P}_k)\sigma_k , \tag{7a}$$

$$|M_{ij}^{(\pm\kappa)}|^2 = M_{ii}^{(\pm\kappa)} M_{jj}^{(\pm\kappa)} , \tag{7b}$$

$$|Z_{ij}^{(0)}|^2 = \frac{1}{2}(1 + \vec{P}_i \vec{P}_j)\sigma_i \sigma_j , \quad Z_{ii}^{(0)} = \sigma_i , \tag{7c}$$

$$|Z_{ij}^{(\kappa)}|^2 = H_{ij} + (\vec{\kappa}\vec{P}_i)(\vec{\kappa}\vec{P}_j)\sigma_i \sigma_j , \quad Z_{ii}^{(\kappa)} = \vec{\kappa}\vec{P}_i \sigma_i \tag{7d}$$

$$|Y_{ij}^{(0)}|^2 = H_{ij} , \quad Y_{ii}^{(0)} = 0 , \tag{7e}$$

where $\vec{\kappa}$ and $\vec{P}$ are defined as

$$\vec{\kappa} \equiv \left\{ \frac{2\,\mathrm{Im}\,w}{1+|w|^2} , \frac{2\,\mathrm{Re}\,w}{1+|w|^2} , \frac{1-|w|^2}{1+|w|^2} \right\} \tag{7f}$$

$$\vec{P}_k \equiv \left\{ \frac{2\,\mathrm{Im}(f_k g_k^*)}{\sigma_k} , \frac{(2\,\mathrm{Re}(f_k^* g_k)}{\sigma_k} , \frac{|f_k|^2 - |g_k|^2}{\sigma_k} \right\} . \tag{7g}$$

Therefore, since the sum rule (2) is equivalent to

$$\sum_{k=1}^{3} c_k F_k^{(\pm\kappa)} = 0 \tag{8a}$$

and also to

$$c_1 c_2 Y_{12}^{(0)} = c_2 c_3 Y_{23}^{(0)} = c_3 c_1 Y_{31}^{(0)}, \quad (8b)$$

we obtain directly the equalities (4a) [see (7e)] . Then, by straightforward calculus, using (8a), (7b,c,d) and (6b,c) we get

$$\mathrm{Re}N_{ij} = (2c_i c_j)^{-1} [c_k^2 N_{kk} - c_i^2 N_{ii} - c_j^2 N_{jj}], \quad (9a)$$

$$N_{ij} \equiv M_{ij}^{(\pm\kappa)}, Z_{ij}^{(0)}, Z_{ij}^{(\kappa)},$$

$$(9b)$$

$$c_i^2 c_j^2 [\mathrm{Im} M_{ij}^{(\pm\kappa)}]^2 = -\frac{1}{4}\lambda_\kappa^{(\pm)} = \{[-H - \frac{1}{4}\lambda(\sigma)]^{1/2} \pm \eta_\kappa [H - \frac{1}{4}\lambda(\vec{\kappa}\cdot\vec{P}\sigma)]^{1/2}\}^2,$$

$$(9c)$$

$$c_i^2 c_j^2 [\mathrm{Im} Z_{ij}^{(0)}]^2 = -H - \frac{1}{4}\lambda(\sigma) = \frac{1}{4}\{[-\frac{1}{4}\lambda_\kappa^{(+)}]^{1/2} + \epsilon_\kappa [-\frac{1}{4}\lambda_\kappa^{(-)}]^{1/2}\}^2,$$

$$(9d)$$

$$c_i^2 c_j^2 [\mathrm{Im} Z_{ij}^{(\kappa)}]^2 = H - \frac{1}{4}\lambda(\vec{\kappa}\cdot\vec{P}\sigma) = \frac{1}{4}\{[-\frac{1}{4}\lambda_\kappa^{(+)}]^{1/2} - \epsilon_\kappa [-\frac{1}{4}\lambda_\kappa^{(-)}]^{1/2}\}^2,$$

where η_κ and ϵ_κ are defined as

$$\eta_\kappa \equiv \mathrm{sign}[\mathrm{Im} Z_{ij}^{(0)} \mathrm{Im} Z_{ij}^{(\kappa)}] = \mathrm{sign}\{-\lambda_\kappa^{(+)} + \lambda_\kappa^{(-)}\}, \quad (9e)$$

$$\epsilon_\kappa \equiv \mathrm{sign}[\mathrm{Im} M_{ij}^{(+\kappa)} \mathrm{Im} M_{ij}^{(-\kappa)}] = \mathrm{sign}\{-8H - \lambda(\sigma) + \lambda(\vec{\kappa}\cdot\vec{P}\sigma)\}. \quad (9f)$$

We note that the equalities (9a) are equivalent to

$$c_i^2 c_j^2 [(\mathrm{Re} N_{ij})^2 - N_{ii}N_{jj}] = \frac{1}{4}\lambda[c_1^2 N_{11}, c_2^2 N_{22}, c_3^2 N_{33}]. \quad (9a')$$

Therefore, from (9a',b,c,d,) and the identity

$$\lambda_\kappa^{(+)} + \lambda_\kappa^{(-)} = 2\lambda(\sigma) + 2\lambda(\vec{\kappa}\cdot\vec{P}\sigma), \quad (10)$$

we obtain the results (4a,b,c,d,e) and (5a,b,c), while the bounds (5d) are derived from the triangle inequalities applied to the bilinear form $Y_{ij}^{(0)}$ defined by eq. (6d). Now, from the above results, also we obtain the following interesting consequences:

(i) *The lower and upper bounds (5a,b,c) are exactly saturated on the zero trajectories of* $\mathrm{Im}N_{ij}$ *and* $\mathrm{Re}N_{ij}$, $N_{ij} \equiv M_{ij}^{(\pm\kappa)}$, $Z_{ij}^{(\kappa)}$, $Z_{ij}^{(0)}$ *respectively. The bounds (5a,b,c) are degenerated if and only if* $|M_{ij}^{(\pm\kappa)}| = 0$, $|Z_{ij}^{(\kappa)}| = 0$, $|Z_{ij}^{(0)}| = 0$, *respectively. The signs* η_κ , ϵ_κ *and the zero trajectories of* $\mathrm{Im}N_{ij}$ *are independent of channel indices* ij *[see (9b,c,d,e,f)] .*

(ii) *If* $\Lambda_\kappa(\eta_\kappa)$ *and* $E_\kappa(\epsilon_\kappa)$ *are the regions from the physical domain where* η_κ *and* ϵ_κ */defined by (9e,f)/ respectively take a constant value* (±) *then the bounds* $-\lambda_\kappa^{(+)} \geq 0$ *in* $\Lambda_\kappa(-)$ *and* $-\lambda_\kappa^{(-)} \geq 0$ *in* $\Lambda_\kappa(+)$ *are equivalent to the bound*

$$2[-4H - \lambda(\sigma)]^{1/2} [4H - \lambda(\vec{\kappa}\cdot\vec{P}\sigma)]^{1/2} \leq -\lambda(\sigma) - \lambda(\vec{\kappa}\cdot\vec{P}\sigma) \quad (11a)$$

while the bound $4H \leq -\lambda(\sigma)$ *in* $E_\kappa(-)$ *and the bound* $4H \geq -\lambda(\vec{\kappa}\cdot\vec{P}\sigma)$ *in* $E_\kappa(+)$ *regions are both equivalent to the bound*

$$[-\lambda_\kappa^{(+)}]^{1/2} [-\lambda_\kappa^{(-)}]^{1/2} \leq -\lambda(\sigma) - \lambda(\vec{\kappa}\cdot\vec{P}\sigma). \quad (11b)$$

The bounds $-\lambda_\kappa^{(+)} \geq 0, -\lambda_\kappa^{(-)} \geq 0$, $4H \leq -\lambda(\sigma), \lambda(\vec{\kappa}\cdot\vec{P}\sigma) \leq 4H$ *in the complementary regions* $\Lambda_\kappa(+)$, $\Lambda_\kappa(-)$, $E_\kappa(+)$, $E_\kappa(-)$, , *respectibely, are all weaker than the bound*

$$\lambda(\vec{\kappa}\cdot\vec{P}\sigma) \leq -\lambda(\sigma). \quad (11c)$$

These statements can be proved observing that eq. (9b,c,d) and (10) imply:

$$(12a)$$

$$-\lambda_\kappa^{(\pm)} = -\lambda(\sigma) - \lambda(\vec{\kappa}\cdot\vec{P}\sigma) \pm 2\eta_\kappa [-4H - \lambda(\sigma)]^{1/2} [4H - \lambda(\vec{\kappa}\cdot\vec{P}\sigma)]^{1/2},$$

$$H = -\frac{1}{8}\lambda(\sigma) + \frac{1}{8}\lambda(\vec{\kappa}\cdot\vec{P}\sigma) - \frac{\epsilon_\kappa}{8}[-\lambda_\kappa^{(+)}]^{1/2}[-\lambda_\kappa^{(-)}]^{1/2}. \quad (12b)$$

We note that the regions $\Lambda_\kappa(+)$ are separated from $\Lambda_\kappa(-)$ by the zero trajectories of $\mathrm{Im}\, Z_{ij}^{(0)}$ and $\mathrm{Im}\, Z_{ij}^{(\kappa)}$ while the regions $E_\kappa(+)$ and $E_\kappa(-)$ are separated by the zero trajectories of $\mathrm{Im} M_{ij}^{(\pm\kappa)}$.

(iii) *The exact saturation of one of the bounds* $-\lambda_\kappa^{(\pm)} \geq 0$, $4H \leq -\lambda(\sigma)$ *and* $\lambda(\vec{\kappa}\cdot\vec{P}\sigma) \leq 4H$, *implies a relation of equivalence for the other three bounds.* Indeed, from (12a,b) (10) and (9e,f) we obtain

$$\lambda_\kappa^{(\pm)} = 0 \rightarrow -4H - \lambda(\sigma) = 4H - \lambda(\vec{\kappa}\cdot\vec{P}\sigma) = -\frac{1}{4}\lambda_\kappa^{(\mp)}, \tag{13a}$$

$$4H = -\lambda(\sigma) \rightarrow 4H - \lambda(\vec{\kappa}\cdot\vec{P}\sigma) = -\lambda_\kappa^{(+)} = -\lambda_\kappa^{(-)}, \tag{13b}$$

for all $\vec{\kappa}$, and

$$4H = \lambda(\vec{\kappa}\cdot\vec{P}\sigma) \rightarrow -4H - \lambda(\sigma) = -\lambda_\kappa^{(+)} = -\lambda_\kappa^{(-)}. \tag{13c}$$

The constraints on the experimental data and on the amplitude analysis, when the bounds are exactly saturated, and a hierarchy of the bounds are given in the table. We note, of course, that the bounds of class III are more stringent that the bounds of the classes II and I, respectively.

(iv) *The exact saturation of the upper bound (5b) and (5c) /or of the lower bounds (5b,c)/ simultaneously implies the mirror symmetry:* $\vec{\kappa}\cdot\vec{P}_i = -\vec{\kappa}\cdot\vec{P}_j$. These consequences are obtained expliciting the constraints: $\lambda_\kappa^{(+)} = \lambda_\kappa^{(-)}$ and

$$c_k^2 N_{kk} = c_i^2 N_{ii} + c_j^2 N_{jj}, \quad N_{ij} = Z_{ij}^{(0)},\ Z_{ij}^{(\kappa)}.$$

The above results improve, in the most general form, the constraints previously obtained [4-5] and are sufficient to obtain any constraints on differential observables by specializing the unit vectors $\vec{\kappa}$ in a given spin reference frame. An interesting result is obtained for $\vec{\kappa} = \vec{P}_i \times \vec{P}_j / |\vec{P}_i \times \vec{P}_j|$. For example, from (5b,c) we obtain:

$$\max_{(i\neq j\neq k)} \{ c_k^4 \frac{\sigma_k^2}{|\vec{P}_i \times \vec{P}_j|^2} [\vec{P}_k\cdot(\vec{P}_i\times\vec{P}_j)]^2 \} \leq 4H \leq -\lambda(\sigma). \tag{14a}$$

The lower bound (14a) is exactly saturated on the zero trajectories of $\mathrm{Re}\, Z_{ij}^{(0)}$ while the upper bound (14a) on the zero trajectories of $\mathrm{Im}\, Z_{ij}^{(0)}$. Therefore, the bounds (14a) are degenerated if and only if $|Z_{ij}^{(0)}| = 0$. We note that the lower bound (14a) is equivalent to

$$c_1^2 c_2^2 c_3^2 \sigma_1\sigma_2\sigma_3 |\vec{P}_1\cdot(\vec{P}_2\times\vec{P}_3)| \leq 4H \min_{(ij)} \{[c_i^2 c_j^2 \sigma_i \sigma_j - H]^{1/2}\}. \tag{14b}$$

Finally, we remark that a large class of inequalities can be derived using Young's inequality [14]

$$ab \leq \int_0^a \Phi(x)\,dx + \int_0^b \Phi^{-1}(y)\,dy, \tag{15}$$

where $y = \Phi(x)$ is any continuous strictly increasing function of x for $x \geq 0$ with $\Phi(0) = 0$ and $\Phi^{-1}(y)$ is the function to $\Phi(x)$, a and b are combinations of $-\lambda_\kappa^{(\pm)}$, $\lambda(\sigma)$, , H, $\lambda(\vec{\kappa}\cdot\vec{P}\sigma)$ such that $a \geq 0$, $b \geq 0$. The sign of equality in (15) holds if and only if $b = \Phi(a)$.

3. *Constraints on Integrated Cross-Sections and on Average Polarizations*

In this section we derive all the bounds on integrated cross-sections and on average polarization projections which are implied by the sum rule (2) alone. More concretely we shall prove that the sum rule (2) alone implies that the equalities (4a) and each bound listed in the table have an "integrated" analog.

For this we start with the integrals

$$I_p[F] = [\int_D |F|^p d\mu]^{1/p} \tag{16}$$

defined for any $1 < p < +\infty$ when F are the functions $F^{(\pm\kappa)}$ defined by (6a), D is a region from the physical domain and μ is a positive measure defined on the physical domain. Now, using the properties of the integrals $I_p[F]$ (see ref. [6]) and the sum rules (8a) we obtain:

$$|c_i| I_p[F_i^{(\pm\kappa)}] \leq |c_j| I_p[F_j^{(\pm\kappa)}] + |c_k| I_p[F_k^{(\pm\kappa)}] \qquad (17)$$

valid for any $1<p<+\infty$ any $\vec{\kappa}$ and $i \neq j \neq k = 1,2,3$. . Therefore, if we define the *generalized integrated cross-sections:*

$$\bar{\Sigma}_\ell^{(\pm\kappa n)} \equiv \{ \int_D [\,|F_\ell^{(\pm\kappa n)}|^2\,]^n \, d\mu \}^{1/n} \quad \frac{1}{2} < n < +\infty \qquad (18)$$

and if we choose $p = 2n$ from the inequalities (17) we obtain

$$|c_i| [\bar{\Sigma}_i^{(\pm\kappa n)}]^{1/2} \leq |c_j| [\bar{\Sigma}_j^{(\pm\kappa n)}]^{1/2} + |c_k| [\bar{\Sigma}_k^{(\pm\kappa n)}]^{1/2} \qquad (19)$$

valid for any n, any $\vec{\kappa}$ and $i \neq j \neq k = 1,2,3$. Now, let us define

$$\bar{\sigma}_\ell = \int_D \sigma_\ell \, d\mu, \quad \vec{\bar{P}}_\ell = \frac{1}{\bar{\sigma}_\ell} \int_D \vec{P}_\ell \, \sigma_\ell \, d\mu, \qquad (20a)$$

$$\lambda(\bar{\sigma}) \equiv \lambda(c_1^2 \bar{\sigma}_1, c_2^2 \bar{\sigma}_2, c_3^2 \bar{\sigma}_3),$$

$$(20b)$$

$$\lambda(\vec{\kappa} \cdot \vec{\bar{P}} \bar{\sigma}) \equiv \lambda[c_1^2 \vec{\kappa} \cdot \vec{\bar{P}}_1 \bar{\sigma}_1, c_2^2 \vec{\kappa} \cdot \vec{\bar{P}}_2 \bar{\sigma}_2, c_3^2 \vec{\kappa} \cdot \vec{\bar{P}}_3 \bar{\sigma}_3],$$

$$(20c)$$

$$\lambda_{\vec{\kappa}}^{(\pm)} \equiv \lambda[c_1^2(1 \pm \vec{\kappa} \cdot \vec{\bar{P}}_1) \bar{\sigma}_1, c_2^2(1 \pm \vec{\kappa} \cdot \vec{\bar{P}}_2) \bar{\sigma}_2, c_3^2(1 \pm \vec{\kappa} \cdot \vec{\bar{P}}_3) \bar{\sigma}_3],$$

$$\lambda_{\vec{\kappa}}^{(\pm n)} \equiv \lambda[c_1^2 \bar{\Sigma}_1^{(\pm\kappa n)}, c_2^2 \bar{\Sigma}_2^{(\pm\kappa n)}, c_3^2 \bar{\Sigma}_3^{(\pm\kappa n)}], \qquad (20d)$$

$$\bar{H}_{ij} = \frac{1}{2}(1 - \vec{\bar{P}}_i \vec{\bar{P}}_j) \bar{\sigma}_i \bar{\sigma}_j . \qquad (20e)$$

Then, we can prove that the inequalities (19) alone imply

$$\bar{H} \equiv c_1^2 c_2^2 \bar{H}_{12} = c_2^2 c_3^2 \bar{H}_{23} = c_3^2 c_1^2 \bar{H}_{31}, \qquad (21)$$

$$0 \leq -\frac{1}{4} \lambda_{\vec{\kappa}}^{(\pm)} \leq \min_{(ij)} \{ c_i^2 c_j^2 (1 \pm \vec{\kappa} \cdot \vec{\bar{P}}_i)(1 \pm \vec{\kappa} \cdot \vec{\bar{P}}_j) \bar{\sigma}_i \bar{\sigma}_j \}, \qquad (22a)$$

$$\max_{(ij)} \{ -c_i^2 c_j^2 (\vec{\kappa} \cdot \vec{\bar{P}}_i)(\vec{\kappa} \cdot \vec{\bar{P}}_j) \bar{\sigma}_i \bar{\sigma}_j \} \leq \frac{1}{4} \lambda(\vec{\kappa} \cdot \vec{\bar{P}} \bar{\sigma}) \leq \bar{H}, \qquad (22b)$$

$$\bar{H} \leq -\lambda(\bar{\sigma}) \leq \min_{(ij)} \{ c_i^2 c_j^2 \bar{\sigma}_i \bar{\sigma}_j \}, \qquad (22c)$$

$$\bar{\Omega}_\kappa^{(-)} \leq \bar{H} \leq \bar{\Omega}_\kappa^{(+)} \qquad (22d)$$

(for all $\vec{\kappa}$),

where $\bar{\Omega}_\kappa^{(\pm)}$ are defined by the relations (5e) and by the substitutions $\sigma \to \bar{\sigma}$, $\vec{P} \to \vec{\bar{P}}$, $\ell = i, j$. Indeed, since the inequalities (19) are equivalent to

$$0 \leq -\frac{1}{4} \lambda_{\vec{\kappa}}^{(\pm n)} \leq \min_{(ij)} \{ c_i^2 c_j^2 \bar{\Sigma}_i^{(\pm\kappa n)} \bar{\Sigma}_j^{(\pm\kappa n)} \} \qquad (23)$$

such that taking $n = 1$ we obtain the bounds (22a). We note that the upper bounds (23) are obtained from the definition (3a) and the lower bound (23), since according to (3a), we can write

$$\lambda(x,y,z) = (x-y-z)^2 - 4yz = (y-x-z)^2 - 4xz = (z-x-y)^2 - 4xy.$$

Next, using the bounds (22a) for $\vec{\kappa} \equiv \vec{\bar{P}}_i$, $i = 1,2,3$, we obtain

$$-\lambda_{\bar{P}_1}^{(-)} \equiv -\lambda[0, c_2^2(1 - \vec{\bar{P}}_1 \cdot \vec{\bar{P}}_2) \bar{\sigma}_2, c_3^2(1 - \vec{\bar{P}}_1 \cdot \vec{\bar{P}}_3) \bar{\sigma}_3] = 0, \qquad (24a)$$

$$-\lambda_{\bar{P}_2}^{(-)} \equiv -\lambda[c_1^2(1 - \vec{\bar{P}}_1 \cdot \vec{\bar{P}}_2) \bar{\sigma}, 0, c_3^2(1 - \vec{\bar{P}}_2 \cdot \vec{\bar{P}}_3) \bar{\sigma}_3] = 0, \qquad (24b)$$

which imply the equalities (21), and

$$\lambda^{(+)}_{\bar{P}_i} \equiv \lambda[c_1^2(1+\vec{\bar{P}}_i\cdot\vec{\bar{P}}_1)\bar{\sigma}_1, c_2^2(1+\vec{\bar{P}}_i\cdot\vec{\bar{P}}_2)\bar{\sigma}_2, c_3^2(1+\vec{\bar{P}}_i\cdot\vec{\bar{P}}_3)\bar{\sigma}_3] =$$

$$= 4[\lambda(\bar{\sigma}) + 4\bar{H}] \leq 0 \qquad (24c)$$

from which we prove the bounds (22c).

In a similar way, using the lower bound (22a) for $\vec{\kappa}' \equiv 2(\vec{\kappa}\cdot\vec{\bar{P}}_i)\vec{\kappa} - \vec{\bar{P}}_i$

we get

$$\lambda^{(+)}_{\bar{\kappa}'} = 4(\vec{\kappa}\cdot\vec{\bar{P}}_i)^2\{\lambda(\vec{\kappa}\cdot\vec{\bar{P}}\bar{\sigma}) - 4\bar{H}\} \leq 0 \qquad (24d)$$

which implies the bounds (22b).

The inequalities (22d) can be proved using the definitions

$$\bar{\sigma}_\ell = |\bar{f}_\ell|^2 + |\bar{g}_\ell|^2, \ \bar{\sigma}_\ell\bar{P}_\ell = \{2\mathrm{Im}(\bar{f}_\ell\bar{g}_\ell^*), 2\mathrm{Re}(\bar{f}_\ell\bar{g}_\ell^*), |\bar{f}_\ell|^2 - |\bar{g}_\ell|^2\}, \qquad (25)$$

and $\bar{F}_\ell^{(\pm\kappa)}$ and $\bar{Y}_{\ell m}^{(0)}$ defined by (6a,d) and by the substitutions: $f_\ell \to \bar{f}_\ell$, , $g_\ell \to \bar{g}_\ell$. Then, we obtain the inequalities:

$$\frac{1}{4}\{|\bar{F}_i^{(+\kappa)}||\bar{F}_j^{(-\kappa)}| - |\bar{F}_i^{(-\kappa)}||\bar{F}_j^{(+\kappa)}|\}^2 \leq |\bar{Y}_{ij}^{(0)}|^2 \leq \qquad (26a)$$

$$\leq \frac{1}{4}\{|\bar{F}_i^{(+\kappa)}||\bar{F}_j^{(-\kappa)}| + |\bar{F}_i^{(-\kappa)}||\bar{F}_j^{(+\kappa)}|\}^2,$$

which imply the bounds (22d) since

$$c_i^2 c_j^2 |\bar{Y}_{ij}^{(0)}|^2 = \bar{H}, \ |\bar{F}_\ell^{(\pm\kappa)}|^2 = (1 \pm \vec{\kappa}\cdot\vec{\bar{P}}_\ell)\bar{\sigma}_\ell . \qquad (26b)$$

Now, let us consider Young's inequality (15) for $y = x^{p-1}$, then, we can write

$$a^{1/p}\, b^{1/q} < \frac{a}{p} + \frac{b}{q}, \qquad (27)$$

for any $a \geq 0$, $b \geq 0$, $1 < p < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$. The sign of equality holds in (27) if and only if $a = b$. Therefore, specializing a, b and p, from (27), (22a,b,c,d) and (23) we obtain a number of interesting inequalities. For example, for $p = q = 2$ and $a \equiv -\lambda^{(+)}_{\bar{\kappa}}$, $b \equiv -\lambda^{(-)}_{\bar{\kappa}}$; $a \equiv -4\bar{H} - \lambda(\bar{\sigma})$, $b \equiv 4\bar{H} - \lambda(\vec{\kappa}\cdot\vec{\bar{P}}\bar{\sigma})$, we get

$$[-\lambda^{(+)}_{\bar{\kappa}}]^{1/2}\, [-\lambda^{(-)}_{\bar{\kappa}}]^{1/2} \leq -\lambda(\bar{\sigma}) - \lambda(\vec{\kappa}\cdot\vec{\bar{P}}\bar{\sigma}), \qquad (28a)$$

$$2[-4\bar{H} - \lambda(\bar{\sigma})]^{1/2}\, [4\bar{H} - \lambda(\vec{\kappa}\cdot\vec{\bar{P}}\bar{\sigma})]^{1/2} \leq -\lambda(\bar{\sigma}) - \lambda(\vec{\kappa}\cdot\vec{\bar{P}}\bar{\sigma}), \qquad (28b)$$

which are the "integrated" analogs of the bounds (11a,b).

We note that in the derivation of the inequality (28a) we have used the identity

$$\lambda^{(+)}_{\bar{\kappa}} + \lambda^{(-)}_{\bar{\kappa}} = 2\lambda(\bar{\sigma}) + 2\lambda(\vec{\kappa}\cdot\vec{\bar{P}}\bar{\sigma}). \qquad (28c)$$

Now, let us consider the bounds

$$|8\bar{H} + \lambda(\bar{\sigma}) - \lambda(\vec{\kappa}\cdot\vec{\bar{P}}\bar{\sigma})| \leq -\lambda(\bar{\sigma}) - \lambda(\vec{\kappa}\cdot\vec{\bar{P}}\bar{\sigma}), \qquad (29a)$$

$$\frac{1}{2}|\lambda^{(+)}_{\bar{\kappa}} - \lambda^{(-)}_{\bar{\kappa}}| \leq -\lambda(\bar{\sigma}) - \lambda(\vec{\kappa}\cdot\vec{\bar{P}}\bar{\sigma}), \qquad (29b)$$

which are natural consequence of the bounds (22a,b,c). Then, one can see that the bounds (28a,b) are equivalent to (29a,b), respectively, if and only if the equalities (4b,c) [or (12a,b)] have an integrated analog. Here we can prove only implications: $\lambda^{(+)}_{\bar{\kappa}} = \lambda^{(-)}_{\bar{\kappa}} \to 4\bar{H} = -\lambda(\bar{\sigma})$ for all $\vec{\kappa}$ simultaneously.

Therefore, we have proved that *each bound on the differential cross-sections and on polarization projections, listed in the table, has an integrated analog*.

Next, let us consider the bound $4\bar{H} \leq -\lambda(\bar{\sigma})$ written in the equivalent form:

$$[c_i^2\bar{\sigma}_i - c_j^2\bar{\sigma}_j]^2 + 4\bar{H} \leq 2c_k^2\bar{\sigma}_k[c_i^2\bar{\sigma}_i + c_j^2\bar{\sigma}_j - \frac{1}{2}c_k^2\bar{\sigma}_k] \qquad (30a)$$

$i \neq j \neq k = 1,2,3$. This bound requires that if

$$c_k^2\bar\sigma_k\,[\,c_i^2\bar\sigma_i + c_j^2\bar\sigma_j - \frac{1}{2}c_k^2\bar\sigma_k\,] \underset{s\to+\infty}{\to} 0 . \quad (30b)$$

then

$$c_i^2\bar\sigma_i - c_j^2\bar\sigma_j \underset{s\to+\infty}{\to} 0, \quad \vec{\bar{P}}_i - \vec{\bar{P}}_j \underset{s\to+\infty}{\to} 0. \quad (30c)$$

and conversely, $\bar\sigma_k$ cannot vanish for $s\to+\infty$ if one of the above relations (30c) does not hold for $s\to+\infty$. On the other hand, from the bounds (28a,b), which are equivalent to

$$[\,c_i^2\bar\sigma_i - c_j^2\bar\sigma_j\,]^2 + [\,c_i^2\bar\sigma_i\,(\vec\kappa\cdot\vec{\bar{P}}_i) - c_j^2\bar\sigma_j\,(\vec\kappa\cdot\vec{\bar{P}}_j)\,]^2 \leq \quad (31a)$$

$$\leq 2c_k^2\bar\sigma_k\,[1 + (\vec\kappa\cdot\vec{\bar{P}}_k)(\vec\kappa\cdot\vec{\bar{X}})][c_i^2\bar\sigma_i + c_j^2\bar\sigma_j - \frac{1}{2}c_k^2\bar\sigma_k] - \bar{A} ,$$

where

$$\vec\kappa\cdot\vec{\bar{X}} \equiv \frac{c_i^2\bar\sigma_i\,(\vec\kappa\cdot\vec{\bar{P}}_i) + c_j^2\bar\sigma_j\,(\vec\kappa\cdot\vec{\bar{P}}_j) - \frac{1}{2}c_k^2\bar\sigma_k(\vec\kappa\cdot\vec{\bar{P}}_k)}{c_i^2\bar\sigma_i + c_j^2\bar\sigma_j - \frac{1}{2}c_k^2\bar\sigma_k} , \quad (31b)$$

$$\bar{A} \equiv \max\{[-\lambda_{\kappa}^{(+)}]^{1/2}\,[-\lambda_{\kappa}^{(-)}]^{1/2}\,,\ 2[4\bar{H} - \lambda(\vec\kappa\cdot\vec{\bar{P}}\bar\sigma)]^{1/2}[-4\bar{H} - \lambda(\bar\sigma)]^{1/2}\} \quad (31c)$$

we obtain the Pomeranchuk-type theorem (30c) assuming that

$$2c_k^2\bar\sigma_k\,(c_i^2\bar\sigma_i + c_j^2\bar\sigma_j - \frac{1}{2}c_k^2\bar\sigma_k)[1+(\vec\kappa\cdot\vec{\bar{P}}_k)(\vec\kappa\cdot\vec{\bar{X}})] - \bar{A} \underset{s\to+\infty}{\to} 0 . \quad (31d)$$

Next, let us consider the inequality

$$2|\,\mathrm{Re}N_{ij}\,| \leq |F_i^{(+\kappa)}||F_j^{(+\kappa)}| + |\,F_i^{(-\kappa)}||F_j^{(-\kappa)}| , \quad (32a)$$

$$N_{ij} \equiv Z_{ij}^{(0)},\ Z_{ij}^{(\kappa)} ,$$

which follows from the definitions (6a,b,c), Then, using the Hölder inequality we can write

$$2\,|\int \mathrm{Re}N_{ij}\,d\mu| \leq I_p[F_i^{(+\kappa)}]I_q[F_j^{(+\kappa)}] + I_{p'}[F_i^{(-\kappa)}]I_{q'}[F_j^{(-\kappa)}] , \quad (32b)$$

where $p > 1, p' > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\frac{1}{p'}+\frac{1}{q'} = 1$ and the integrals $I_\ell[F]$, $\ell = p, q, p', q'$ are defined by the relation (16). Therefore, from (32b), (16), (18) and (9a) we obtain

$$4c_i^2c_j^2\bar{N}_{ii}\bar{N}_{jj} + \lambda[\bar{N}] \leq$$

$$\leq c_i^2c_j^2\{[\bar\Sigma_i^{(+\kappa n)}]^{1/2}[\bar\Sigma_j^{(+\kappa m)}]^{1/2} + [\bar\Sigma_i^{(-\kappa n')}]^{1/2}[\bar\Sigma_j^{(-\kappa m')}]^{1/2}\}^2 ,$$

where $\bar{N}_{\ell\ell} \equiv \bar\sigma_\ell$, $(\vec\kappa\cdot\vec{\bar{P}}_\ell)\bar\sigma_\ell$, $\lambda(\bar{N}) \equiv \lambda(\bar\sigma)$, $\lambda(\vec\kappa\cdot\vec{\bar{P}}\bar\sigma)$ respectively, for $n > \frac{1}{2}$, $n' > \frac{1}{2}$ and $\frac{1}{2n}+\frac{1}{2m} = 1$, $\frac{1}{2n'}+\frac{1}{2m'} = 1$.

Finally, we remark that other interesting constraints can be derived if we write explicitly the bounds (23) and (33) on $\bar\Sigma^{(\pm\kappa n)}$ for $n \geq 2$ and $\vec\kappa$ conveniently chosen. These constraints can be very useful to check the sum rule (2) directly from the coefficients obtained by polynomial fits of the differential (polarized or unpolarized) cross-sections.

4. Conclusions

In this paper we have investigated the constraints on differential, $\bar\Sigma^{(\pm\kappa n)}$-integrated cross-sections and on polarization projections (in arbitrary directions) of three reactions related by internal symmetry. So, using the definition (6a) and the bilinear forms: $M_{ij}^{(\pm\kappa)}$, $Z_{ij}^{(0)}$, $Z_{ij}^{(\kappa)}$ [see (6b,c,d)] , in sect 2, *we have proved that the sum rule (2) alone implies the equalities (4a,b,c,d,e) and the* *bounds (5a,b,c,d,e) valid for any direction* $\vec{\kappa}$ *in the spin space at any energy and scattering angle.* The exact saturation of these bounds is obtained in terms of [$\mathrm{Im}\,N_{ij}$, $\mathrm{Re}\,N_{ij}$] zero trajectories [$N_{ij} \equiv M_{ij}^{(\pm\kappa)}, Z_{ij}^{(0)}, Z_{ij}^{(\kappa)}$] or equivalently in terms of [$n\pi$, $(n+\frac{1}{2})\pi$] phase contours. The zero trajectories of $\mathrm{Im}\,N_{ij}$ are all independent of channel indices (i, j) and the bilinear forms $Z_{ij}^{(0)}$ are invariant under rotations of spin reference frame. A hierarchy of the bounds is established and the constraints on the experimental data and on amplitude analysis when the bounds are exactly saturated are given in the table. These results are sufficient to obtain certain tests of the sum rule (2) and to determine the breaking parameters when the experimental data for differential cross-sections and polarization vectors for all three reactions are available. Moreover, these results enable us to understand the small differences between elastic differential cross-sections, at high energies and fixed transfer momentum, in terms of the small charge-exchange differential cross-sections [see the bounds (30a) and (31a) in terms of differential observables] .

Next, using the properties of the $I_p[F]$ -integrals (16) /see ref. /6/ / and definition (18) of the generalized $\Sigma^{(\pm\kappa n)}$ -integrated cross-sections, in sect. 3, we have derived the bounds (23), such that, starting with these results, in sect. 3, *we have proved the equalities (21)* and the bounds (22a,b,c,d) which are the integrated analogs of the equalities (4a) and the bounds (5a,b,c,d), respectively. Hence, we have proved that: *the sum rule (2) alone implies that each bound on the differential observables, listed in the table, has an integrated analog.* These results improve the result (44a,b) from ref. /6/ since the bound $4\bar{H} < -\lambda(\bar{\sigma})$ is equivalent to the bound (30a) and the bounds (28a,b) are equivalent to the bound (31a). We remark that the bounds (30a) and (31a), respectively, can be improved using the unitarity-analyticity bound (1) from ref. /14/ .

Finally, we note that all the results obtained in this paper can be extended to the cases when the 0-spin particles are replaced by unpolarized J-spin particles and also to the three body final states ($0\,\frac{1}{2} \to 0'0'\frac{1}{2}'$) reactions.

Table

The hierarchy of the bounds and the constraints on experimental data and on amplitude analysis when the bounds are exactly saturated

class		The bounds		Exact saturation of bound implies constraints on: experimental data		amplitudes
I	1	$-\lambda(\sigma) \geq 0$		$\lambda(\sigma)=0,\ \vec{P}_1=\vec{P}_2=\vec{P}_3$		$\mathrm{Im}\,N_{ij}=0$ [1]
II	1	$4\Omega_x^{(-)} \leq -\lambda(\sigma)$		$4\Omega_x^{(-)}=4H=-\lambda(\sigma),\ \lambda_x^{(+)}=\lambda_x^{(-)}$		$\mathrm{Im}\,Z_{ij}^{(0)}=0$
	2	$\lambda(\vec{x}\cdot\vec{P}\sigma) \leq -\lambda(\sigma)$		$4H=-\lambda(\sigma)=\lambda(\vec{x}\cdot\vec{P}\sigma),\ \lambda_x^{(\pm)}=0$		$\mathrm{Im}\,N_{ij}=0$ [1]
	3	$\lambda(\vec{x}\cdot\vec{P}\sigma) \leq 4\Omega_x^{(+)}$		$4\Omega_x^{(+)}=4H=\lambda(\vec{x}\cdot\vec{P}\sigma),\ \lambda_x^{(+)}=\lambda_x^{(-)}$		$\mathrm{Im}\,Z_{ij}^{(x)}=0$
III	1	[2] $-\lambda_x^{(+)} \geq 0,\ \Lambda_x^{(-)}$	$[-4H-\lambda(\sigma)]^{1/2}[4H-\lambda(\vec{x}\cdot\vec{P}\sigma)]^{1/2}$	$-4H-\lambda(\sigma)=-\frac{1}{4}\lambda_x^{(-)}$	$4H-\lambda(\vec{x}\cdot\vec{P}\sigma)=$	$\mathrm{Im}\,M_{ij}^{(+x)}=0$
	2	[2] $-\lambda_x^{(-)} \geq 0,\ \Lambda_x^{(+)}$	$\leq -\lambda(\sigma)-\lambda(\vec{x}\cdot\vec{P}\sigma)$	$-4H-\lambda(\sigma)=-\frac{1}{4}\lambda_x^{(+)}$	$=-4H-\lambda(\sigma)$	$\mathrm{Im}\,M_{ij}^{(-x)}=0$
	3	[2] $\lambda(\vec{x}\cdot\vec{P}\sigma) \leq 4H,\ E_x^{(+)}$	$[-\lambda_x^{(+)}]^{1/2}[-\lambda_x^{(-)}]^{1/2}$	$-4H-\lambda(\sigma)=-\lambda_x^{(-)}$	$\lambda_x^{(+)}=\lambda_x^{(-)}$	$\mathrm{Im}\,Z_{ij}^{(x)}=0$
	4	[2] $4H \leq -\lambda(\sigma),\ E_x^{(-)}$	$\leq -\lambda(\sigma)-\lambda(\vec{x}\cdot\vec{P}\sigma)$	$4H-\lambda(\vec{x}\cdot\vec{P}\sigma)=-\lambda_x^{(+)}$		$\mathrm{Im}\,Z_{ij}^{(0)}=0$

1) N_{ij} $M_{ij}^{(\pm\kappa)}$, $Z_{ij}^{(0)}$, $Z_{ij}^{(\kappa)}$.
2) These bounds, in the complementary regions, are weaker than the bound of class II.2 (see the text).

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