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REPRESENTATION OF SUPERCONFORMAL ALGEBRA. TWO- AND THREE-POINT GREEN FUNCTIONS OF SCALAR SUPERFIELDS

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Представления суперконформной алгебры. Двухточечные и трехточечные функции Грина скалярных суперполей

Рассмотрено некоторое множество представлений суперконформной алгебры. Для скалярных суперполей найдены конечные преобразования под действием суперконформной группы, двухточечные и трехточечные функции Грина. Найдены также ограничения на аномальные размерности суперполей, при выполнении которых трехточечные функции могут быть отличными от нуля.

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Representation of Superconformal Algebra. Two- and Three-Point Green Functions of Scalar Superfields

Some set of representations of the superconformal algebra is considered. The finite superconformal transformations for scalar superfields are obtained. Two and three point Green's functions are constructed. Restrictions on the anomalous dimensions of the superfields are obtained in order the three point functions do not vanish.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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Introduction

Some years ago in Golfand-Lichtman's paper [1] an algebra was invented which includes, in addition to Poincare algebra, four-generators forming together a four-dimensional spinor. For these spinor generators some anticommutation relations hold instead of ordinary commutation relations.

In the same paper some interesting features were mentioned of the theories invariant with respect to the algebras of such a kind. Similar algebras were considered also by Akulov, Volkov [2]. Recently these algebras have been re-discovered first in the context of the dual model [3], and finally have been formulated independently of the latter [4]. Theories invariant under the action of similar algebras were called supersymmetrical, and the symmetry itself supersymmetry. As a matter of fact, in the papers [4,5] a more extended algebra was considered which contains, as a subalgebra, the Lie algebra of conformal group.

With some exceptions [6] this generalization remains unnoticed by theorists. (They consider basically the minimal extensions of the Poincare group algebra by spinor generators).

This paper is aimed to construct some representations of extended conformal algebra, on the one hand, and to find superconformally invariant two- and three-point functions of some (scalar) superfields transforming irreducibly with respect to this algebra, on the other hand.

1. We consider a 24-generator generalized Lie algebra with the following commutation and anticommutation relations [5]:

$$[J_{AB}, J_{CD}] = i(g_{AD}J_{BC} + g_{BC}J_{AD} - g_{AC}J_{BD} - g_{BD}J_{AC}),$$

$$[\pi, J_{AB}] = 0,$$

$$[Q_\alpha, \pi] = i(\beta_7)^\beta_\alpha Q_\beta,$$

$$[Q_\alpha, J_{AB}] = i(\gamma_{AB})^\beta_\alpha Q_\beta,$$

$$\{Q_\alpha, Q_\beta\} = 2(\gamma^{AB}A)_{\alpha\beta}J_{AB} - 3(\beta_7 A)_{\alpha\beta}\,\pi. \quad (1.1)$$

The generators J_{AB} ($A, B, C, D = 0,1,2,3,5,6$) from the Lie algebra of the pseudo-rotation group $SO(4,2)$ which is locally isomorphic to the conformal group (g_{AB} is the metric tensor: $g_{AB}=0$, $A \neq B$; $g_{AA}=(-1,1,1,1;1,-1)$). The generators of supertransformations Q_α form an eight-dimensional spinor. π is the generator of γ_5-transformations.

The 8 x 8 matrices γ_{AB} generate a representation of $SO(4,2)$. We use the following direct product realization of γ_{AB}, β_7 and A:

$$\gamma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu].1 = \sigma_{\mu\nu}.1; \quad \gamma_{\mu 5} = -\frac{i}{2}\gamma_\mu\,\sigma_2; \quad \gamma_{\mu 6} = \frac{1}{2}\gamma_\mu.\sigma_1;$$

$$\gamma_{56} = \frac{1}{2}I.\sigma_3,$$

$$\beta_7 = \gamma_5.\sigma_3,$$

$$A = \gamma^0.\sigma_1.$$

Here γ_μ are the Dirac matrices in the Majorana representation, $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$; σ_i are the usual Pauli matrices and I, 1 are four- and two-dimensional unit matrices.

In what follows, the transformations generated by the algebra (1.1) will be called conformal supersymmetry, and the algebra itself - superconformal algebra.

For convenience we rewrite the algebra (1,1) in a somewhat different form. For this purpose we introduce, instead of Q_α, the following set of generators:

$$Q^+ = (1 + i\beta_7)Q, \quad Q^- = (1 - i\beta_7)Q. \quad (1.2)$$

They satisfy the following commutation and anticommutation relations:

$$[Q^\pm_\alpha, \pi] = \pm Q^\pm_\alpha; \quad [Q^\pm_\alpha, J_{AB}] = i(\gamma_{AB})^\beta_\alpha Q^\pm_\beta,$$

$$\{Q^+_\alpha Q^+_\beta\} = \{Q^-_\alpha Q^-_\beta\} = 0, \quad (1.3)$$

$$\{Q^+_\alpha, Q^-_\beta\} = 4(\gamma^{AB}A(1+i\beta_7))_{\alpha\beta}J_{AB} + 6i(A(1+i\beta_7))_{\alpha\beta}\,\pi.$$

In order to write the algebra (1.3) in the Minkowsky space we write Q as:

$$Q = \begin{pmatrix} S \\ -T \end{pmatrix}, \quad (1.4)$$

where S, T are four-dimensional spinors. Hence, for Q^+ and Q^- we obtain:

$$Q = \begin{pmatrix} S^+ \\ -T^- \end{pmatrix}, \quad Q = \begin{pmatrix} S^- \\ -T^+ \end{pmatrix}, \quad (1.5)$$

where

$$S^\pm = (1 \pm i\gamma_5)S, \quad T^\pm = (1 \pm i\gamma_5)T. \quad (1.6)$$

The ordinary conformal group generators may be expressed in terms of the generators $J_{\mu\nu}$ in the following way:

$$M_{\mu\nu} = J_{\mu\nu}, \quad D = J_{65},$$
$$P_\mu = J_{6\mu} + J_{5\mu}, \quad K_\mu = J_{6\mu} - J_{5\mu}. \quad (1.7)$$

We do not write out here the commutation relations (1.1), (1.3) in terms of $S^{\pm}$, $T^{\pm}$ and conformal group generators. They may be obtained from (1.1) and (1.3) by taking into account the definitions (1.4), (1.5) and (1.7)/5/.

2. To construct a realization of the algebra (1.1) we make use of the induced representation method. Let us choose as a "little" or stability subalgebra C the algebra generated by $M_{\mu\nu}$ (Lorentz subgroup generators), K_{μ} (special conformal transformations), S^- and T^+.

The commutation relations (1.3) show, that S^- and T^+ form a Grassman algebra which is an ideal of the "little" subalgebra C . This makes it possible to consider inducing representations of stability subalgebra with $S^- = T^+ = 0$.

Let the vector $\langle 0|$ belong to the space of such a representation, i.e.,

$$\langle 0|S^- = \langle 0|T^+ = 0 \tag{2.1}$$

and besides

$$\begin{aligned} \langle 0|M_{\mu\nu} &= \langle 0|\Sigma_{\mu\nu}, \\ \langle P|K_{\mu} &= \langle 0|k_{\mu}. \end{aligned} \tag{2.2}$$

Denote as $\Psi(x_\mu, \rho, \lambda, \theta^+, \theta^-)$ the operator * :

$$\Psi = e^{i\rho D + i\lambda\pi}\, e^{i\tilde{\theta}^+ S^+ + i\tilde{\theta}^- T^-}\, e^{i x_\mu P^\mu} . \tag{2.3}$$

---

*Our notations are as follows. The spinor conjugated to θ is denoted as $\tilde{\theta}$ and is determined by the expression: $\tilde{\theta} \equiv \theta\gamma^0$ (no complex conjugation!); ∂_θ stands for $\frac{\partial}{\partial\theta}$; $\tilde{\partial}_\theta \equiv \partial_\theta \gamma^0 = -\frac{\partial}{\partial\tilde{\theta}}$; $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$; $x\partial \equiv x_\mu \partial^\mu$.

We construct here the representation of the algebra (1.3) realized in the space of the functions f defined as follows:

$$f(x_\mu, \rho, \lambda, \theta^+_\alpha, \theta^-_\beta) = \langle 0|\Psi(x_\mu, \rho, \lambda, \theta^+_\alpha, \theta^-_\beta) . \tag{2.4}$$

Four-dimensional spinors θ^+ and θ^- form the Grassman algebra. They anticommute with the operators S^+ and T^- and commute with all the other quantities in eq. (2.3). θ^+ and θ^- could be obtained from one four-dimensional Majorana spinor by projecting with $(1+i\gamma_5)$ and $(1-i\gamma_5)$, respectively.

Hence, the following equalities

$$\begin{aligned} &(1 \pm i\gamma_5)\theta^{\pm} = 2\theta^{\pm}; \quad \tilde{\theta}^{\pm}(1 \pm i\gamma_5) = 2\tilde{\theta}^{\pm}, \\ &(1 \mp i\gamma_5)\theta^{\pm} = \tilde{\theta}^{\pm}(\mp i\gamma_5) = 0. \end{aligned} \tag{2.5}$$

hold.

Given an element A of the algebra (1.3), we denote the representation of this element in the space of the functions (2.4) as $\hat{A}$. It operates as follows:

$$\hat{A}f - f'(x_\mu, \rho, \lambda, \theta^+, \theta^-) = \langle 0|\Psi A. \tag{2.6}$$

It is readily proved that:

$$\begin{aligned} -i\partial_\mu f &= \langle 0|\Psi P_\mu, \\ -i\partial^+_{\theta\alpha} f &= \langle 0|\Psi S^+_\alpha. \end{aligned} \tag{2.7}$$

The operators ∂^+_θ and ∂^-_θ are defined by the anticommutation relations:

$$\{\theta^+_\alpha, \tilde{\partial}^+_{\theta\beta}\} = -\frac{1}{2}(1+i\gamma_5)_{\alpha\beta}; \quad \{\theta^-_\alpha, \tilde{\partial}^-_{\theta\beta}\} = -\frac{1}{2}(1-i\gamma_5)_{\alpha\beta} \tag{2.8}$$

(all other anticommutators vanish). This operators may be considered as left derivatives in the Grassman algebras generated by θ^+ and θ^-, respectively. They form the Grassman algebras in a sense dual to that of θ^+ and θ^-. From the definition (2.6) and eq. (2.7) it may be deduced that

$$\hat{P}_\mu = -i\partial_\mu ; \quad \hat{S}^+ = -i\partial_\theta^+ . \tag{2.8'}$$

Other generators entering into eq. (2.3) could be obtained in a quite analogous manner by differentiating f with respect to θ^-, ρ and λ:

$$\hat{T}^- = -i\partial_\theta^- - i(x\cdot\gamma)\partial_\theta^+ , \tag{2.9}$$

$$\hat{D} = -i\frac{\partial}{\partial\rho} - ix\partial - \frac{i}{2}\tilde{\theta}^+\partial_\theta^+ + \frac{i}{2}\tilde{\theta}^-\partial_\theta^- , \tag{2.10}$$

$$\hat{\pi} = -i\frac{\partial}{\partial\lambda} + \tilde{\theta}^+\partial_\theta^+ + \tilde{\theta}^-\partial_\theta^- . \tag{2.11}$$

What one still needs is the realization of the generators of the stability subalgebra. Equation (2.6) suggests that for the purpose we must commute the operator A with Ψ. Then we use eqs. (2.1) and (2.3) to obtain the following results:

$$\hat{M}_{\mu\nu} = \Sigma_{\mu\nu} + i(x_\mu\partial_\nu - x_\nu\partial_\mu) + i\tilde{\theta}^+\sigma_{\mu\nu}\partial_\theta^+ + i\tilde{\theta}^-\sigma_{\mu\nu}\partial_\theta^- ; \tag{2.12}$$

$$\hat{K}_\mu = e^\rho k_\mu + 2ix^\nu[-i\Sigma_{\mu\nu} + \tilde{\theta}^+\sigma_{\mu\nu}\partial_\theta^+ + \tilde{\theta}^-\sigma_{\mu\nu}\partial_\theta^- + g_{\mu\nu}(\frac{\partial}{\partial\rho} + \frac{1}{2}\tilde{\theta}^+\partial_\theta^+ - \frac{1}{2}\tilde{\theta}^-\partial_\theta^-)] + i\tilde{\theta}^+\gamma_\mu\partial_\theta^- + i(2x_\mu x_\nu - g_{\mu\nu}x^2)\partial^\nu ; \tag{2.13}$$

$$\hat{S}^- = -8i\Sigma_{\mu\nu}(\sigma^{\mu\nu}\theta^-) + 8(\gamma^\nu\theta^+)\partial_\nu - 8\theta^-\frac{\partial}{\partial\rho} + 12i\theta^-\frac{\partial}{\partial\lambda} + 8(\tilde{\theta}^-\theta^-)\partial_\theta^- ; \tag{2.14}$$

$$\hat{T}^+ = (x\cdot\gamma)\hat{S}^- + e^\rho 8ik_\nu\gamma^\nu\theta^- - 8i\Sigma_{\mu\nu}\sigma^{\mu\nu}\theta^+ + 8(\tilde{\theta}^+\theta^+)\partial_\theta^+ + 8\theta^+\frac{\partial}{\partial\rho} + 12i\theta^+\frac{\partial}{\partial\lambda} - 8(\tilde{\theta}^+\gamma_\mu\theta^-)\gamma^\mu\partial_\theta^- . \tag{2.15}$$

We consider only inducing representations of the stability subalgebra C with $k_\mu = 0$ (it is the case that corresponds to physically significant representations of conformal group). In this case, as is easily seen, the operators $\frac{\partial}{\partial\rho}$ and $\frac{\partial}{\partial\lambda}$ commute with all the generators of the superconformal algebra. Therefore, we may introduce the functions:

$$f_{(d,z)}(x_\mu, \theta^+, \theta^-, \rho, \lambda) = e^{d\rho - iz\lambda}\,\phi_{(d,z)}(x_\mu, \theta^+, \theta^-) , \tag{2.16}$$

where d and z are arbitrary complex numbers.

In the spaces $R^\Sigma_{(d,z)}$ formed by the functions $\phi_{(d,z)}(x_\mu, \theta^+, \theta^-)$ are induced representations of the algebra (2.8)–(2.15). The representation acting in $R^\Sigma_{(d,z)}$ is characterized by two complex "quantum numbers" d, z and by inducing representation Σ of the stability subalgebra (the Lie algebra of the Lorentz group in the case).

Consider the scalar representation, i.e.,

$$\Sigma_{\mu\nu} = 0 . \tag{2.17}$$

It acts in the space $R^0_{(d,z)}$. It may be observed that the condition

$$z = \frac{2}{3}d \tag{2.18}$$

implies the reducibility of $R^0_{(d,z)}$ which contains in this case an invariant subspace R^0_d formed by the functions:

$$\phi_{(d,\frac{3}{2}d)}(x_\mu, \theta^+, 0) \equiv \Phi_d(x_\mu, \theta^+) \tag{2.19}$$

independent of θ^-.

In the space R^0_α an irreducible representation of the superconformal algebra (2.8)–(2.15) is realized. The generator of this representation are:

$$\hat{M}_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu + \tilde{\theta}^+ \sigma_{\mu\nu} \partial_\theta^+); \quad \hat{P}_\mu = -i\partial_\mu ;$$

$$\hat{K}_\mu = i[(2x_\mu x^\nu - g^\nu_\mu x^2)\partial_\nu + 2x_\mu (d + \tfrac{1}{2}\tilde{\theta}^+ \partial_\theta^+) + 2x^\nu \tilde{\theta}^+ \sigma_{\mu\nu} \partial_\theta^+ ;$$

$$\hat{D} = -i(d + x\partial + \tfrac{1}{2}\tilde{\theta}^+ \partial_\theta^+ ; \quad \hat{\pi} = -\tfrac{2}{3} d + \tilde{\theta}^+ \partial_\theta^+ ;$$

$$\hat{S}^+ = -i\partial_\theta^+ ; \quad \hat{S}^- = 8\partial_\nu \gamma^\nu \theta^+ ; \quad \hat{T}^- = -ix_\mu \gamma^\mu \partial_\theta^+ ;$$

$$\hat{T}^+ = 8[2d\theta^+ + x_\mu \partial_\nu \gamma^\mu \gamma^\nu \theta^+ + (\tilde{\theta}^+ \theta^+)\partial_\theta^+]. \qquad (2.20)$$

This series of scalar representations of the superconformal algebra (labelled by scale dimension d) was obtained in /6/ within the framework of an algebra larger than (1.1).

Expand the functions $\Phi_d(x_\mu, \theta^+)$ in powers of θ^+ and observe that the series must terminate in the second order:

$$\Phi_d(x, \theta^+) = A(x) + \tilde{\theta}^+ \Psi(x) + F(x)\tilde{\theta}^+ \theta^+ . \qquad (2.21)$$

There exists only one linearly independent term quadratic in θ because of the identity:

$$\theta^+ \tilde{\theta}^+ = -\frac{1}{4}(\tilde{\theta}^+ \theta^+)(1 + i\gamma_5) . \qquad (2.22)$$

Therefore the superfield (2.21) consists of two scalar fields A(x) and F(x) with scale dimensions d and d + 1, respectively, and a spinor field $\Psi(x)$ with a dimension $d + \frac{1}{2}$. The supermultiplet (2,21) corresponds to Wess-Zumino's scalar supermultiplet /4/.

3. Algebra (2.20) generates finite transformations of superfields. In order to write down these transformations it is necessary to find the transformations of superfield arguments x_μ and θ^+. We have

$$x' = e^{i\alpha A} x_\mu e^{-i\alpha A}, \qquad \theta^{+\prime}_a = e^{i\alpha A} \theta^+_a e^{-i\alpha A}, \qquad (3.1)$$

where A is some of the generators of (2.20). In fact it is necessary to evaluate x′ by (3.1) only when A is some of supersymmetry generators (S^+, S^-, T^+, T^-) since the transformations of x and θ^+ under the other generators are well known. For example, the special conformal transformations of x and θ^+ are

$$x'_\mu = (1 - 2cx + c^2 x^2)^{-1}(x - cx^2),$$
$$\theta^{+\prime}_a = (1 + (x\cdot\gamma)(c\cdot\gamma))_a^{\ \beta} \theta^+_\beta . \qquad (3.2)$$

The dilatation and γ_5-transformation of x and θ^+ :

$$D: x'_\mu = \lambda x_\mu \qquad \theta^{+\prime} = \sqrt{\lambda}\,\theta^+ \qquad (3.3)$$

$$\pi: x'_\mu = x_\mu \qquad \theta^{+\prime} = e^{i\delta}\theta^+ \qquad (3.4)$$

are obvious too.

It is not difficult to obtain global transformations in all other cases by applying the formula

$$e^A B e^{-A} = \Sigma \frac{1}{n!}[A[A \ldots [A, B] \ldots]]. \qquad (3.5)$$

It should be noted that the series (3.5) has terms only with $n \le 2$ because of nilpotence properties of S^+, S^-, T^+, T^- parameters. In such a way we find the following global transformations:

a) T^+_a transformation (with a (finite) parameter ϵ^+_a)

$$(g^{-1}_{T^+} x)_\mu \equiv x'_\mu = [1 - 64(\tilde{\epsilon}^+ \epsilon^+)(\tilde{\theta}^+ \theta^+)]x_\mu + 8i\tilde{\epsilon}^+(x\cdot\gamma)\gamma_\mu \theta^+$$

$$(g^{-1}_{T^+}\theta^+) \equiv \theta^{+\prime} = \theta + 8i(\tilde{\theta}^+ \theta^+)\epsilon^+ \qquad (3.6)$$

b) T^- transformation (a parameter ϵ^-)

$$
\begin{aligned}
(g^{-1}_{T^-} x)_\mu &\equiv x'_\mu = x_\mu ; \\
(g^{-1}_{T^-} \theta^+) &\equiv \theta^{+\prime} = \theta^+ - (x \cdot \gamma)\, \epsilon^-
\end{aligned} \tag{3.7}
$$

c) S^+ transformation (a parameter τ^+)

$$
\begin{aligned}
(g^{-1}_{S^+} x)_\mu &\equiv x'_\mu = x_\mu \\
(g^{-1}_{S^+} \theta^+) &\equiv \theta^{+\prime} = \theta^+ + \tau^+
\end{aligned} \tag{3.8}
$$

d) S^- transformation (a parameter τ^-)

$$
\begin{aligned}
(g^{-1}_{S^-} x)_\mu &\equiv x'_\mu + 8i\tilde{\tau} - \gamma_\mu \theta^+ \\
(g^{-1}_{S^-} \theta^+) &\equiv \theta^{+\prime} = \theta^+ .
\end{aligned} \tag{3.9}
$$

Now we are in a position to determine the global transformations of the superfield. The result is:

$$
e^{i\tilde{\epsilon}^+ T^+} \Phi(x,\theta^+)\, e^{-i\tilde{\epsilon}^+ T^+} = (1 + 16i\tilde{\epsilon}^+\theta^+)^d\, \Phi(g^{-1}_{T^+} x, g^{-1}_{T^+}\theta^+) ,
$$

$$
e^{icK} \Phi(x,\theta^+)\, e^{-icK} =
$$

$$
= (1 - 2(cx) + c^2 x^2)^d\, \Phi\left(\frac{x - x^2 c}{1 - 2cx + c^2 x^2}, (1 + (x\gamma)(c\gamma))\,\theta^+\right),
$$

$$
\lambda^{iD}\, \Phi(x,\theta^+)\, \lambda^{-iD} = \lambda^{+d}\, \Phi(\lambda x, \sqrt{\lambda}\,\theta^+) ,
$$

$$
e^{i\delta\pi}\, \Phi(x,\theta^+)\, e^{-i\delta\pi} = e^{-\frac{2}{3} i d \delta}\, \Phi(x, e^{i\delta}\theta) . \tag{3.10}
$$

For all other generators $e^{i\lambda A}\, \Psi(x,\theta^+)\, e^{-i\lambda A} = \Psi(g^{-1}_A x, g^{-1}_A \theta^+)$.

In the derivation of (3.10) we have used the identity

$$
e^{16id\,[\tilde{\epsilon}^+\theta^+ + 4i(\tilde{\epsilon}^+\epsilon^+)(\tilde{\theta}^+\theta^+)]} = (1 + 16i\tilde{\epsilon}^+\theta^+)^d . \tag{3.11}
$$

The reader can check (3.11) expanding both the sides of eq. (3.11) in powers of ϵ^+ and θ^+ and using the nilpotence of both of them. The remaining eqs. (3.10) can be derived straightforward.

We end this section considering the algebra conjugated to the algebra (2.20). It may be obtained by considering the stability subalgebra C' with the generators S^+ and T^- instead of S^- and T^+. In this case the corresponding scalar representation characterized by numbers d and z turns to have an invariant subspace $R^{0}_{d}{}^*$ when

$$
z = -\frac{2}{3} d . \tag{3.12}
$$

One may assume that the space $R^{0}_{d}{}^*$, where the representation conjugated to the representation (2.20) acts, consists of the functions complex conjugated to the functions (2.21)

$$
\Phi^*_d(x,\theta^+) = A^*(x) + \tilde{\theta}^- \Psi^*(x) + F^*(x)\, \tilde{\theta}^-\theta \tag{3.13}
$$

(star denotes the complex conjugation). All the generators of the conjugated algebra (excepting π) may be obtained from the algebra (2.20) by means of the substitutions:

$$
\theta^+ \to \theta^- ; \quad \partial^+_\theta \to \partial^-_\theta ; \quad \hat{S}^\pm \to \hat{S}^\mp ; \quad \hat{T}^\pm \to \hat{T}^\mp \tag{3.14}
$$

and $\hat{\pi}$ is expressed by

$$
\hat{\pi} = \frac{2}{3} d - \tilde{\theta}^- \partial_{\tilde{\theta}} . \tag{3.15}
$$

Now we calculate the two- and three-point functions of the scalar superfields (2,21) and (3.17), provided that the physical vacuum is invariant with respect to the superconformal algebra (1.3).

Consider the following two-point functions:

$$
\Delta^{++}_{d_1 d_2}(x_1, x_2;\, \theta^+_1, \theta^+_2) \equiv \langle \Phi_{d_1}(x_1,\theta^+_1)\, \Phi_{d_2}(x_2,\theta^+_2) \rangle_0
$$

$$\Delta^{+-}_{d_1 d_2}(x_1, x_2; \theta_1^+, \theta_2^-) \equiv \langle \Phi_{d_1}(x_1, \theta_1^+)\, \Phi^*_{d_2}(x_2, \theta_2^-) \rangle_0$$

$$\Delta^{--}_{d_1 d_2}(x_1, x_2; \theta_1^-, \theta_2^-) \equiv \langle \Phi^*_{d_1}(x_1, \theta_1^-)\, \Phi^*_{d_2}(x_2, \theta_2^-) \rangle_0 \qquad (4.1)$$

Denote $A^i_{d_i}$ any arbitrary generator of the algebra (2.20) acting in the space of the functions with the arguments x_i and θ_i^+ $(i = 1, 2)$ and the corresponding generator of the conjugated algebra (3.16) as $A^i_{d_i}{}^*$. The invariance conditions for the functions (4.1) may be written then uniquely as:

$$(A^1_{d_1} + A^2_{d_2})\, \Delta^{++}_{d_1 d_2}(x_1, x_2; \theta_1^+, \theta_2^+) = 0$$

$$(A^1_{d_1} + A^{2*}_{d_2})\, \Delta^{+-}_{d_1 d_2}(x_1, x_2; \theta_1^+, \theta_2^-) = 0$$

$$(A^{1*}_{d_1} + A^{2*}_{d_2})\, \Delta^{--}_{d_1 d_2}(x_1, x_2; \theta_1^-, \theta_2^-) = 0 \qquad (4.2)$$

(scale dimensions d_1 and d_2 are assumed to be real). Let us show first that the equations (4.2) give only the following solutions for Δ^{++} and Δ^{--}:

$$\Delta^{++}_{d_1 d_2} = N\delta^4(x_1 - x_2)(\tilde{\theta}_1^+ - \tilde{\theta}_2^+)(\theta_1^+ - \theta_2^+) \quad (d_1 + d_2 = 3) \qquad (4.3)$$

and

$$\Delta^{++}_{d_1 d_2} = \Delta^{--}_{d_1 d_2} = 0 ,$$

when $d_1 + d_2 \neq 3$.

Consider the first of the equations (4.2). Translational $(A^1_{d_1} = -i\partial_1, A^2_{d_2} = -i\partial_2)$ and $S^+(A^1_{d_1} = -i\partial_{\theta_1^+}, A^2_{d_2} = -i\partial_{\theta_2})$ invariances imply that

$$\Delta^{++} = \Delta^{++}_{d_1 d_2}(x_1 - x_2; \theta_1^+ - \theta_2^+) ,$$

S^--invariance $(A^1_{d_1} = 8(\gamma\cdot\partial_1)\theta_1^+, A^2_{d_2} = 8(\gamma\cdot\partial_2)\theta_2^+)$ gives then the following expression for the considered two-point function:

$$\Delta^{++}_{d_1 d_2} = f(x_1 - x_2)\, \delta(\theta_1^+ - \theta_2^+) + C(\theta_1^+ - \theta_2^+) \qquad (4.4)$$

(δ-function $\delta(\theta_1^+ - \theta_2^+)$ is defined in /8/. In our case: $\delta(\theta_1^+ - \theta_2^+) \approx (\tilde{\theta}_1^+ - \tilde{\theta}_2^+)(\theta_1^+ - \theta_2^+)$).

T^--invariance gives more restrictions:

$$C(\theta_1^+ - \theta_2^+) = \text{const} \equiv M ,$$
$$f(x_1 - x_2) = N\delta^4(x_1 - x_2) .$$

At last T^+-invariance results in (4.3). The latter of the equations (4.2) has an analogous solution.

Turn now to the second of the equations (4.2). Substituting here $A^i = S^{i+}_{d_i}$ and $S^{i-}_{d_i}$ we obtain for the two-point function $\Delta^{+-}_{d_1 d_2}$ the expression

$$\Delta^{+-}_{d_1 d_2}(x; \theta_1^+ \theta_2^-) = D(x) - 8i \frac{\partial D(x)}{\partial x^\nu} \tilde{\theta}_2^- \gamma^\nu \theta_1^+ + 16\Box D(x)(\tilde{\theta}_1^+ \theta_1^+)(\tilde{\theta}_2^- \theta_2^-) , \qquad (4.5)$$

where $x = x_1 - x_2$ and $D(x)$ is an arbitrary function of x. This function is fixed then up to a multiplicative constant by T^+ and T^- invariance, and turns out to be zero if $d_1 \neq d_2$ and $c(x^2)^{-d}$ if $d_1 = d_2 = d$. Therefore

$$\Delta^{+-}_{d_1 d_2}(x; \theta_1^+, \theta_2^-) = 0 \qquad d_1 \neq d_2$$

$$\Delta^{+-}_{dd}(x; \theta_1^+, \theta_2^-) = C[(x^2)^{-d} + 16i\, d(x^2)^{-d-1} \tilde{\theta}_2^-(x\cdot\gamma)\theta_1^- + 64d(d-1)(x^2)^{-d-1} (\tilde{\theta}_1^+ \theta_1^+)(\tilde{\theta}_2^- \theta_2^-) . \qquad (4.6)$$

From the commutation relations (1.3) it may be seen that the invariance of the two-point function with respect to the spinorial generators implies automatically the invariance with respect to the whole superconformal algebra. So there is no need to verify the invariance of the two-point functions (4.3) and (4.6) with respect to rotations, dilatation, special conformal transformations and π. Substituting into the expression (4.1) for $\Delta^{+-}_{d_1 d_2}$ the decompositions of superfields Φ_{d_1} and $\Phi^*_{d_2}$ in powers of θ^+ and θ^- given by eq. (2.21) and (3.17) and then comparing the expression obtained with the equality (4.6) we may identify the first term in the r.h.s. of eq. (4.6) with the two-point function of a scalar field $A(x)$. In doing so, the second term gives the two-point function of a spinor field $\Psi(x)$ and the third term that of a scalar field $F(x)$. Note that if the field $A(x)$ has canonical dimension $(d=1)$ the last term in the r.h.s. of eq. (4.6) vanishes.

The three-point functions may be obtained in an analogous manner. As in the case of the two-point functions the coefficients in the decomposition of three-point functions of the superfields in powers of θ are conformally invariant three-point functions of ordinary fields $A(x)$, $\Psi(x)$ and $F(x)$. The distinction of our results from that of paper /7/ is the following: the three-point functions prove to be different from zero only when some restrictions of the dimensions d_1, d_2, d_3 hold. Denote

$$\Gamma^0_{d_1 d_2 d_3} \equiv \langle \Phi_{d_1}(x_1, \theta_1^+)\, \Phi_{d_2}(x_2, \theta_2^+)\, \Phi^*_{d_3}(x_3, \theta_3^-) \rangle_0 , \qquad (4.7)$$

$$\Gamma^+_{d_1 d_2 d_3} \equiv \langle \Phi_{d_1}(x_1, \theta_1^+)\, \Phi_{d_2}(x_2, \theta_2^+)\, \Phi_{d_3}(x_3, \theta_3^-) \rangle_0 . \qquad (4.8)$$

It turns out that

$$\Gamma^0_{d_1 d_2 d_3} \neq 0$$

only in the case

$$d_1 + d_2 = d_3 \qquad (4.9)$$

and $\Gamma^+_{d_1 d_2 d_3} \neq 0$

implies

$$d_1 + d_2 + d_3 = 3 . \qquad (4.10)$$

To understand how the condition (4.9) arises let us write out the most general expression for $\Gamma^0_{d_1 d_2 d_3}$ compatible with the Poincare and $S^\pm$ invariance:

$$\Gamma^0_{d_1 d_2 d_3} = A - 8i \frac{\partial A}{\partial x^\mu_{13}} \tilde\theta_1^+ \gamma^\mu \theta_3^- - 8i \frac{\partial A}{\partial x^\mu_{23}} \tilde\theta_2^+ \gamma^\mu \theta_3^- +$$

$$+ 16 \Box_{13} A (\tilde\theta_1^+ \theta_1^+)(\tilde\theta_3^- \theta_3^-) + 16 \Box_{23} A (\tilde\theta_2^+ \theta_2^+)(\tilde\theta_3^- \theta_3^-) +$$

$$+ 16 \frac{\partial A}{\partial x^\mu_{13}\, \partial x^\mu_{23}} (\tilde\theta_1^+ \gamma^\mu \gamma^\nu \theta_2^+)(\tilde\theta_3^- \theta_3^-) , \qquad (4.11)$$

where A is an arbitrary function of $x_{13} \equiv x_1 - x_3$ and $x_{23} \equiv x_2 - x_3$. To satisfy the $T^\pm$ invariance it is necessary (and sufficient) that all the coefficients in the decomposition of $\Gamma^0_{d_1 d_2 d_3}$ in powers of θ are conformally covariant functions. It is easily seen, however, that $A(x_{13}, x_{23})$ and, e.g., $\Box_{13} A$ may simultaneously be conformally invariant only in the case (4.9). The explicit expressions of the functions (4.7) and (4.8) are

$$\Gamma^0_{d_1 d_2\, d_3 = d_1 + d_2}(x_{13}, x_{23}; \theta_1^+, \theta_2^+, \theta_3^-) = C (x^2_{13})^{-d_1} (x^2_{23})^{-d_2} \times$$

$$\times \left[1 + 16 d_1 i x_{13}^{-2} \tilde\theta_1^+ (x_{13} \cdot \gamma)\, \theta_3^- + 16 d_2 i x_{23}^{-2} \tilde\theta_2^+ (x_{23} \cdot \gamma)\, \theta_3^- - \right.$$

$$- 64 d_1 (d_1 - 1) x_{13}^{-2} (\tilde\theta_1^+ \theta_1^+)(\tilde\theta_3^- \theta_3^-) - 64 d_2 (d_2 - 1) x_{23}^{-2} (\tilde\theta_2^+ \theta_2^+)(\tilde\theta_3^- \theta_3^-) -$$

$$\left. - 128\, i d_1 d_2 (x_{13}^{-2} x_{23}^{-2})(\tilde\theta_3^- \tilde\theta_3^-)(\tilde\theta_1^+ (x_{13} \cdot \gamma)(x_{23} \cdot \gamma)\, \theta_2^+) \right] .$$

$$\Gamma^+_{d_1 d_2 d_3}(x_{13}, x_{23}, x_{12}; \theta_1^+, \theta_2^+, \theta_3^+) =$$

$$= C (x^2_{23})^{d_1 - 2} (x^2_{13})^{d_2 - 2} (x^2_{12})^{d_3 - 2} \times$$

$$\times [\tilde{\theta}_1^{+}(x_{13}\cdot\gamma)(x_{23}\cdot\gamma)\tilde{\theta}_2^{+} - \frac{1}{2} x_{23}^2 \tilde{\theta}_1^{+}\theta_1^{+} + \text{ciclic permutations}(1,2,3) \quad (d_1 + d_2 + d_3 = 3)$$

Degenerate solutions:

$$\Gamma^{+} = N\delta^4(x_{12})\tilde{\theta}_{12}^{+}\theta_{12}^{+} \approx \Delta_{d_1 d_2}^{++}(x_{12}, \theta_{12}), \qquad d_1 + d_2 = 3, \; d_3 = 0,$$

$$\Gamma^{+} = N\delta^4(x_{23})\tilde{\theta}_{23}^{+}\theta_{23}^{+}, \qquad d_2 + d_3 = 3, \; d_1 = 0,$$

$$\Gamma^{+} = N\delta^4(x_{13})\tilde{\theta}_{13}^{+}\theta_{13}^{+}, \qquad d_1 + d_3 = 3, \; d_2 = 0: \tag{4.12}$$

At last, we draw attention to the following trivial fact, resulting from the condition (4.9): conformally-invariant three-point function (4.7) of three superfields with canonical dimensions vanishes. In this case only the function (4.8) may be different from zero.

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