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CANONICAL FIELD QUANTIZATION IN AN EXTERNAL TIME-DEPENDENT GRAVITATIONAL FIELD: RESTRICTIONS FROM THE GENERAL PROPERTIES OF FEYNMAN PROPAGATOR

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CANONICAL FIELD QUANTIZATION IN AN EXTERNAL TIME-DEPENDENT GRAVITATIONAL FIELD: RESTRICTIONS FROM THE GENERAL PROPERTIES OF FEYNMAN PROPAGATOR

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Каноническое квантование поля в нестатическом внешне-гравитационном поле: ограничения, вытекающие из общих свойств фейнмановского пропагатора

Для квантового скалярного поля, взаимодействующего с гравитационным полем однородной изотропной замкнутой Вселенной, исследуются средние по циклическим состояниям различных представлений канонических коммутационных соотношений. Показано, что свойства регулярности этих функций такие же как у фейнмановского пропагатора только в представлениях образующих некоторый класс унитарной эквивалентности.

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Canonical Field Quantization in an External Time-Dependent Gravitational Field: Restrictions from the General Properties of Feynman Propagator

The Green functions of the quantum scalar field interacting with gravitation of the homogeneous isotropic closed Universe are studied. They have been determined as an expectation value of the time-ordered product of two field operators in the cyclic states of various, in general, unitary-nonequivalent representations of canonical commutation relations. The regularity properties of these functions are shown to be the same as of the Feynman propagator obtained in [9,10] for arbitrary Riemannian space-time only in the representations that form a class of unitary equivalence.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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I. Introduction

The quantum theory of the field interacting with the time-dependent external (i.e., given classic) field or source is of great interest both for considerations of particular physical systems and as a linear quantum field model with real processes of particle creation and annihilation.

However, to describe such a system in general there is no unique or distinguished way analogous to the familiar Fock second quantization for free quantum fields. The origin of this non-uniqueness is the time variation of external conditions and following from this energy non-conservation for the quantum field.

Till we deal with static external conditions the traditional scheme of canonical quantization enables us to construct the especial representation of canonical commutation relations (CCR) in the Fock space in which the Hamiltonian of the quantum field is diagonal. It is just the Fock representation. If we formally change the creation operators C_i^+ and the annihilation operators C_i^- of the Fock representation by means of a Bogolubov transformation to new $\tilde{C}_i^{\pm}$'s:

$$\tilde{C}_i^{\pm} = \overset{\pm}{A}_i{}^j C_j^+ + \overset{\pm}{B}_i{}^j C_j^- \qquad (1)$$

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x) Here i, j are collective indices for all possible quantum numbers.

then we obtain a new representation in the space with the cyclic vector $\tilde{\bar{\Phi}}_0$ defined by the equation

$$\tilde{C}_i \tilde{\bar{\Phi}}_0 = 0$$

in which the Hamiltonian is not diagonal. In this sense the new representation is not as good as the original Fook one and at the same time it is not unitary equivalent without strong restrictions on the coefficients $\overset{\pm}{A}_i{}^j$, $\overset{\pm}{B}_i{}^j$.

Under time-dependent external conditions there is no representation in which the Hamiltonian is diagonal for all moments of time. One can only construct a set of equally good (or equally bad) quasi-Fock representations of which "creation" and "annihilation" operators are connected by transformations of the type of Eq. (1). So in this case the problem arises of choice of a representation in which the time development of the system should be considered. If the external field (or source) vanishes or at least becomes static in the remote past and future then there exist two special representations such that the Hamiltonian is diagonal for $t \to +\infty$ in one representation and for $t \to -\infty$ in the other. In this case there is a olear corpuscular interpretation of initial and final state vectors for $t \to \mp\infty$ respectively and the problem is reduced to determination of the unitary connection between them, i.e.,to the calculation of S-matrix if it exists (see, e.g.[1-4]). However, the S-matrix approach is inadequate for description of the interaction with external gravitation

in the framework of general relativity. In this case the problem of choice of a representation of CCR becomes unavoidable.

One might attempt to extract information on the dynamics of the quantum-field in time-dependent external conditions from quantum Green functions, for example, from the Feynman propagator $G_F(x, x')$. For the free field (noninteracting at all) these functions may be determined without reference to the canonical quantization but an analogous procedure in the case under consideration does not lead to any unique result.

The most general way[4] to determine $G_F(x,x')$ or the causal Green function $\mathcal{D}_C(x,x') = i\,G_F(x,x')$ is based on the analytic continuation into the complex plane of the so-called elementary solution[5] $G^{(1)}(x,x')$ of the field equations. However, the elementary solution itself is determined up to an arbitrary two-point regular solution $G_0(x,x')$ of the field equations, i.e.,only its singularities are determined uniquely and the same may be said about $G_F(x,x')$. If the external field vanishes sufficiently rapidly for $t \to \pm\infty$ one may hope to fix $G_0(x,x')$ by the requirement of asymptotic coincidence of $G_F(x,x')$ with its expression for the free field. In the general case $G_0(x,x')$ stays arbitrary. A priori the arbitrariness in $G_F(x,x')$ is not necessarily equivalent to that in the choice of CCR representation. The present paper is devoted just to comparison of the canonical formalism and the Green function in this aspect for a particular case of interaction of the quantum scalar field with an external gravitational field. As the latter we take the cosmological field of the homogeneous isotropic closed universe. In general relativistic terms we consider the quantum theory of a scalar field in the closed homogeneous isotropic space-time (denote it as $H\mathcal{J}_{1,3}$). Apparently, in this case the S-matrix approach is the most in appropriate one. At the same time the system is of "applied" interest since in a vicinity of cosmological (big-bang) singularities the quantum pair creation by the nonstationary gravitational field had possibly an essential influence on evolution of the Universe [6,8]. Construction and comparison of some classes of representations of CCR is considerably simplified in $H\mathcal{J}_{1,3}$ in view of existence of the group of motions isomorphic to O(4). At last, as the space is closed we may use only discrete quantum numbers for description of quasi-one-particle states and thus avoid "volume" divergences when comparing various representations.

In Sec.2 we discuss what is the general form that the Feynman propagator should have in an arbitrary curved space-time and to what extent the propagator is arbitrary. In Sec.3 the canonical field quantization in $H\mathcal{J}_{1,3}$ is discussed briefly. It results in a great class of O(4) invariant quasi-Fock representations of CCR which in general are not unitary equivalent. In every representation one may introduce a propagator as the expectation value of the two time-ordered Heisenberg field operators in the cyclic state. Then in Sec.4 properties of these expectation values are compared with the general requirements for the Feynman propagator of Sec.2 and there we prove that the representations in which the requirements are fulfilled form a class of unitary equivalence. This is the main result of the paper.

2. Structure of the Feynman propagator in the curved space-time

In papers [5,9] for the Feynman propagator of scalar field in an arbitrary curved space-time ($V_{1,3}$) with signature -2 the following general expression is given

$$G_F(x,x') = \frac{1}{2\pi^2}\left\{\frac{\Delta^{1/2}}{\Omega - i0} + v\,\ln(\Omega - i0) + w\right\} \tag{2}$$

Ω, Δ, v and w being symmetric functions of x and x', $x, x' \in V_{1,3}$, i.e., $\Omega = \Omega(x,x') = \Omega(x,x')$, etc. They are defined as follows:

Ω is the geodesic interval between x and x' (i.e., one half the square of distance along the geodesic line). For a fixed x' the equation

$$\Omega(x,x') = 0 \tag{3}$$

determines the light conoid of the point x' that is the locus of all null geodesics radiating from x'. We shall denote it by $\mathcal{LC}(x')$. Ω satisfies the equation

$$\partial_\alpha \Omega\, \partial^\alpha \Omega = 2\Omega, \quad \partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} \tag{4}$$

and the initial condition $\lim\limits_{x \to x'} \Omega(x,x') = 0$. Here and further all differentiations refer to the variable x.

The function $\Delta(x,x')$ obeys the equation

$$\nabla^{\alpha}(\Delta \partial_\alpha \Omega) = 4\Delta , \qquad (5)$$

where ∇_α is the covariant derivative. The initial condition for Δ is $\lim_{x\to x'} \Delta(x,x') = 1$.

Equations for v and w result from a field equation. As the latter we prefer to adopt the conformal-covariant (for m^2 =0) generalization of special relativistic Fock-Klein-Gordon equation [10,11]

$$(\Box + \frac{R}{6} + m^2)\varphi = 0 , \quad \Box\varphi \equiv \nabla^{\alpha}\partial_\alpha \varphi \quad (c = \hbar = 1) \qquad (6)$$

R being the scalar curvature x) . Then

$$(\Box + \frac{R}{6} + m^2) v = 0 , \qquad (7)$$

$$\partial^{\alpha}\Omega\,\partial_\alpha(\Delta^{-\frac{1}{2}} v) + \frac{1}{2}\Delta^{-\frac{1}{2}} v + \frac{1}{2}\Delta^{-\frac{1}{2}}(\Box + \frac{R}{6} + m^2)\Delta^{1/2} +$$
$$+ \Delta^{-1/2}\Omega(\Box + \frac{R}{6} + m^2) w = 0 . \qquad (8)$$

Expression (2) is obtained by means of the correct (i.e., resulting in the same singularities on $\mathcal{L}C(x')$ as in the flat space-time case) analytic continuation of the elementary solution [5] to complex plane in Ω (see also [9]). Of course, the analytic continuation is correct only provided $v(x,x')$ and $w(x,x')$ are sufficiently regular on $\mathcal{L}C(x')$. Thus, $v(x,x')$

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x) However, the result of the paper is valid for the theory with the traditional scalar field equation $(\Box + m^2)\varphi = 0$

and $w(x,x')$ should be sought in the form

$$v(x,x') = \sum_{n=0}^{\infty} v_n(x,x')\,\Omega^n(x,x') \qquad (9)$$

$$w(x,x') = \sum_{n=0}^{\infty} w_n(x,x')\,\Omega^n(x,x') \qquad (10)$$

with v_0 and w_0 regular on $\mathcal{L}C(x')$. Then substitution of these series into Eqs. (7) and (8) gives a system of recurrent equations for $v_0,\dots, v_n,\dots$ and for $w_1,\dots, w_n,\dots$:

$$\partial^{\alpha}\Omega\,\partial_\alpha(\Delta^{-1/2} v_0) + \Delta^{-1/2} v_0 = -\frac{1}{2}\Delta^{-1/2}(\Box + \frac{R}{6} + m^2)\Delta^{1/2} \qquad (11)$$

. . . .

$$\partial^{\alpha}\Omega\,\partial_\alpha(\Delta^{-1/2} v_n) + (n+1)\Delta^{-1/2} v_n = -\frac{\Delta^{-1/2}}{2n}(\Box + \frac{R}{6} + m^2) v_{n-1} \qquad (12)$$

$$\partial^{\alpha}\Omega\,\partial_\alpha(\Delta^{-1/2} w_n) + (n+1)\Delta^{-1/2} w_n = -\frac{\Delta^{-1/2}}{2n}(\Box + \frac{R}{6} + m^2)(w_{n-1} - \frac{1}{n} v_{n-1}) - \Delta^{-1/2} v_n \qquad (13)$$

All v_n's are determined uniquely by system (13) taken as a whole because the integration constant for v_n is fixed by the equation for v_{n+1} . On the contrary, one may choose w_0 at will except for its regularity on $\mathcal{L}C(x')$. Transition from the given w_0 to another is equivalent to addition of a regular (on $\mathcal{L}C(x')$) two-point solution of Eq. (6) to an originally chosen $G_F(x,x')$.

The vector $\partial_\alpha\Omega$ is tangent to the geodesic connecting x with x' . Therefore Eqs. (11)-(13) may be integrated along every geodesic as an ordinary differential equation with a canonical parameter as the independent variable (see,e.g.,[9]).

We cite the result for $w_n(x,x')$:

$$w_n(x(\tau),x') = -\frac{\Delta^{1/2}}{\tau^{n+1}}\int_0^{\tau} d\tilde\tau\, \tilde\tau^{\,n} \Delta^{-1/2}(x(\tilde\tau),x')\Big\{\frac{1}{2}\big(\Box+\frac{R}{6}+m^2\big)\big(w_{n-1}-\frac{1}{n}\, \mathcal{J}_{n-1}\big) - v_n\Big\}_{x=x(\tilde\tau)} \quad (14)$$

the integration being performed along the geodesic. At last it should be noted that all formulae and assertions of the section are valid only in such a domain of $V_{1,3}$ where any point x can be connected with x' by a geodesic and this one is unique.

Is there any reasonable restriction on $w_0(x,x')$ in addition to its regularity on $\mathcal{L}C(x')$? In principle, it might have singularities in some points that do not belong to $\mathcal{L}C(x')$. From general conditions on $G_F(x,x')$ and from Eq. (14) it follows that any singularities off $\mathcal{L}C(x')$ must be excluded in the domain of existence and uniqueness of geodesics. Equation (14) sets the minimal order of regularity of w_0 which is necessary for regularity of $w(x,x')$ in the domain. For the integral in Eq. (14) to be convergent on any geodesic radiating from x' the function (of $\tilde\tau$) $\Box w_{n-1}$ may, at most, have a singularity of the type

$$\frac{f(\tilde\tau)}{(\tilde\tau-\tau'')^{\mu}} \qquad \mu<1 \qquad \tau''=\mathrm{const} \quad .$$

On a given geodesic of $V_{1,3}$ with metric tensor $g_{\alpha\beta}(x)$ one has

$$\frac{1}{2}(\tilde\tau-\tau'')^2\, \frac{dx^{\alpha}(\tilde\tau)}{d\tilde\tau}\cdot\frac{dx^{\beta}(\tilde\tau)}{d\tilde\tau}\, g_{\alpha\beta}(x(\tilde\tau)) = \Omega(\tilde x, x'')$$

where x'' is the point at which $\tilde\tau=\tau''$, i.e., $x''=x(\tau'')$, $\tilde x = x(\tilde\tau)$. So if the geodesic connecting x and x' passes through x'' we are to have

$$\Box[w_0(x,x')] = \mathcal{F}_1(x,x')\,\Omega^{-\frac{\mu}{2}}(x,x'') + \mathcal{F}_2(x,x'),$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are some functions regular at $x=x''$. This condition is fulfilled by the substitution

$$w_0(x,x') = \mathcal{F}_3(x,x')\,\Omega^{1-\frac{\mu}{2}}(x,x'') + \mathcal{F}_4(x,x'),$$

$\mathcal{F}_3$ and $\mathcal{F}_4$ being any functions with regular second derivatives at x'' . It is easy to be convinced of that such a behaviour of $w_0(x,x')$ guarantees regularity of all w_n's at x'' . Obviously, this result may be formulated also so that $\partial_\alpha w(x,x')$ is allowed to have, along a geodesic, a singularity of the form

$$\partial_\alpha w(x,x') \sim \mathcal{F}(x,x')\,\Omega^{-\frac{\mu}{2}}(x,x'') \qquad \mu<1, \qquad (15)$$

x'' being a point on the geodesic.

3. Representations of CCR for a scalar field in $H\mathcal{J}_{1,3}$

The metric form of a closed homogeneous isotropic space-time may be written as

$$ds^2 = b^2(\eta)\,(d\eta^2 - h_{ij}(\xi)\,d\xi^i d\xi^j) \qquad i,j=1,2,3,$$

where $b(\eta)$ is a function of time-like coordinate η, ξ^i are some curvilinear coordinates on a three-dimensional sphere (S_3) of the unit radius.

Canonical quantization of the scalar field, obeying Eq. (6), in $HJ_{1,3}$ has been performed in paper [12] and now we only reproduce its main results with slight changes in notations. The field operator φ may be represented by separation of variables in the form

$$\varphi = \frac{1}{b(\eta)} \sum_{s=0}^{\infty} \sum_{\sigma=1}^{(s+1)^2} u_{s\sigma}(\eta)\, \mathcal{P}_{s\sigma}[k(\xi)]. \tag{16}$$

$\mathcal{P}_{s\sigma}[k(\xi)]$ are (c-number) harmonic polynomials of power s in homogeneous coordinates $k^a(\xi)$, $a = 1,2,3,4$ on S_3. In other words $\mathcal{P}_{s\sigma}$ is a spherical function on S_3. σ is a collective index for two quantum numbers that enumerate basis elements in the $(s+1)^2$-dimensional space of harmonic polynomials of fixed power s; we shall not need more details of the basis. The operator $u_{s\sigma}$ satisfies the equation

$$\ddot{u}_{s\sigma} + [(s+1)^2 + m^2 b^2(\eta)]\, u_{s\sigma} = 0\ , \tag{17}$$

where the dots mean differentiation with respect to η.

At an initial moment of time η_0 we impose the canonical commutation relations between $u_{s\sigma}(\eta_0)$ and $\dot{u}_{s\sigma}(\eta_0)$ which are equivalent to the relations between $\varphi(\eta_0, \xi)$ and $\partial_0 \varphi(\eta_0, \xi)$. As the form of $b(\eta)$ is not fixed we may assume

$\eta_0 = 0$ without loss of generality. Then, in notations $q_{s\sigma} = u_{s\sigma}(0)$ and $p_{s\sigma} = \dot{u}_{s\sigma}(0)$ we have

$$[p_{s\sigma}, p_{s'\sigma'}] = 0, \quad [q_{s\sigma}, q_{s'\sigma'}] = 0$$
$$[q_{s\sigma}, p_{s'\sigma'}] = i\,\delta_{ss'}\,\delta_{\sigma\sigma'}\ . \tag{18}$$

Then $u_{s\sigma}(\eta)$ may be represented as

$$u_{s\sigma}(\eta) = \frac{\sqrt{S_0}}{2}\, q_{s\sigma}(u_s^+ + u_s^-) + i\,\frac{p_{s\sigma}}{2\sqrt{S_0}}\,(u_s^+ - u_s^-), \tag{19}$$

where $S_0 = b_0^{-1}\sqrt{(s+1)^2 + m^2 b_0^2}$, $b_0 \equiv b(0)$, and $u_s^{\pm} = u_s^{\pm}(\eta)$ are two complex conjugate c-number solutions of Eq. (6) defined by the initial conditions

$$u_s^{\pm}(0) = \frac{1}{\sqrt{S_0}}\ , \quad \dot{u}_s^{\pm}(0) = \pm i\sqrt{S_0}\ .$$

Using Eq. (19) we represent the field operator φ in the form analogous to the expansion of the free field in the flat space-time in positive and negative frequency solutions

$$\varphi(x) = \frac{1}{\sqrt{2}\, b(\eta)} \sum_{s,\sigma} \{ C^+_{s\sigma} u_s^+(\eta) + C^-_{s\sigma} u_s^-(\eta) \}\, \mathcal{P}_{s\sigma}[k(\xi)], \tag{20}$$

where the operators

$$C^{\pm}_{s\sigma} = \frac{1}{\sqrt{2}}\left(\sqrt{S_0}\, q_{s\sigma} \mp \frac{i}{\sqrt{S_0}}\, p_{s\sigma}\right)$$

satisfy the usual commutation relations for creation (C^+)

and annihilation C^- operators. We realize $C^{\pm}_{s\sigma}$ as operators of creation and annihilation in a quasi Fock space Φ with a O(4)-invariant cyclic vector $|0\rangle$ defined by the equation

$$C^-_{s\sigma}|0\rangle = 0 . \qquad (21)$$

However the representation of CCR thus obtained is not the unique O(4)-invariant one. To find others we turn to expansion (20). The functions

$$\Psi^{\pm}_{s\sigma}(\eta,\xi) = \frac{1}{b(\eta)}\, u^{\pm}_{s\sigma}(\eta)\, \mathcal{Y}_{s\sigma}[k(\xi)]$$

form a basis in the space of solutions of Eq. (6) with the usual Klein-Gordon scalar product

$$(\Psi_1,\Psi_2) = i\, b^2(\eta) \int_{S_3} dS_3(\xi) \left(\Psi_1^* \frac{\partial \Psi_2}{\partial \eta} - \frac{\partial \Psi_1^*}{\partial \eta} \Psi_2 \right), \qquad (22)$$

where the asterisk denotes the complex conjugation. In this space a representation of the group O(4) is realized which is reducible since for a fixed S each of the functions $\Psi^+_{s\sigma}$ and $\Psi^-_{s\sigma}$ separately form bases of two equivalent irreducible representations of O(4). As a consequence of their equivalence the linear transformations exist which mix $\Psi^+_{s\sigma}$ and $\Psi^-_{s\sigma}$ and conserve scalar product (22) while commute with operators of the reducible group representation in the space of

solutions of Eq. (6). Correspondingly, there exists the Bogolubov transformation of $C^{\pm}_{s\sigma}$ (i.e., a transformation that preserves the usual commutation relations for $C^{\pm}_{s\sigma}$) which commutes with all generators of the group representation. Its general form is

$$C^+_{s\sigma}(\lambda_s) = \frac{C^+_{s\sigma} + \lambda_s C^-_{s\sigma}}{\sqrt{1-|\lambda_s|^2}}$$
$$|\lambda_s| < 1 \qquad (23)$$
$$C^-_{s\sigma}(\lambda_s) = \frac{C^-_{s\sigma} + \lambda_s^* C^+_{s\sigma}}{\sqrt{1-|\lambda_s|^2}}$$

Obviously $C^{\pm}_{s\sigma} = C^{\pm}_{s\sigma}(0)$.

These $C^{\pm}_{s\sigma}(\lambda_s)$ are the creation and annihilation operators in a quasi Fock space with the cyclic vector $|0,\{\lambda_s\}\rangle$ defined by the equation

$$C^-_{s\sigma}(\lambda_s)|0,\{\lambda_s\}\rangle = 0 .$$

So any choice of an infinite sequence $\{\lambda_s\} = \{\lambda_0, \lambda_1, \dots\}$ gives rise to a quasi Fock representation of CCR. In particular our original representation corresponds to the sequence $\{0\}$. In general, representations corresponding to different sequences are unitary non-equivalent. Two representations defined by sequences $\{\lambda_s\}$ and $\{\lambda_s'\}$ are unitary equivalent if the series $\sum_{s=0}^{\infty}(s+1)^2|\lambda_s-\lambda_s'|^2$ converges [12]. Thus in spite of the considerable symmetry of the system we have a large arbitrariness in the choice of representations of CCR.

How would the situation change if the external gravitational field were static? Suppose $b(\eta) = b_0 = const$, then $\Psi^{+}_{s\sigma} \sim e^{i\sqrt{s_0}\,\eta}\, \mathcal{P}_{s\sigma}$, and the group of motion of space-time is trivially extended to the direct product $T_0 \times O(4)$, where T_0 is the one-parameter group of time translations. Representations of $T_0 \times O(4)$ in subspaces separately spanned by $\Psi^{+}_{s\sigma}$ and $\Psi^{-}_{s\sigma}$ for fixed s are irreducible and non-equivalent in view of different signs of the eigenvalues of the generator $i \frac{\partial}{\partial \eta}$. Consequently there is no Bogolubov transformation of the form Eq. (23) which commutes with all the generators of representation of the group in Φ .
In this sense the original representation of CCR is unique.
Such a uniqueness occurs also in the more general case when the group of motion consists of T_0 merely , i.e. the metric is static without any spatial symmetry, see, e.g., [13].

4. The cyclic state expectation values of two time-ordered field operators vs. the Feynman propagator

Now we shall search for the O(4)-invariant quasi-Fock representations of CCR in which the cyclic state expectation value of time-ordered product of two scalar field operators

$$G(x,x'|\{\lambda_s\}) = \langle 0,\{\lambda_s\}|\,T(\varphi(x)\,\varphi(x'))\,|\{\lambda_s\},0\rangle \qquad (24)$$

can be reduced to the form of $G_F(x,x')$ of Sec.2. In the

representation defined by a sequence $\{\lambda_s\}$ we use the expansion in $C^{\pm}_{s\sigma}(\lambda_s)$ analogous to Eq. (20) and substitute it into Eq. (24). Then, application of the summation theorem for harmonic polynomials (see, e.g. [14] v.1, p.163)

$$\sum_{\sigma=1}^{(s+1)^2} \mathcal{P}_{s\sigma}[k(\xi)]\, \mathcal{P}_{s\sigma}[k(\xi')] = \frac{(s+1)\sin(s+1)\gamma}{2\pi \sin\gamma},$$

where

$$\gamma = \pm \arccos[k_a(\xi)\, k^a(\xi')],$$

gives

$$G(x,x'|\{\lambda_s\}) = G(x,x'|\{0\}) + \tilde{G}(x,x'|\{\lambda_s\}) + \tilde{\tilde{G}}(x,x'|\{\lambda_s\}), \qquad (25)$$

where

$$G(x,x'|\{0\}) = \sum_{s=0}^{\infty} (s+1)\sin(s+1)\gamma\, \{\theta(\eta-\eta')\, u_s^-(\eta)\, u_s^+(\eta') + \theta(\eta'-\eta)\, u_s^+(\eta)\, u_s^-(\eta')\}\, A(\eta,\eta',\gamma) \qquad (26)$$

$$\tilde{G}(x,x'|\{\lambda_s\}) = A(\eta,\eta',\gamma) \sum_{s=0}^{\infty} \frac{(s+1)\sin(s+1)\gamma}{1-|\lambda_s|^2}\, 2\,\mathrm{Re}[\lambda_s\, u_s^-(\eta)\, u_s^-(\eta')], \qquad (27)$$

$$\tilde{\tilde{G}}(x,x'|\{\lambda_s\}) = A(\eta,\eta',\gamma) \sum_{s=0}^{\infty} (s+1) \frac{|\lambda_s|^2 \sin(s+1)\gamma}{1-|\lambda_s|^2}\, 2\,\mathrm{Re}[u_s^+(\eta)\, u_s^-(\eta')] \qquad (28)$$

with

$$A(\eta,\eta',\gamma) = [4\pi^2\, b(\eta)\, b(\eta')\, \sin\gamma]^{-1}.$$

We will show $G(x,x'|\{0\})$ to have the same singularities on $\mathcal{L}C(x')$ as $G_F(x,x')$ has. To this end we make use of the uniformly convergent series of interactions in powers of $(s+1)^{-1}$ for $u_s^{\pm}(\eta)$ obtained in [13]. We write out explicitly just the terms which can give rise to divergences of series (26) and so singularities in $G(x,x'|\{\lambda_s\})$:

$$u_s^{\pm}(\eta) = e^{\pm i(s+1)\eta}\left\{\frac{1}{(s+1)^{1/2}} \pm \frac{im^2}{2(s+1)^{3/2}}\int_0^{\eta} b^2(\eta')d\eta' + O(s^{-5/2})\right\}.$$

After substitution of this expression into Eq. (26) and writing out expliitly only divergent terms we have

$$G(x,x'|\{0\}) = A(\eta,\eta';\gamma)\sum_{s=1}^{\infty} e^{-is|\eta-\eta'|}\sin s\gamma\left[1-\frac{im^2}{2s}\left|\int_{\eta'}^{\eta} d\eta'' \, b^2(\eta'')\right|\right]+$$

$$+\,G_1(x,x'|\{0\}).$$

The series in the first term diverges everywhere but it has a generalized sum which may be calculated by means of regularization via the substitution $|\eta-\eta'| \to |\eta-\eta'| - i\epsilon,\ \epsilon>0.$ The result is

$$G(x,x'|\{0\}) = \left[8\pi^2 b(\eta)b(\eta')\right]^{-1}\left\{\frac{-1}{\rho-i\epsilon_1} - \frac{im^2}{2\sin\gamma}\ln(\rho-i\epsilon_1)\left|\int_{\eta'}^{\eta} dy\, b^2(y)\right|+\right.$$

$$\left.+\,G_2(x,x')\right\},\ \epsilon_1>0,\ (\text{but } \epsilon_1\neq\epsilon) \quad (29)$$

Here

$$\rho(x,x') = \cos\gamma - \cos(\eta-\eta') \quad (30)$$

a regular function. It is easy to see that the surface $\rho(x,x')=0$ is just $\mathcal{L}C(x')$.

What is a connection between the geodesic interval $\Omega(x,x')$ and $\rho(x,x')$? Suppose that

$$\Omega(x,x') = b(\eta)b(\eta')\mu^{1/2}(x,x')\rho^{\nu}(x,x'), \quad (31)$$

where $\mu(x,x')\neq 0$ when $\rho(x,x')=0$ in a finite neighbourhood of x'. Substitution of this expression into Eq. (4) leads necessarily to $\nu=1$ and to an equation for $\mu(x,x')$ that may be represented in the form

$$\partial^{\alpha}\Omega\,\partial_{\alpha}\mu + \mu^{3/2}\frac{b(\eta')}{b(\eta)}\left[\cos\gamma+\cos(\eta-\eta')+2\frac{\dot b(\eta)}{b(\eta)}\right]-2\mu = \quad (32)$$

$$=\Omega\,\mathcal{F}_1(\eta-\eta',\gamma,b,\dot b,\mu,\partial_{\alpha}\mu).$$

If $x\in\mathcal{L}C(x')$, i.e., $\Omega(x,x')=0$ this is an equation along a null geodesic. In the case of $HJ_{1,3}$ Eq. (5) for Δ may be reduced to the same form of Eq. (32) up to replacement of $\mathcal{F}_1(\eta-\eta',\gamma,b,\dot b,\mu,\partial_{\alpha}\mu)$ by some $\mathcal{F}_2(\eta-\eta',\gamma,b,\dot b,\ddot b,\Delta,\partial_{\alpha}\Delta)$. So $\mu(x,x')$ and $\Delta(x,x')$ satisfy the same equation on $\mathcal{L}C(x')$ and under the condition $\lim_{x\to x'}\mu(x,x')=1$ these functions coincide with each other there. Note that $\Delta(x,x')>0$ and consequently $\mu^{1/2}(x,x')>0$ in a neighbourhood of $\mathcal{L}C(x')$

in the domain under consideration since $\lim_{x \to x'} \Delta(x,x') = 1$ and $\Delta(x,x') = 0$ only for the points of intersection of geodesics radiating from x', i.e., for points which are outside the domain. Therefore we may write now

$$G(x,x'|\{0\}) = (8\pi^2)^{-1}\left\{ \frac{\Delta^{1/2}}{\Omega - i0} - \frac{i m^2 b(\eta) b(\eta')}{2 \sin \gamma} \ln(\Omega - i0) \left| \int_{\eta'}^{\eta} b^2(y)\, dy \right| + \right.$$

$$\left. + G_3(x,x') \right\}, \quad \gamma > 0 .$$

Hence it is seen that $G(x,x'|\{0\})$ has the same singularities as $G_F(x,x')$ (Eq. (2)), and we are left to require that $\tilde{G}(x,x'|\{\lambda_s\})$ and $\tilde{\tilde{G}}(x,x'|\{\lambda_s\})$ should have the regularity properties of $w(x,x')$.

We start with series (27) and write out explicitly the "dangerous" terms

$$\tilde{G}(x,x'|\{\lambda_s\}) = \frac{1}{2} A(\eta,\eta',\gamma) \sum_{s=0}^{\infty} \frac{\sin(s+1)\gamma}{1-|\lambda_s|^2} \left\{ \mathrm{Re}\,\lambda_s \cos(s+1)(\eta+\eta') - \right.$$

$$\left. - \mathrm{Im}\,\lambda_s \sin(s+1)(\eta+\eta') + O\left(\frac{\lambda_s}{s+1}\right) \right\} . \tag{31}$$

Comparison with $G(x,x'|\{0\})$ shows that if the sequence $\{\lambda_s\}$ were arbitrary (but $|\lambda_s| < 1$) $\tilde{G}(x,x'|\{\lambda_s\})$ might have singularities on the light conoids of some points $x'' \not\equiv x'$. For example, if $\lim_{s \to \infty} \arg \lambda_s = 0$ then the singularity can be on $\mathcal{LC}(x'')$ x'' having the coordinates, $-\eta', \xi'$.

According to Sec.2 we have to impose such conditions on behaviour of λ_s for $s \to \infty$ that exclude any singularities from $G(x,x'|\{\lambda_s\})$ but in principle admit singularities of the form of Eq. (15) for $\partial_\alpha G(x,x'|\{\lambda_s\})$. To this end we consider the derivative with respect to γ or η of the trigonometric series in Eq. (33). According to the known theorem, see, e.g. [15], following from the Parseval equality, the sum of the derivative is a square-integrable function of the variables $\eta + \eta' \pm \gamma$ if the series $\sum_{s=0}^{\infty} (s+1)^2 |\lambda_s|^2$ converges and vice versa. This means that at most $\partial_\alpha G(x,x'|\{\lambda_s\})$ may have singularities of the form

$$\partial_\alpha G(x,x'|\{\lambda_s\}) \sim \mathrm{const}\, (\eta + \eta' \pm \gamma - C)^{-\frac{1}{2}}, \quad u < 1 ,$$

C being a constant. This condition coincides with the one of Eq. (15) in view of Eq. (31) with $\nu = 1$ and definition of ρ by Eq. (30) .

Now we recall that representations of CCR defined by sequences $\{\lambda_s\}$ satisfying the condition

$$\sum (s+1)^2 |\lambda_s|^2 < \infty$$

form a class of unitary equivalence that contains the one defined by $\{0\}$ and as we have seen only in this class $G(x,x'|\{\lambda_s\})$ satisfies the minimal requirements on $G_F(x,x')$ of Sec.2. In spite of the particular character of the considered external field the result seems to be interesting for it provides a possible way to essential constraints on the choice of representation of CCR in the general case.

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