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Замечания о сингулярных решениях уравнения Дирака

В работе рассмотрены сингулярные решения уравнения Дирака. Они построены лоренц-ковариантным образом из функций, пропорциональных статическим мультипольным полям скалярного и (или) электромагнитного полей, и регулярных решений уравнений Дирака. Предлагается регуляризационная процедура, которая устраняет расходимости полной энергии, импульса и момента количества движения рассматриваемого спинорного поля.

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Remarks about Singular Solutions to the Dirac Equation

In the paper the singular solutions of the Dirac equation are investigated. They are constructed, in the Lorentz-covariant way, of functions, proportional to the static multipole functions of scalar and (or) electromagnetic fields, and of regular solutions of the Dirac equation. The regularization procedure excluding divergences of total energy, momentum and angular momentum of the considered spinor field is presented.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. Preliminaries and Results

There are analysed bispinors that are solutions to the free particle Dirac equation everywhere but a single point and

a) have (generally) a moving singularity point ,
b) are eigenfunctions of operators H_D (the Dirac energy operator), $\vec{J}^2$, J_z (the operator of the total angular momentum and its z-component, respectively) and the parity operator π in a reference system in which the singularity point is at rest and placed at the origin of the pseudoorthogonal system of coordinates.

The point which is excluded from the definition region of the Dirac eq. is the singularity point of the solutions in question.

It is shown (sec. 3,4) that these solutions may be regarded as linear combinations of products of invariant matrices, constructed out of special solutions to the Maxwell and/or d'Alembert equations, with plane wave solutions to the Dirac free particle equation.

Canonical energy momentum tensor (EMT) constructed (sec. 5) of the bispinors mentioned above may be expressed in terms of a massless scalar field and/or Maxwell field EMT and via terms which appear due to the existence of the spin of the Dirac field. Just the latter terms make the contribution of the "orbital" angular momentum density to the total angular momentum density to be nonzero in the rest (with respect to the singular point) reference frame.

In sec. 6 there is suggested a regularisation procedure which makes (in the case $J = 1/2$) the integral value of the energy and momentum and of the total angular momentum (spin) of the bispinor field discussed to be finite and have the correct value. The regularisation procedure is based on an assumption according to which the Dirac field interacts in a nonpolynomial way with the scalar field which defines (together with the Coulomb field) the solutions studied in secs. 3,4. The regularisation procedure admits geometrical interpretation.

Some of the results involved in this paper were published in a preliminary form in /7/.

2. Notation Conventions

Metrics: $(\eta_{\mu\nu}) = \mathrm{diag}\,(+1, +1, +1, -1)$; $\mu, \nu = 1, \dots, 4$; $i, j = 1, 2, 3$.
Units: $\hbar = c = 1$
Representation for γ-matrices:

$$\gamma^4 \equiv \beta \equiv \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \qquad \gamma^i \equiv \beta \alpha^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad \sigma_i \ \dots\dots \text{ Pauli matrices}$$

$$\gamma_5 \equiv i \gamma^1 \gamma^2 \gamma^3 \gamma^4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2 \eta^{\mu\nu}$$

$$[\gamma^\mu, \gamma^\nu] \equiv \gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu \qquad \gamma^{\mu\nu} \equiv \frac{1}{2i} [\gamma^\mu, \gamma^\nu] \qquad \Sigma_k \equiv \gamma_5\, \alpha_k .$$

Plane wave solutions to the Dirac equation

$$(P + m_0)\, w_\sigma^{(\epsilon)} (\vec{k}, x) = 0$$

$$P \equiv \gamma^\mu P_\mu \ , \quad P_\mu \equiv -i \frac{\partial}{\partial x^\mu} \equiv -i \partial_\mu \ , \quad \sigma = \pm \frac{1}{2} \ , \tag{2.1}$$

with positive energy $(\epsilon = 1)$ and negative energy $(\epsilon = -1)$

$$w_\sigma^{(\epsilon)} (\vec{k}, x) = \sqrt{\left(\frac{1 + \gamma}{2}\right)} \cdot \left\{ \frac{1 + \gamma_5}{2} + \epsilon \frac{1 - \gamma_5}{2} \right\} (1 + \pi \gamma_5) \begin{pmatrix} \chi_\sigma \\ 0 \end{pmatrix} e^{i \epsilon k x} ,$$

$$\gamma \equiv (1 - \vec{v}^2)^{-1/2}, \quad \vec{v} = \vec{k} (1 + \vec{k}^2)^{-1/2} \ , \quad k x \equiv k_\mu x^\mu = \vec{p}\vec{x} - |F| t \ ,$$

$$\chi_\sigma = \begin{pmatrix} 1/2 + \sigma \\ 1/2 - \sigma \end{pmatrix} \ , \quad \pi \equiv \frac{\gamma}{1 + \gamma} \vec{v} \vec{\sigma} \ , \qquad \vec{v} \dots \text{const} \tag{2.2}$$

Proper Lorentz transformation matrix: (L^μ_ν) $\quad x^\mu \equiv L^\mu_\nu x'^\nu$.

Clebsch-Gordan coefficients /4/: $\quad C(L\ \ell\ J; M - m, m)$
Other conventions: $\{U^\mu\} \equiv \{\gamma \vec{v}, \gamma\}$

$$\underset{(0)}{F}(\phi) \equiv \phi \ , \qquad \underset{(1)}{F}(\phi) \equiv \gamma^\mu \partial_\mu \phi \ ,$$

$$\underset{(1)}{F}(A) \equiv \gamma^\mu A_\mu \ , \qquad \underset{(2)}{F}(A) \equiv \frac{1}{2} \gamma^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) \ ,$$

$$\underset{(i)}{\tilde{F}}(X) \equiv \gamma_5 \underset{(i)}{F}(X) \ , \ i = 0, 1, 2, \quad X = \phi, A \ , \tag{2.3}$$

$$\Box \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu \ .$$

Electromagnetic field tensor: $f_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$,

$$f_{4i} = -E_i \qquad f_{ij} = \epsilon_{ijk} H_k .$$

3. Construction of Singular Solutions to the Dirac Equation

Let $g^0(x)$ be a solution to the equation

$$\Box g^0(x) = \frac{1}{\gamma}\delta^3(\vec{x} - \vec{v}x^4 - \vec{\xi}) \tag{3.1}$$

which tends to zero at space infinity

$$g^0(x) \to 0 \quad \text{for} \quad |\vec{x}| \to \infty, \ |x^4| < \infty, \ |\vec{\xi}| < \infty . \tag{3.2}$$

The eq. (3.1) may be written in another form

$$P \underset{(1)}{F}(g^0) = i\frac{1}{\gamma}\delta^3(\vec{x} - \vec{v}x^4 - \vec{\xi}) \equiv \underset{(0)}{J} \tag{3.3}$$

or

$$P \underset{(1)}{\tilde{F}}(g^0) = -i\frac{1}{\gamma}\gamma_5\,\delta^3(\vec{x} - \vec{v}x^4 - \vec{\xi}) \equiv \underset{(0)}{\tilde{J}} . \tag{3.4}$$

Equation (3.1) or, equivalently (3.3) or (3.4), is the equation determining the scalar field of moving (with velocity $\vec{v}$) point source.

We shall also investigate equations

$$\Box U^\mu g^0(x) = -j^\mu(\vec{x} - \vec{v}x^4 - \vec{\xi}) , \tag{3.5}$$

$$\{j^\mu(\vec{\rho})\} \equiv \{\vec{v}\,\delta^3(\vec{\rho}) , \ \delta^3(\vec{\rho})\} \tag{3.6}$$

that one gets multiplying (3.1) by U^μ .

Taking into account easily verified relation

$$U^\mu \partial_\mu g^{(0)}(x) = 0 \tag{3.7}$$

we may rewrite eq. (3.5) into the form

$$P \underset{(2)}{F}(Ug^0) = \gamma^\mu j_\mu(\vec{x} - \vec{v}x^4 - \vec{\xi}) \equiv \underset{(1)}{J} \tag{3.8}$$

or, equivalently,

$$P \underset{(2)}{\tilde{F}}(Ug^0) = -\gamma_5 \underset{(1)}{J} \equiv \underset{(1)}{\tilde{J}} . \tag{3.9}$$

The components of four-vector

$$A^\mu \equiv U^\mu g^0(x) \tag{3.10}$$

are easily recognised to be functions proportional to the Lienard-Wiechert potentials. Equation (3.9) together with its Hermitean conjugate are algebraic consequence of non-homogeneous Bargmann-Wigner equations [1,2] for zero rest-mass vector field and are therefore equivalent to non-homogeneous Maxwell equations (in this connection see also [3]). The same statement is true concerning eq. (3.8) and its Hermitean conjugate.

Multiplying eq. (3.4) or (3.9) by the plane wave solution to the free particle Dirac equation and taking into account relations

$$[\gamma^\mu, \underset{(1)}{\tilde{F}}(g^0)]\,P_\mu w^{(\epsilon)}(x) = 0 , \tag{3.11}$$

$$[\gamma^\mu, \underset{(2)}{\tilde{F}}(A)]\,P_\mu w^{(\epsilon)}(x) = 0 \tag{3.12}$$

we get an interesting result that bispinors

$$\underset{(1)}{\tilde{F}}(g^0)\,w^{(\epsilon)} \equiv \underset{(1)}{\Psi}_\epsilon , \tag{3.13}$$

$$\underset{(2)}{\tilde{F}}(A)\,w^{(\epsilon)} \equiv \underset{(2)}{\Psi}_\epsilon \tag{3.14}$$

are singular solutions to the nonhomogeneous Dirac eq.

$$(P + m_0)\underset{(a)}{\Psi}_\epsilon = -\underset{(a-1)}{\tilde{J}}\,w^{(\epsilon)} , \quad a = 1, 2 . \tag{3.15}$$

The validity of eqs. (3.11-12) is the consequence of the relation (3.7) and of algebraic properties of γ -matrices. An assertion analogous to (3.15) with $\underset{(a-1)}{\tilde{J}} \to \underset{(a-1)}{J}$ on the r.h.s and with $\underset{(a)}{\Psi} \to \underset{(a)}{F}w$ on the l.h.s. is not valid.

Bispinors $\underset{(a)}{\Psi}$ have one singular point and there exists a reference system in which (3.13-14) coincide with bispinors, that one gets from usual quantum mechanical stationary singular (with radial part equal to $k^\ell n_\ell(kr)$, n_ℓ ... spherical Neumann function, $k \equiv |\vec{k}|$) solutions to the free particle Dirac equation by $k \to 0$. When $k \neq 0$ this coincidence does not hold. In order to keep in mind this difference, the bispinors (3.13-14) will be referred to as hybrid bispinors.

Equation (3.15) is not quantum mechanical equation and accepting it as physically reasonable we have to point out what are the advantages (if any) of its solutions.

Fields g^0 and/or

$$f_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.16}$$

may be regarded as classical self-fields of some (maybe hypothetical) particles. To each type of just mentioned classical fields (or to their superposition) we may put into accordance hybrid bispinors. This connection is not one-to-one because as we shall see later, bispinor (3.13) is equal to (3.14) up to a phase factor.

The questions which now arise are:

1) Do some hybrid bispinors correspond to other solutions to the d'Alembert and/or Maxwell equations?
2) The canonical energy momentum tensor (EMT) constructed out of hybrid solutions differs from canonical EMT of a scalar and Maxwell field. Are there some plausible arguments in favour of the first one? In other words, is it reasonable to ascribe EMT of the hybrid bispinor field to the self field of spinning particle?

The answer to the first of this questions is affirmative and will be presented in sec. 4. The second question is more fundamental. The answer, which will be given in secs. 5 and 6, states that EMT of hybrid bispinor field leads to the angular momentum density which has the same type of singularity as the energy and momentum density. Therefore, the same regularization procedure which leads to the correct value of space integral of the total angular momentum density and its components gives the correct value of the total energy (i.e., m_0 in the rest system of the particle; the constant m_0 being the same as in (3.15)). This

is not the case if one starts with EMT of scalar and/or Maxwell field of a point source.

Accepting special solutions to the classical field equations as functions describing (with known degree of accuracy) self-field of some particles and exploiting for construction of conserved quantities functions (hybrid bispinors in our case) which are solutions of another equation, is of course logically inconsistent, in spite of the fact that between both kinds of fields there is a close connection.

The clue for solving the above mentioned inconsistency gives probably the observation that the equation defining the regularised hybrid bispinors may be looked upon as the Dirac equation describing electron which interacts with a scalar field. Accepting this point of view we may not give some importance to the way how the unregularised bispinor may be constructed. The singular solutions after regularisation may be regarded as quantum-mechanical wave functions.

4. Hybrid Bispinors Associated with Static Multipole Fields

In order to simplify further discussion we shall suppose in this section that all sources are placed at the origin of the pseudo-Cartesian coordinate system. Instead of non-homogeneous scalar field and Maxwell equations we shall consider the homogeneous one and work with functions that are their static, decreasing at space infinity solutions everywhere but the single point placed at the origin.

We look for 4x4 matrices W, which are a linear (with real coefficients $M_{(iX)}$, $\tilde{M}_{(iX)}$) combination of eight matrices (2.3)

$$\begin{gathered} W = \sum_{(i,X)} \left(M_{(i,X)} F_{(i)}(X) + \tilde{M}_{(i,X)} \tilde{F}_{(i)}(X) \right), \quad i = 0,1,2, \\ X = \phi, A; \quad M_{(0,A)} = \tilde{M}_{(0,A)} = M_{(2,\phi)} = \tilde{M}_{(2,\phi)} = 0, \end{gathered} \tag{4.1}$$

and for which hybrid bispinors W_w are solutions to the free particle Dirac equation everywhere but a single point (origin).

Every singular static solution of scalar field and Maxwell eqs. may be expressed as a linear combination of irreducible (under the group of 3-dimensional rotations) tensors of rank L -static multipole fields [4]

$$\vec{F}_{Lm} \equiv \vec{\nabla}\, \phi_{Lm}, \qquad \vec{E}_{Lm} \equiv -\vec{\nabla}\, A^4_{Lm}, \qquad \vec{H}_{Lm} \equiv \mathrm{rot}\, \vec{A}_{Lm} = \vec{E}_{Lm}, \tag{4.2}$$

where

$$\phi_{Lm} = -A^4_{Lm} = \frac{1}{r^{L+1}} Y_{Lm}(\theta,\phi)\ ,$$

$$\vec{\nabla}\, \phi_{Lm} = \sqrt{(2L+1)(L+1)}\, \frac{1}{r^{L+2}} \vec{T}_{L,L+1,m}\ ,$$

$$\vec{A}_{Lm} = \frac{1}{L} \vec{r} \wedge \vec{H}_{Lm} = +i \sqrt{\frac{L+1}{L}}\, \frac{1}{r^{L+1}} \vec{T}_{LLm}\ , \tag{4.3}$$

and

$$\vec{T}_{L,\ell,m} \equiv \sum_{n=-1}^{1} C(\ell\ 1\ L;\, m-n, n)\, Y_{\ell\, m-n}\, \vec{\xi}_n, \tag{4.4}$$

(the notation used is based on that in [4]). Operator W constructed out of multipole fields (4.2) with fixed L, m is a component of irreducible tensor operator W_{Lm}.

Now we are looking for sets of constants M for which bispinors

$$\Psi^M_{\epsilon,\tilde{\omega},J} = \sum_{\sigma=-1/2}^{1/2} C(J-\tfrac{1}{2}, \tfrac{1}{2}, J;\, n-\sigma, \sigma)\, W_{J-1/2, M-\sigma}\, w^{(\epsilon)}_\sigma(\vec{k}=0, x) \tag{4.3}$$

are eigenfunctions of operators

$$H_D \equiv \vec{\alpha}\vec{P} + m_0 \beta, \tag{4.4}$$

with the eigenvalue $E = \epsilon m_0$

$$\vec{J}^2,\ J_z \tag{4.5}$$

with the eigenvalues $J(J+1)(J=\frac{1}{2},\frac{3}{2},\ldots)$, and $M(M=J, J-1, \ldots -J)$ respectively, and parity operator π, the eigenvalues of which depend on quantum numbers J and $\tilde{\omega} = \pm 1$ in the following way [5]:

$$\pi \Psi^M_{\epsilon,\tilde{\omega},J} = (-1)^{J+\frac{\tilde{\omega}}{2}}\, \Psi^M_{\epsilon,\tilde{\omega},J}\ . \tag{4.6}$$

We adopt phase conventions

$$\overset{C}{\Psi}{}^M_{\epsilon,\tilde{\omega},J} = (-1)^M\, \Psi^{-M}_{-\epsilon,-\tilde{\omega},J}, \qquad \overset{C}{w}{}^{(\epsilon)}_\sigma = (-1)^\sigma\, w^{(-\epsilon)}_{-\sigma} \tag{4.7}$$

between charge-conjugated bispinors,

$$\overset{C}{\Psi} \equiv C\bar{\Psi}^T, \qquad \overset{C}{w} \equiv C\bar{w}^T, \tag{4.8}$$

(T ..... symbol for transposed matrix).
The first of equations (4.8) is valid if the matrix

$$\overset{C}{W}_{J-\frac{1}{2}, M-\sigma} \equiv C(\gamma^4 W^+_{J-\frac{1}{2}, M-\sigma}\gamma^4)^T C^{-1} \tag{4.9}$$

coincides with $W_{J-\frac{1}{2}, -M+\sigma}$ up to the phase factor

$$\overset{C}{W}_{J-\frac{1}{2}, M-\sigma} = (-1)^{M-\sigma} W_{J-\frac{1}{2}, -M+\sigma}\ . \tag{4.10}$$

This is the case when

$$M_{(1\phi)} = M_{(1A)} = M_{(2A)} = \tilde{M}_{(0\phi)} = 0\ . \tag{4.11}$$

Now it may be easily verified that (4.3) has the desired property in two cases only:

C1/ $\vec{H} = 0$, $M_{(0\phi)} = \tilde{M}_{(1A)} = 0$, $M_{(1\phi)}, \tilde{M}_{(2A)} \ldots$ arbitrary

C2α/ $\vec{E} = 0$, $M_{(0\phi)} = 0$, $\tilde{M}_{(1A)} = 2m_0 \tilde{M}_{(2A)}$, $M_{(1,\phi)} = 0$,

C2β/ $\vec{E} = 0$, $M_{(0\phi)} = \tilde{M}_{(1A)} = \tilde{M}_{(2A)} = 0$, $\tilde{M}_{(1\phi)} \ldots$ arbitrary

If $\vec{E}=0$ in the case C1, then this possibility coincides with $C2\beta$.

Using connection (4.13) between spherical vector harmonics (4.4) and spherical spinors

$$\mathcal{Y}^{M}_{LJ} \equiv \sum_{\sigma} C(L\,\tfrac{1}{2}\,J;\,M-\sigma,\sigma)\,Y_{L\,M-\sigma}\,\chi_{\sigma} \tag{4.12}$$

we may write down the hybrid bispinors in a more customary form. The above-mentioned connection reads

$$\sum_{\sigma=-1/2}^{1/2} C(J'\,\tfrac{1}{2}\,J;\,M-\sigma,\sigma)\,\vec{\sigma}\,\vec{T}_{J'L\,M-\sigma}\,\chi_{\sigma} = K(L,J')\,\mathcal{Y}^{M}_{LJ}\,, \tag{4.13}$$

where

$$K(J+\tfrac{1}{2},J+\tfrac{1}{2}) = -\sqrt{\left(\frac{2J+3}{2J+1}\right)}\,,$$

$$K(J+\tfrac{1}{2},J-\tfrac{1}{2}) = \sqrt{\left(\frac{4J}{2J+1}\right)}\,,$$

$$K(J-\tfrac{1}{2},J+\tfrac{1}{2}) = -\sqrt{\left(\frac{4(J+1)}{2J+1}\right)}\,,$$

$$K(J-\tfrac{1}{2},J-\tfrac{1}{2}) = \sqrt{\left(\frac{2J-1}{2J+1}\right)}\,,$$

$$K(J+\tfrac{3}{2},J+\tfrac{1}{2}) = K(J-\tfrac{3}{2},J-\tfrac{1}{2}) = 0. \tag{4.14}$$

Equalities (4.13) may be verified by using properties of the Wigner coefficients /4/.

The customary form of singular solutions to the Dirac equation is

$$\Psi^{M}_{\epsilon,\tilde{\omega},J} = N(\epsilon,\tilde{\omega}) \begin{pmatrix} \dfrac{1}{r^{J+1+\frac{\tilde{\omega}}{2}}}\,\mathcal{Y}^{M}_{J+\frac{\tilde{\omega}}{2},J}\left[\dfrac{1+\epsilon}{2} + \dfrac{iJ}{m_0}\,\dfrac{(1-\epsilon)(1+\tilde{\omega})}{4}\right] \\ \dfrac{1}{r^{J+1-\frac{\tilde{\omega}}{2}}}\,\mathcal{Y}^{M}_{J+\frac{\tilde{\omega}}{2},J}\left[\dfrac{1-\epsilon}{2} - \dfrac{iJ}{m_0}\,\dfrac{(1+\epsilon)(1-\tilde{\omega})}{4}\right] \end{pmatrix} e^{-i\epsilon m_0 t}\,, \tag{4.15}$$

The constants of integration $N(\epsilon,\tilde{\omega})$ are in our case expressed in terms of constants $\tilde{M}_{(i,X)}$:

$$N(\epsilon,\tilde{\omega}) = 2J(-\epsilon\tilde{M}_{(1\phi)} + i\tilde{M}_{(2A)})\,\frac{1+\epsilon\tilde{\omega}}{2} - i\epsilon\tilde{M}_{(1A)}\,\frac{1-\epsilon\tilde{\omega}}{2}\,. \tag{4.16}$$

The sets of solutions with $(\epsilon=\pm 1, \tilde{\omega}=\pm 1)$ (coherent choice of signs) correspond to the case C1 (and $C2\beta$), sets $(\epsilon=\pm 1, \tilde{\omega}=\mp 1)$ correspond to $C2\alpha$).

All solutions with exception of those with

$$\epsilon = \pm 1, \quad \omega = \pm 1, \quad J = \tfrac{1}{2} \tag{4.17}$$

are square integrable in a domain lying outside a sphere with radius $\rho > 0$ and with the centre placed at origin.

If we admit all but (4.17) solutions (4.15) we see that there do not exist solutions constructed out of the magnetic monopole field. This is the consequence of assumed reality of constants $M_{(iX)}$, $\tilde{M}_{(iX)}$.

Solutions with $J > 1/2$ probably do not correspond to any really existing objects (particles). The reason for this conviction is that states with higher than 1/2 value of quantum number J should be very likely states with energy $E(J)$

$$E(J > \tfrac{1}{2}) > E(J = \tfrac{1}{2}) = m_0\,. \tag{4.18}$$

But if we do not suppose coherent with (4.18) dependence on J of the constant m_0 in Dirac eq., the bispinor (4.15) is not longer solution to the Dirac equation for $J > 1/2$. Solutions with $J = 1/2$ are supposed to correspond to electron when $(\epsilon=+1, \tilde{\omega}=+1)$ and to positron when $(\epsilon=-1, \tilde{\omega}=-1)$.

5. *Energy Momentum Tensor and Angular Momentum Density of the Hybrid Bispinor Field*

In this section we shall give explicit expressions for components of the canonical EMT and angular momentum density of the hybrid bispinor field $(\epsilon=1, \tilde{\omega}=1, J=1/2)$ in terms of the Lienard-Wiechert fields $\vec{E}$, $\vec{H}$ of a uniformly moving electron, i.e. we suppose $\tilde{M}_{(1\phi)} = 0$.

The explicit dependence of $\Psi_\sigma \equiv \Psi^\sigma_{11\frac{1}{2}}$ on $\vec{E}, \vec{H}$ is

$$\Psi_\sigma = \tilde{M}_{(2A)}(-\alpha\vec{H} + i\vec{\Sigma}\vec{E})\, w_\sigma^{(+1)}(\vec{k}, x) . \tag{5.1}$$

Using (5.1) we may easily verify that

$$\mathcal{T}^4_4 \equiv \frac{1}{2}\{\bar{\Psi}_\sigma \gamma^4 P_4 \Psi_\sigma + h.c.\} =$$

$$= \tilde{M}^2_{(2A)}[-m_0\gamma\{\gamma(\vec{E}^2 - \vec{H}^2)\} + \gamma^2 \frac{1}{r'^6}\epsilon_{ijk} x'^j v_k w_\sigma^+ \Sigma_i w_\sigma] , \tag{5.2}$$

where h.c. stands for the Hermitean conjugated term and

$$x'^j = (L^{-1})^j_\beta\, x^\beta$$

(L) ..... the matrix of proper Lorentz transformation, connecting special reference frame chosen with the rest frame. The second term in square brackets in (5.2) is the odd function of coordinates x'^j and is equal to zero in the rest frame of the electron, the term in curly brackets (multiplied by $\tilde{M}^2_{(2A)}$) is just

$$\Psi^+\Psi = \tilde{M}^2_{(2A)}\{\quad\} . \tag{5.3}$$

By direct calculation it may be proved that the following relations for various components of EMT

$$\mathcal{T}^4 \doteq \frac{1}{2}\{\bar{\Psi}_\sigma \gamma^4 P_i \Psi_\sigma + h.c.\} \doteq \frac{\tilde{M}^2_{(2A)}}{r^6}\epsilon_{ijk} x^j \langle S_k \rangle ,$$

$$S_k \equiv \frac{1}{2}\Sigma_k , \qquad \langle S_k \rangle \equiv \chi^+_\sigma S_k \chi_\sigma , \tag{5.4}$$

$$\mathcal{T}^i_4 \doteq \mathcal{T}^i_j \doteq 0 \tag{5.5}$$

are valid in Σ' reference frame (the special choice of coordinate system is expressed by dots over equality signs). (5.4) being the odd function of x^j, the total momentum of the field (5.1) is zero in Σ'. But the orbital momentum density

$$\ell_i \doteq \epsilon_{ijk} x^j \mathcal{T}^4_k = \frac{1}{2}\{\Psi^+ L_i \Psi + h.c.\} \tag{5.6}$$

is an even function

$$\ell_i \doteq \frac{2\tilde{M}^2_{(2A)}}{r^6}\{r^2\delta_{i3} - x^i x^3\} , \tag{5.7}$$

and gives a nonzero (and even infinite if a suitable regularization procedure is not accepted) contribution to the space integral of the total angular momentum density.

The total angular momentum density

$$\mathcal{J}_i \doteq \frac{1}{2}\{\Psi^+_\sigma J_i \Psi_\sigma + h.c.\} \doteq \Psi^+_\sigma \Psi_\sigma \langle S_i \rangle , \tag{5.8}$$

has the same type of singularity as (5.2) and (5.7) and in both cases the same regularization procedure, if it exists, should lead to the correct value of the total energy, momentum and spin (the integral of the total angular momentum density).

In a more general case, when we do not neglect a possible contribution of the scalar field $\phi = g^0$ to the self-field of electron (i.e., when we take $M_{(1\phi)} \neq 0$, $\phi \doteq 1/r$) we may easily prove that the 4-4 component of EMT in the special reference frame is

$$\mathcal{T}^4_4 \doteq -m_0\{\tilde{M}^2_{(1\phi)}(\vec{\nabla}\phi)^2 + \tilde{M}^2_{(2A)}\vec{E}^2\} . \tag{5.9}$$

Eq. (5.4) with $\tilde{M}^2_{(2A)}$ replaced by $\tilde{M}^2_{(1\phi)} + \tilde{M}^2_{(2A)}$ and eq. (5.5) are also valid.

6. *Regularisation through a Nonpolynomial Interaction of the Dirac Field with a Massless Scalar Field*

Let us suppose that the Dirac field interacts with a scalar field Φ in a way similar to that analysed recently in connection with the infinity suppression problem in quantum field theory [6].

The Lagrangian density of the Dirac field is supposed to be

$$\overset{*}{\mathfrak{L}}(\overset{*}{\Psi}) \equiv e^{\rho\Phi}\, \mathfrak{L}(\overset{*}{\Psi}) \qquad \rho \dots \text{const.} \tag{6.1}$$

$$\mathfrak{L}(\overset{*}{\Psi}) \equiv \frac{1}{2}\{\, \overset{*}{\bar{\Psi}}(P + m_0)\overset{*}{\Psi} + \text{h.c.}\}\,. \tag{6.2}$$

The stars over the Ψ's in (6.2) indicate that $\overset{*}{\Psi}$ should now be a solution to the equation

$$(P - i\frac{\rho}{2} F_{(1)}(\Phi) + m_0)\overset{*}{\Psi} = 0\,, \tag{6.3}$$

which is the Euler-Lagrange equation derived from the variational principle

$$\delta_{\overset{*}{\bar{\Psi}}} \int \overset{*}{\mathfrak{L}}\, d^4x = 0\,. \tag{6.4}$$

The substitution

$$\overset{*}{\Psi} = e^{-\frac{\rho}{2}\Phi}\, \Psi \tag{6.5}$$

transforms (6.3) into the free particle Dirac equation. It is easily proved that Lagrangian density (6.1) as well as canonical EMT

$$\overset{*}{\mathcal{T}}{}^{\mu}_{\nu}(\overset{*}{\Psi}) \equiv e^{\rho\Phi}\, \mathcal{T}^{\mu}_{\nu}(\overset{*}{\Psi}) \tag{6.6}$$

transforms into $\mathfrak{L}(\Psi)$ and $\mathcal{T}^{\mu}_{\nu}(\Psi)$, respectively, by substituion (6.5), i.e.,

$$\begin{aligned} \overset{*}{\mathfrak{L}}(\overset{*}{\Psi}) &= \mathfrak{L}(\Psi)\,, \\ \overset{*}{\mathcal{T}}{}^{\mu}_{\nu}(\overset{*}{\Psi}) &= \mathcal{T}^{\mu}_{\nu}(\Psi)\,. \end{aligned} \tag{6.7}$$

EMT (6.6) is a conserved quantity as a consequence of validity of equation of motion (6.3). Now we shall prove

that when special assumption concerning thus far arbitrary scalar function Φ is adopted, we have at our disposal another conserving quantity

$$\mathcal{T}^{\mu}_{\nu}(\overset{*}{\Psi}) = e^{-\rho\Phi}\, \mathcal{T}^{\mu}_{\nu}(\Psi)\,. \tag{6.8}$$

The assumption just mentioned reads

$$\Phi = \phi = \frac{1}{r}\,, \qquad \rho > 0\,. \tag{6.9}$$

The conservation law

$$\partial_{\mu}\mathcal{T}^{\mu}_{\nu}(\overset{*}{\Psi}) = 0 \tag{6.10}$$

is valid because

$$\phi_{,\mu}\mathcal{T}^{\mu}_{\nu}(\Psi) = -\eta^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma}\{p_{\nu}\phi_{,\mu}\bar{w}\gamma^{\mu}w + \partial_{\nu}(\phi_{,\mu}\bar{w}\gamma^{\mu}w)\} \tag{6.11}$$

and relations

$$\phi_{,\mu}\,\bar{w}\gamma^{\mu}w = 0$$

as well as

$$\partial_{\mu}\mathcal{T}^{\mu}_{\nu}(\Psi) = 0$$

are fulfilled identically.

The point is that the total energy and the space integral of the total angular momentum density of the field $\overset{*}{\Psi}$ is finite and have the correct value if (6.8) is regarded to be the correct energy momentum tensor density, the singular ($\epsilon = \pm 1$, $\tilde{\omega} = \pm 1$, $J = 1/2$) bispinors are taken for Ψ's and the constants N, ρ are appropriately chosen. Moreover, the four quantities

$$P_{\mu} \equiv \int \mathcal{T}^{4}_{\mu}(\overset{*}{\Psi})\, d^3x \tag{6.12.}$$

are components of the total energy-momentum four-vector, as may be easily proved using the transformation properties of the EMT

$$\mathcal{T}^{4}_{\mu}(x) = L^{4}_{\rho} L^{\sigma}_{\mu} \mathcal{T}'^{\rho}_{\sigma}(L^{-1}x)\,, \tag{6.13}$$

and the fact that all but 4-4 components of the EMT

$\mathcal{T}'^{\rho}_{\sigma}$ are zeros or odd functions of variables x'^{i} in the rest frame Σ' and therefore

$$P_\mu = \frac{1}{\gamma} L^4_\rho L^\sigma_\mu \int \mathcal{T}'^{\rho'}_{\sigma}(x')\, d^3x' = U_\mu \int \mathcal{T}'^{4}_{4}(x')\, d^3x',$$

$$\{U^\mu\} = \{\vec{v}\,\gamma, \gamma\}. \qquad (6.14)$$

It follows that

$$P_\mu = p_\mu = U_\mu\, m_0,$$

if

$$\int \mathcal{T}^4_4(\overset{*}{\Psi})\, d^3x \doteq \int e^{-\frac{\rho}{r}} \frac{m_0 |N|^2}{r^4}\, d^3x. \qquad (6.15)$$

The desired connection between two arbitrary constants N, ρ is therefore

$$N = e^{i\xi} \sqrt{\left(\frac{\rho}{4\pi}\right)}, \qquad (6.16)$$

where $(i\xi)$ is an arbitrary phase factor.

As an interesting fact we point out that if

$$\rho = r_e \equiv \frac{e^2}{m_0},$$

i.e., if we put the regularisation constant ρ to be equal to the classical radius r_e of the electron, and if $\tilde{M}_{(1\phi)} = \tilde{M}_{(2A)}$ (i.e., if $\xi = \pi/4$) the integrand of (6.15) becomes the sum of two equal parts energy densities of the Coulomb and quasi-Coulomb fields, multiplied by the regularisation function $\exp(-r_e/r)$. The term "quasi-Coulomb field" is used for the static massless scalar field of a point source with the charge f, $|f| = |e|$.

Because Ψ is also the eigenfunction of operators $\bar{J}^2$, J_z we get

$$\int \overset{*}{\Psi}{}^+_\sigma \bar{J}^2 \overset{*}{\Psi}_\sigma\, d^3x \doteq \frac{3}{4}, \quad \int \overset{*}{\Psi}{}^+_\sigma J_i \overset{*}{\Psi}_\sigma\, d^3x \doteq \delta_{i3}\,\sigma$$

as it should be. If we use the regularised EMT in (5.6) and integrate over spatial variables, we get for the components of the "orbital" angular momentum

$$\int \ell_i(\overset{*}{\Psi}_\sigma)\, d^3x \doteq \delta_{i3} \cdot \frac{4}{3}\sigma.$$

The bispinor $\overset{*}{\Psi}_\sigma$ has no singularity, is square integrable function and solves eq. (6.3) in the whole Euclidean 3-dimensional space (with no point excluded). Equation (6.3) may be regarded as the Dirac equation of a free particle in a Riemannian space with conformally flat metrics

$$\overset{*}{g}_{\mu\nu} = e^{\frac{2}{3}\rho\phi}\, \eta_{\mu\nu}$$

($\eta_{\mu\nu}$ metric tensor of Minkowski space).

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References

1. *V. Bargmann, E.P. Wigner. Proc. Nat.* **Acad.** *Sci., (USA), 34, 211 (1948).*
2. *D.Lurie. Particles and Fields (Interscience Pub. 1968).*
3. *A.B.Pestov. JINR Preprint, P2-5798, Dubna, 1971.*
4. *M.E.Rose. Elementary Theory of Angular Momentum, John. Wiley (1967).*
5. *A.Messiah. Quantum Mechanics II. North. Holland P.C. 1962.*
6. *C.J. Isham, A.Salam, J.Strathdee. Phys. Rev., D5, No. 10, 2548 (1972).*
7. *M.Uhlíř. Acta Polytechnica IV. Vědecká konference, 149 (1973); Acta Polytechnica IV. No. 1, 65 (1974).*

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