ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

ДУБНА

E2 - 8646

M.Havlíček, W.Lassner

CANONICAL REALIZATIONS OF THE LIE ALGEBRAS gl(n,R) AND sl(n,R).

I. FORMULAE AND CLASSIFICATION

1975

E2 - 8646

M.Havlíček, W.Lassner

CANONICAL REALIZATIONS OF THE LIE ALGEBRAS gl(n,R) AND sl(n,R).

I. FORMULAE AND CLASSIFICATION

Submitted to "Reports on Mathematical Physics".

Гавличек М.,Ласснер В. **E2 - 8646**

Канонические реализации алгебр Ли $gl(n,R)$ и $sl(n,R)$. I. Формулы и классификация

В работе генераторы алгебр Ли $gl(n,R)$ и $sl(n,R)$ рекуррентно выражены через полиномы квантово-механических канонических переменных q_i и p_i. Эти реализации антиэрмитовы, операторы Казимира в них кратны единице и в зависимости от числа использованных канонических пар зависят от k $(k-1$ для $sl(n,R))$, $k=2,\ldots,n$ свободных действительных параметров.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований
Дубна 1975

Havlíček M., Lassner W. **E2 - 8646**

Canonical Realizations of the Lie Algebras $gl(n,R)$ and $sl(n,R)$. I. Formulae and Classification

The generators of the Lie algebra of the general linear group $gl(n,R)$ and of the special linear group $sl(n,R)$ are, recurrently, expressed through polynomials in the quantum canonical variables p_i and q_i. These realizations are skew-hermitean, the Casimir operators are realized by constant multiples of identity element and, in dependence of the number of the canonical pairs used, they depend on k $(k-1$ for $sl(n,R))$, $k=2,\ldots,n$ free real parameters.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research
Dubna 1975

1. *INTRODUCTION*

Realizations of classical Lie algebras in the Weyl algebra or in an associated quotient division ring have been considered in the last years from different points of view. Besides the study of the Weyl algebra and different algebraic structures associated with the Weyl algebra itself and the presentation of various classes of realizations of Lie algebras in these algebraic structures there was the problem of minimal canonical realizations which has been treated successfully. The minimal number N_{min} of canonical pairs q_i and p_i which are needed for faithful realizations of classical Lie algebras are determined almost completely [1,2,3]. As to the Weyl algebra the number N_{min} equals n (= rank) for the algebras A_n and C_n and

$$N_{min} = \begin{matrix} 2n-2 \text{ or } 2n-1 & \text{for } B_n \\ 2n-3 \text{ or } 2n-2 & \text{for } D_n \end{matrix}$$

(the uncertainty can be removed for algebras with small dimensions).

It is further shown that minimal canonical realizations are Schur-realizations, i.e., all Casimir operators are realized by multiples of identity element. As, however, the number of canonical pairs is as small as possible the set of minimal canonical realizations is not too rich and some "degenerations" among them can be expected. It is shown in [3], e.g., that with exception of some low-dimensional cases, in any realization of the Lie algebras $B_n(D_n)$ in the Weyl algebra by means of $2n-1(2n-2)$ canonical pairs only one independent Casimir operator exists at most.

To remove this degeneration one has to try to enrich the set of realizations and to this purpose the Weyl algebra with N_{min} pairs must be enlarged. The simple change of the number of canonical pairs may be combined with the algebraic extension of the Weyl algebra, e.g., its embedding in quotient division ring or in the matrix Weyl algebra.

It may happen, however, that removing degeneration the realization will not be further a Schur-realization as in non-minimal realizations this property needs not to be necessarily fulfilled. The question whether "non-degenerated" sets of Schur-realizations exist had been positively solved in /5/ for the Lie algebra $o(n,m)$ in the framework of matrix canonical realizations.

In this paper we deal with the same problem for Lie algebras $gl(n,R)$ or $sl(n,R)$, respectively. In contrast to the case of $o(n,m)$ the investigated realizations of these algebras are contained in the Weyl algebra with an appropriate number of canonical pairs. We give a family of $(d+1)$-parametric classes, $d=1,2,\dots,n-1$, of realizations in the Weyl algebra by means of $N(d)=\frac{d}{2}(2n-d-1)$ canonical pairs. These realizations possess the following "good" properties.

(i) The Casimir operators are multiples of the identity element.

(ii) The realizations are "inequivalent" (non-related) up to endomorphisms of the Weyl algebra.

(iii) The realizations are skew-hermitean.

(iv) The realizations of the $(d+1)$-parameter set possess Casimir operators the eigenvalues of which can be polynomially expressed by $(d+1)$ independent symmetric functions of the parameters.

The last property will be considered in detail in the second part of this paper where the Casimir operators of the given realizations are studied.

2. *BASIC NOTIONS*

The *Weyl algebra* W_{2N} is the associative algebra

over C with identity generated by $2N$ elements q_i and p_i, $i=1,2,\dots,N$, which satisfy the relations

$$p_i q_j - q_j p_i = \delta_{ij} I \qquad i,j = 1,2,\dots,N. \tag{1}$$

According to the Poincare-Birkhoff Theorem a basis in W_{2N} is given by the ordered monomials

$$q^k p^l = q_1^{k_1} \dots q_N^{k_N} p_1^{l_1} \dots p_N^{l_N}, \tag{2}$$

i.e., every element x of W_{2N} can be uniquely written in the form

$$x = \sum_{k,l} \alpha_{kl} q^k p^l \qquad \alpha_{kl} \in C. \tag{3}$$

D e f i n i t i o n 1: A canonical realization of a Lie algebra L in W_{2N} is an homomorphism ϕ

$$\phi: L \to W_{2N}.$$

We shall consider this homomorphism in all cases already to be uniquely extended to a homomorphism of the enveloping algebra UL of L into W_{2N}. (If we speak in the following about realizations we mean always canonical realizations).

D e f i n i t i o n 2: The realization ϕ is called to be a *Schur-realization* if every central element z of the enveloping algebra UL is realized by a multiple of the identity

$$\phi(z) = \lambda_z I \qquad \lambda_z \in C.$$

D e f i n i t i o n 3: Two realizations ϕ and ϕ' are called to be *related* if an endomorphism θ of W_{2N} exists such that either

$$\phi'(g) = \theta(\phi(g))$$

or $\phi(g) = \theta(\phi'(g))$ for all $g \in L$.

For possible applications to representation theory we introduce in W_{2N} the involution "+" through the following relations

$$q_i^+ = q_i \,,$$
$$p_i^+ = -p_i \,. \tag{4}$$

A realization ϕ of the real Lie algebra L is then called *skew-hermitean* if

$$\phi(g)^+ = -\phi(g) \qquad \text{for all } g \in L \tag{5}$$

holds.
Besides realizations of the Lie algebra $gl(n,R)$ we consider also those of the subalgebra $sl(n,R)$. The Lie algebra $sl(n,R)$ is simple and a noncompact real form of the complex Lie algebra A_{n-1} of the Cartan classification. The rank of $sl(n,R)$ equals $n-1$. Canonical realizations of $sl(n,R)$ can exist only in W_{2N} with $N > n-1$ (see [1,2]). The standard basis of $gl(n,R)$ is given by the n^2 elements e_{ij} which are, in their $n\times n$-matrix representation, matrices with the matrix elements $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. The commutation relations have the form

$$[e_{ij}\,,\, e_{kl}] = \delta_{jk}\, e_{il} - \delta_{li}\, e_{kj}\,, \qquad i,j,k,l = 1,2,\ldots,n\,. \tag{6}$$

The element

$$e = \sum_{i=1}^{n} e_{ii} \in gl(n,R) \tag{7}$$

commutes with all e_{ij} and the elements

$$a_{ij} = e_{ij} - \frac{e}{n}\delta_{ij} \tag{8}$$

obey the same commutation relations as e_{ij}:

$$[a_{ij}\,,\, a_{kl}] = \delta_{jk}\, a_{il} - \delta_{li}\, a_{kj}\,. \tag{9}$$

Since

$$\sum_{i=1}^{n} a_{ii} = 0 \tag{10}$$

the n^2 elements a_{ij} are not independent; the n^2-1 elements a_{ij} without $a_{nn} = -\sum_{\nu=1}^{n-1} a_{\nu\nu}$ form a basis of the subalgebra $sl(n,R)$ of $gl(n,R)$.
The realization of an element $x \in L$ will be denoted by the same but capital letter.

3. CANONICAL REALIZATIONS OF $gl(n,R)$ AND $sl(n,R)$

Theorem 1: Let $F_{\mu\nu}$, $\mu,\nu = 1,2,\ldots,n-1$, be a canonical realization of generators of $gl(n-1,R)$ fulfilling (6) in W_{2m}. The following formulae define a realization $E_{ij} = E_{ij}(F_{\mu\nu},\alpha)$ of $gl(n,R)$ in $W_{2(n-1+m)}$.

$$E_{\mu\nu} = q_\mu p_\nu + F_{\mu\nu} + \frac{1}{2}\delta_{\mu\nu}\,1$$

$$E_{n\mu} = -p_\mu$$

$$E_{\mu n} = q_\mu\left(q_\nu p_\nu + \frac{n}{2} - i\alpha\right) + q_\nu F_{\mu\nu}$$

$$E_{nn} = -q_\nu p_\nu - \frac{n-1}{2} + i\alpha\,1\,, \quad \alpha \in C \tag{11}$$

(summation over ν).
This realization has the following properties.
(i) The realization is skew-hermitean if α is real and if $F_{\mu\nu}$ are skew-hermitean.
(ii) The realization is a Schur-realization if the realization of $gl(n-1,R)$ is a Schur-realization.
(iii) Two realizations (11) with different values of the parameter α are non-related.

(iv) Two realizations (11) differing only in the realization of $gl(n-1,R)$ are related if and only if these realizations of $gl(n-1,R)$ are related.

In the proof we use two assertions which are easy provable and generalize known properties of W_2 /6/ . (We remark that $[qp, q^k p^l] = (k-l) q^k p^l$ holds in W_2).

Assertion 1: If $x \in W_{2N}$ commutes with p_i (resp q_i) then x does not depend on q_i (resp. p_i).

Assertion 2: Assume that for $x \in W_{2N}$ there holds

$$[q_1 p_1 + \dots + q_{N'} p_{N'}, x] = mx$$

for some $m = 0, \pm 1, \pm 2, \dots$ where $N' \le N$. Then

$$x = \sum_{\substack{k,l \\ k-l=m}} \alpha_{kl} q^k p^l ,$$

where

$$\alpha_{kl} q^k p^l = \alpha_{k_1,\dots,k_{N'},l_1,\dots,l_{N'}} q_1^{k_1} \dots q_{N'}^{k_{N'}} p_1^{l_1} \dots p_{N'}^{l_{N'}}$$

$k - l = k_1 + k_2 + \dots + k_{N'} - l_1 - \dots l_{N'}$ and α_{kl} do not depend on $q_1, \dots, q_{N'}, p_1, \dots, p_{N'}$.

(i) We shall not write here the explicit verification that E_{ij} from (11) satisfy the commutation relations (6) and that they are skew-hermitean for real values of the parameter α and skew-hermitean $F_{\mu\nu}$.

(ii) Let us consider a central element z from the enveloping algebra of $gl(n,R)$. By Z we denote its realization induced by (11),

$$Z = \sum_{k,l} \alpha_{kl} q^k p^l ,$$

where α_{kl} are polynomials in $F_{\mu\nu}$, i.e., $\alpha_{kl} \in W_{2m}$ and where

$$q^k p^l = q_1^{k_1} \dots q_{n-1}^{k_{n-1}} p_1^{l_1} \dots p_{n-1}^{l_{n-1}} .$$

Since

$$[Z, E_{n\mu}] = -[Z, p_\mu] = 0 , \qquad \mu = 1,2,\dots,n-1$$

from assertion 1 it follows that Z does not depend on q_μ, $\mu = 1,2,\dots,n$

$$Z = \sum_\nu \alpha_{0\nu} p^\nu . \tag{12}$$

The relation

$$0 = [Z, E_{nn}] = [Z, \sum_\nu q_\nu p_\nu] ,$$

assertion 2 for $m=0$ and equation (12) simply give

$$Z = \alpha_{00} .$$

It remains to show that $\alpha_{00} = \alpha_{00}(F_{\mu\nu})$ does not depend on $F_{\mu\nu}$. But this is a direct consequence of

$$[\alpha_{00}, E_{\mu\nu}] = [\alpha_{00}, F_{\mu\nu}] = 0 ,$$

because the realization of $gl(n-1,R)$ was assumed to be a Schur-realization.

(iii) and (iv). Let E_{ij} and E'_{ij} be two related realizations (11). That means there exists an endomorphism $\theta \in \mathrm{End}\, W_{2(n-1+m)}$ such that

$$\theta(E_{ij}) = E'_{ij} \qquad \text{for all } i,j = 1,2,\dots,n .$$

We show first that then $\alpha = \alpha'$.

From equation

$$\theta(E_{n\mu}) = E'_{n\mu} = -p_\mu , \quad \mu = 1,2,\dots,n-1$$

we get

$$\theta(p_\mu) = p_\mu . \tag{13}$$

As $\theta(1) = 1$, the equation

$$\theta(E_{nn}) = E'_{nn}$$

can be rewritten in the form

$$(\theta(q_\nu) - q_\nu) p_\nu = \alpha - \alpha' . \tag{14}$$

Since $(\theta(q_\nu) - q_\nu)$ as element of the Weyl algebra cannot have a negative p_ν-degree, equation (14) can hold only in the case $\alpha = \alpha'$. This proves (iii). To show (iv) we assume that $\alpha' = \alpha$ and hence we write equation (14) as

$$\theta(q_\nu p_\nu) = q_\nu p_\nu . \tag{15}$$

Now we use the relation

$$[\theta(p_\mu), \theta(q_\nu)] = \theta([p_\mu, q_\nu]) = [p_\mu, q_\nu] ,$$

which together with equation (13) gives

$$[p_\mu, \theta(q_\nu) - q_\nu] = 0 , \qquad \mu, \nu = 1, 2, \dots, n-1$$

and we can conclude, that according to assertion 1, the elements $\theta(q_\nu) - q_\nu$ do not depend on q_μ. That means $\theta(q_\nu)$ is of q_ν-degree one.

Further the relation

$$[\theta(q_\nu p_\nu), \theta(q_\mu)] = \theta(q_\mu)$$

leads, due to eq. (15), to the relation

$$[q_\nu p_\nu, \theta(q_\mu)] = \theta(q_\mu)$$

and applying assertion 2 for $m = 1$ we see that

$$\theta(q_\mu) = q_\mu . \tag{16}$$

Since the remaining m canonical pairs of W_{2m}, which are used for the realization of $gl(n-1, R)$, commute with q_μ and p_μ, $\mu = 1, \dots, n-1$ we see from (13) and (16) that $\theta(p_\rho)$ and $\theta(p_\rho)$, $\rho = n, n+1, \dots, n-1+m$, commute with q_μ and p_μ, $\mu = 1, 2, \dots, n-1$, and therefore due to assertion 1 the $\theta(q_\rho)$ and $\theta(p_\rho)$, $\rho = n, n+1, \dots, n-1+m$, cannot depend on q_μ and p_μ, $\mu = 1, 2, \dots, n-1$. That means the endomorphism θ of $W_{2(n-1+m)}$ restricted to W_{2m} is an endomorphism $\tilde{\theta}$ of W_{2m}.

From the relation

$$\theta(E_{ij}) = E'_{ij}$$

we get therefore $\qquad \tilde{\theta}(F_{\mu\nu}) = F'_{\mu\nu}$.

On the other hand, if $F_{\mu\nu}$ and $F'_{\mu\nu}$ are related, the endomorphism $\tilde{\theta}$ of W_{2m} for which

$$\tilde{\theta}(F_{\mu\nu}) = F'_{\mu\nu}$$

can be extended to an endomorphism θ of $W_{2(n-1+m)}$ by setting

$$\theta(q_\mu) = q_\mu$$

$$\theta(p_\mu) = p_\mu , \qquad \mu = 1, 2, \dots, n-1 .$$

This yields

$$\theta(E_{ij}) = E'_{ij}$$

and the proof is completed.

Now we use theorem 1 in an iterative manner to construct new realizations of $gl(n, R)$ with more than one parameter. For notation of the new realizations we introduce, by analogy with the representation theory, the notion of "signature".

Definition 4: The $(n+1)$ tuple,, $n \geq 2$
$(d; 0, \dots, 0, \alpha_{n-d}, \dots, \alpha_n)$
with $d = 1, 2, \dots, n-1$ and $\alpha_i \in R, i = n-d, \dots, n$
is called *signature*.

Theorem 2: To every signature $(d; 0, \dots, 0, \alpha_{n-d}, \dots, \alpha_n)$ there corresponds a canonical realization of $gl(n, R)$, $n \geq 2$ in W_{2N} with $N = N(d) = \frac{d}{2}(2n - d - 1)$. This realization is defined as follows.

a) $(1; 0, \dots, 0, \alpha_{n-1}, \alpha_n)$ denotes the realization (11) of $gl(n, R)$ with $\alpha = \alpha_n$ and

$$F_{\mu\nu} = i\alpha_{n-1} \frac{\delta_{\mu\nu} 1}{n-1} .$$

b) $(d; 0, \dots, 0, \alpha_{n-d}, \dots, \alpha_n), d > 1$, denotes the realization (11) of $gl(n, R)$ with $\alpha = \alpha_n$

where the realization of $gl(n-1,R)$ has the signature $(d-1; 0, \dots, 0, \alpha_{n-d}, \dots, \alpha_{n-1})$. The realization with signature $(d;\ 0,\ \dots,\ 0,\ \alpha_{n-d}, \dots, \alpha_n)$ has the following properties.

(i) This realization is skew-hermitean.

(ii) This realization is a Schur-realization.

(iii) Two realizations are related if and only if their signatures are the same.

The proof follows from theorem 1 by simple induction. Note only that the realization of the algebra $gl(n-1, R)$, $F_{\mu\nu} = i\alpha_{n-1}\frac{\delta_{\mu\nu} 1}{n-1}$ (not included in our set) is non-related to a realization of $gl(n-1,R)$ with any signature.

The described realizations have the following two simple properties.

L e m m a 1: (i) In a realization with signature $(d; 0, \dots, 0, \alpha_{n-d}, \dots, \alpha_n)$ the element $E = \sum_{j=1}^{n} E_{jj}$ is given by $E = i \sum_{j=n-d}^{n} \alpha_j 1$.

(ii) If we denote by $E_{ij}^{\tau(\lambda)}$ the generators of the realization with signature $(d; 0, \dots, 0, \alpha_{n-d} - \frac{(n-d)\lambda}{n}, \alpha_{n-d+1} - \frac{\lambda}{n}, \dots, \alpha_n - \frac{\lambda}{n})$ $\lambda \in R$, then

$$E_{ij}^{\tau(\lambda)} = E_{ij}^{\tau(0)} - i\frac{\lambda}{n}\delta_{ij}\ 1 .$$

Proof: For $d = 1$ both the assertions follow immediately from formulae (11). Further we proceed by induction.

Now we shall specify our results for the subalgebra $sl(n,R)$. We denote the set of all signatures by Σ and its subset consisting of all signatures with $\sum_{i=n-d}^{n} \alpha_i = 0$ by Σ_0; clearly $\Sigma_0 \subsetneq \Sigma$. We consider the realization of the Lie algebra $sl(n,R)$ with the basis

$$A_{ij} = E_{ij} - \frac{E}{n}\delta_{ij} \tag{17}$$

(see eq. (8)) where E_{ij} is a realization of $gl(n,R)$ with signature $\tau \in \Sigma$. The realization of $sl(n,R)$ with the generators (17) will be denoted also by the signature τ. As $sl(n,R)$ is a subalgebra of $gl(n,R)$ non-related realizations of $gl(n,R)$ may lead to related realizations of $sl(n,R)$. The question, which realization of $sl(n,R)$ can be omitted, is solved by the following theorem.

T h e o r e m 3: (i) Two realizations of $sl(n,R)$ with signatures from Σ_0 are non-related.

(ii) For any realization of $sl(n,R)$ with signature $\tau \in \Sigma$ there exists a related realization with signature in Σ_0.

Proof: (i) The first assertion of lemma 1 implies $E=0$ for realizations with signatures from Σ_0, therefore, $A_{ij} = E_{ij}$. Hence, the realization of $sl(n,R)$ is the particular case of the realization of $gl(n,R)$ and assertion (iii) of theorem 2 can be applied.

(ii) Denote $\lambda = \sum_{j=n-d}^{n} \alpha_j$ and together with signature

$\tau(0) = (d; 0, \dots, 0, \alpha_{n-d}, \dots, \alpha_n)$

consider the signature

$$\tau(\lambda) = (d; 0, \dots, 0, \alpha_{n-d} - \frac{(n-d)\lambda}{n}, \alpha_{n-d+1} - \frac{\lambda}{n}, \dots, \alpha_n - \frac{\lambda}{n}) .$$

The corresponding realizations of $sl(n,R)$

$$A_{ij}^{\tau(0)} = E_{ij}^{\tau(0)} - \frac{E^{\tau(0)}}{n}\delta_{ij} = E_{ij}^{\tau(0)} - i\frac{\lambda}{n}\delta_{ij} 1$$

and

$$A_{ij}^{\tau(\lambda)} = E_{ij}^{\tau(\lambda)}$$

lie in $W_{2N(d)}$ and due to assertion (ii) of lemma 1 the realizations are the same $A_{ij}^{\tau(0)} = A_{ij}^{\tau(\lambda)}$, i.e., they are trivially related.

4. CONCLUDING REMARKS

1. With exception of skew-hermiticity and its consequences all assertions are valid also for the complex Lie algebras $gl(n,C)$ and $sl(n,C)$.

2. Realizations with signatures $(1; 0, \ldots, 0, \alpha_{n-1}, \alpha_n)$ are minimal realizations of $gl(n,R)$ or (with $\alpha_{n-1} = -\alpha_n$) of $sl(n,R)$ respectively.

3. The relations (11) contain the possibility to obtain further realizations of $gl(n,R)$, different from the studied one, because the $F_{\mu\nu}$'s must not necessarily be canonical realization in W_{2m}. Relations (11) define a realization of $gl(n,R)$ whenever the $F_{\mu\nu}$'s fulfil the commutation relations of $gl(n-1,R)$ and commute with the canonical variables q_μ, p_μ, $\mu = 1, 2, \ldots, n-1$. Of course, if the used realization of $gl(n-1,R)$ will not have such properties as skew-hermiticity, etc., the same properties cannot be expected from the realization of $gl(n,R)$.

Acknowledgements:

The authors would like to thank Professor A.Uhlmann for a critical reading of the manuscript and for comments.

REFERENCES

1. *A.Simoni, F.Zaccaria. Nuovo Cim., 59A, 280 (1969).*
2. *A.Joseph. Comm. Math. Phys., 36, 325 (1974).*
3. *M.Havlíček, P.Exner. On the Minimal Canonical Realizations of the Lie Algebra* $O_C(m)$, *JINR, E2-8089, Dubna, 1974.*
4. *A.Joseph. J. Math. Phys., 13, 351 (1972).*
5. *M.Havlíček, P.Exner. Matrix Canonical Realizations of the Lie Algebra* $O(n,m)$ *I, II, JINR, E2-8533, E2-8700, Dubna, 1975.*
6. *J.Dixmier. Bull. Soc. math. Franc., 96, 209 (1968).*

Received by Publishing Department
on March 3, 1975.