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THE DIRAC EIGENVALUES NEAR UPPER AND LOWER CONTINUUM

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1. Introduction

The spontaneous production of electron-positron pairs by a strong field of a superheavy nucleus has been of a current interest in the heavy ion physics since 1945, when Pomeranchuk and Smorodinsky /1/ studied the problem of a relativistic electron moving in an electric field of a uniformly charged spherical shell. The numerical solution of the corresponding Dirac equation showed that the $1s_{1/2}$ state reaches the lower continuum once the charge Z of the shell becomes greater than $\approx$ 170 /2/,/3/. The physical interpretation of this phenomenon was given in a pioneering work of Zeldovich and Popov /4/. They showed that when such an eigenvalue approaches the lower continuum and the corresponding eigen-state is vacant, a spontaneous (i.e., without energy input) production of two positrons at infinity occurs, while the region surrounding the nucleus carries the charge of two electrons. (In the quantum field theory one prefers to characterize this proces as a spontaneous decay of the neutral electrodynamical vacuum /5/, /6/.) Recently, after the experimental discovery of these spontaneously produced positrons in uranium nuclei collisions /7/, /8/ the theoretical study of such phenomena has become even more important (cf. also /9/,/10/,/11/ and references given therein).

In order to calculate the corresponding cross sections one has to study in great details the behaviour of the eigenvalue (resp. of the corresponding resonance, when the eigenvalue is already absorbed) near the lower continuum /12/,/13/. Herewith it is worthwhile to stress, that this is just the behaviour of the eigenvalue/ resonance of the corresponding <u>Dirac Hamiltonian</u> which has to be studied, since the field corrections like for instance the vacuum polarization remain small /14/. Also an approach based on the Bethe-Salpeter equation is not necessary /15/. So the spontaneous positron creation by a strong external field is easily reducible to a c-number problem, which requires calculations involving the relativistic quantum mechanics only /16/ (cf. also the review article /17/).

A different motivation for the study of the Dirac's eigenvalues near the continuous spectrum comes from the atomic physics. Due to the continued refinement of the experimental techniques the atomic binding energies with relativistic effects have to be calculated with high accuracy /18/,/19/. Hereby the threshold behaviour, i.e., the absorbtion of an eigenvalue by the upper continuum, is of a particular interest.

There is a lot of activity in both the above directions, where the behaviour of the Dirac's eigenvalues near the upper/lower continua is investigated by various (more or less appropriate) numerical techniques (cf. /20/ - /27/ and references given there). On the other hand we know only few works concerning the discrete spectrum of the Dirac Hamiltonian in a mathematically rigorous manner /28/-/32/.

The aim of the present paper is to give a rigorous mathematical approach to the eigenvalue absorbtion based on the Birman-Schwinger principle. Such an approach was already succesfully used by Klaus, Simon and other authors /33/-/37/, who studied the threshold behaviour in the nonrelativistic Schroedinger case. Since the Schroedinger operator is nothing but a nonrelativistic limit of the Dirac one, it is not surprizing that the same method works also in the relativistic case.

The paper is organized as follows: The three-dimensional Dirac operator is discussed in full detail in Sections 2 and 3. In Section 4 we treat briefly the one-dimensional case, while the concluding remarks are contained in Section 5.

2. Dirac operator in three dimensions - the upper continuum case

Consider on $L^2(R^3) \otimes C^4$ the operator

$$H(\lambda) = H_0 + \lambda V, \tag{1}$$

where H_0 denotes the free Dirac Hamiltonian

$$H_0 = -ic\alpha_i \frac{\partial}{\partial x_i} + \beta mc^2 . \tag{2}$$

Here $\alpha_1, \alpha_2, \alpha_3, \beta$ are the usual 4x4 Dirac matrices satisfying the relations

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}, \quad \alpha_i\beta + \beta\alpha_i = 0, \quad i,j = 1,2,3$$

(we use the standard representation with

$$\beta = \begin{pmatrix} I, & 0 \\ 0, & -I \end{pmatrix}$$

for concrete computations), $m \neq 0$ is the particle mas and c denotes the light velocity. In order to ignore inessential techni - calities we suppose $V \leq 0$, $V \in C_0^\infty(R^3)$, $\lambda > 0$ troughout the paper, though the exponential falloff will be sufficient.

It is well known that $\sigma(H_0) = \sigma_{a.c.}(H_0) = R \setminus (-mc^2, mc^2)$. Since the potential V is relatively compact with respect to H_0 (/38/ ch.V,§ 5.4) we get immediately for the continuous spectrum of $H(\lambda)$

$$\sigma_{cont} H(\lambda) = \sigma_{a.c} H(\lambda) = \sigma(H_0) \tag{3}$$

for all λ. The discrete eigenvalues $E_i(\lambda)$ of $H(\lambda)$ are contained in the spectral gap $(-mc^2, mc^2)$. Fix i and consider the λ dependence of $E_i(\lambda)$. In the region $|E_i(\lambda)| < mc^2$ (i.e., E_i being an isolated eigenvalue) the problem is well studied with rigorous results going back at least to Rellich, Prosser and Kato /38/ - /40/: $E_i(\lambda)$ is an analytic function of λ.

Our task here is to study the situation where as $\lambda \searrow \lambda_i^{(+)}$ ($\lambda \nearrow \lambda_i^{(-)}$) some eigenvalue E_i is absorbed by the upper (lower) continuum

$$\lim_{\lambda \to \lambda_i^+} E_i(\lambda) = mc^2 \tag{4a}$$

$$\lim_{\lambda \to \lambda_i^-} E_i(\lambda) = -mc^2. \tag{4b}$$

Since we have taken $V \leq 0$ we call the first case the coupling constant threshold, while to the second case we refer as to a "critical" coupling behaviour.

As is already mentioned in the introduction the coupling constant threshold is well studied in the nonrelativistic case. But, although the critical behaviour of the eigenvalue near the lower continuum is decisive for the spontaneous positron creation in strong external fields, there are not similar results for the Dirac equation. Due to the nonrelativistic limit one can believe that the coupling constant threshold for the Dirac equation will be similar to this of the Schroedinger equation. In any case we must obtain the results of Klaus and Simon /34/ taking the $c \to \infty$ limit. But things become complicated for the critical behaviour, while this phenomenon is of a pure relativistic nature and has no nonrelativistic counterpart.

Let us begin with the relativistic coupling constant threshold characterized by (4a). Our main tool will be the Birman-Schwinger principle /41/,/42/. This principle says:

" $H(\lambda)$ has an eigenvalue $E(\lambda)$ with multiplicity n if and only if the operator $K_E = |V|^{1/2}(H_0 - E)^{-1}|V|^{1/2}$ has an eigenva - lue $1/\lambda$ with multiplicity n."

The operator K_E is usually called the Birman-Schwinger kernel. In our case K_E is a compact integral operator on $L^2(R^3) \otimes C^4$. Let $\mu_i(E)$ denotes the i-th eigenvalue of K_E

$$K_E f_i = \mu_i(E)\, f_i \quad . \tag{5}$$

Using the Birman-Schwinger principle we find that $E_i(\lambda)$ is deter - mined by the equation

$$\lambda \cdot \mu_i(E) = 1 \quad . \tag{6}$$

Since as a rule K_E defines an analytic family of operators for all $E \in \mathbb{C}$, $\mu_i(E)$ is an analytic function of E. So in order to get $E_i(\lambda)$ we have only to inverte the equation (6) :

$$E_i(\lambda) = \mu_i^{-1}(1/\lambda). \tag{7}$$

Moreover the eigenvectors f_i corresponding to $\mu_i(E)$ are defined by the eigenvectors corresponding to $E_i(\lambda)$

$$f_i = |V|^{1/2}\varphi_i , \tag{8}$$

where

$$H(\lambda)\varphi_i = E_i(\lambda)\varphi_i$$

(cf. /43/ for details). Thus using the Birman-Schwinger principle the investigation of $E_i(\lambda)$ is reduced to the investigation of $\mu_i(E)$.

<u>Lemma 1</u>: Any nonzero eigenvalue of K_E is a strictly monotone increasing function of E.

<u>Proof</u>: cf. /42/ $d\mu_i/dE = \left(f_i, |V|^{1/2}(H_0 - E)^{-2}|V|^{1/2} f_i\right) =$

$$= \left((H_0 - E)^{-1}|V|^{1/2} f_i, (H_0 - E)^{-1}|V|^{1/2} f_i\right) > 0 \text{ ,since}$$

$$K_E f_i = \mu_i(E) f_i \quad , \mu_i(E) \neq 0 \quad . \qquad \blacksquare$$

Lemma 1 and the equation (7) imply

$$dE_i(\lambda)/d\lambda = \frac{d}{d\lambda}\mu_i^{-1}(1/\lambda) = -1/\lambda^2 \left(\frac{d}{dE}\mu_i(E)/_{E=1/\lambda}\right)^{-1} \leq 0.$$

So the eigenvalue $E_i(\lambda)$ once "birth" by the upper continuum decreases monotonically with increasing λ and never returns. Moreover $\frac{d}{d\lambda} E_i(\lambda)/_{\lambda=\lambda_i^+}$ is finite and hence the eigenvalue cannot rise from the continuum with an infinite slope.

In order to proceed further we investigate now the Birman-Schwinger kernel in some more detail . Since we are interested in the behaviour of K_E for E near mc^2 , it is convenient to introduce a new operator $L^+_\varkappa$ defined as follows

$$L^+_\varkappa = K_{c(m^2c^2 - \varkappa^2)^{1/2}} \tag{9}$$

(We denote the eigenvalues of $L^+_\varkappa$ as $\nu_i^+(\varkappa)$). So to investigate K_E with E near mc^2 means to investigate $L^+_\varkappa$ with $\varkappa$ near 0.

<u>Lemma 2</u>: $L^+_\varkappa$ forms a holomorphic family of compact operators for $|\varkappa| < mc$. Consequently the eigenvalues $\nu_i^+(\varkappa)$ are analytic functions of $\varkappa$ in this domain.

<u>Proof</u>: Using the fact that $H_0^2 = c^2(-\Delta + m^2c^2)$ we find /44/

$$(H_0 - E)^{-1} = (1/c^2)(H_0 + E)(-\Delta + m^2c^2 - E^2/c^2)^{-1}. \tag{10}$$

So

$$L^+_\varkappa = |V|^{1/2}(H_0 - c\sqrt{m^2c^2 - \varkappa^2})^{-1}|V|^{1/2} =$$

$$= (1/c^2)|V|^{1/2}(H_0 + c\sqrt{m^2c^2 - \varkappa^2})(-\Delta + \varkappa^2)^{-1}|V|^{1/2} \tag{11}$$

depends analytically on $\varkappa$ for $|\varkappa| < mc$, since $(-\Delta + \varkappa^2)^{-1}$ is a well-known integral operator with kernel

$$(-\Delta + \varkappa^2)^{-1}(x,y) = (1/4\pi)\left(\exp(-\varkappa|x - y|)\right)/|x - y| \; .$$

The compactness of $L^+_\varkappa$ follows directly from the formula (11) and from the assumption $V \in C_0^\infty(R^3)$. $\blacksquare$

<u>Lemma 3</u>: $\lambda_i^+ = 1/\nu_i^+$, where ν_i^+ are all nonzero eigenvalues of $L^+_{\varkappa=0}$.

<u>Proof:</u> Since $\nu_i^+ = \mu_i(mc^2)$ we get $\lambda_i^+ \cdot \mu_i(mc^2) = 1$. For $\lambda > \lambda_i^+$, λ near λ_i^+ we find

$$\lambda \cdot \mu_i(E) = 1$$

for some $E < mc^2$, since (Lemma 1) $\mu_i(E)$ is an increasing function of E. Hence $H(\lambda)$ has an eigenvalue $E_i(\lambda)$ near mc^2 for $\lambda > \lambda_i^+$, for which holds $\lim_{\lambda \to \lambda_i^+} E_i(\lambda) = mc^2$. $\blacksquare$

<u>Consequences</u>:

1/ $H(\lambda)$ has no bound states for λ small enough (in contrary to the one-dimensional case) . This follows immediatelly from the bounded - ness of $L^+_{\varkappa=0}$.

2/ For $c \to \infty$ one must (due to the nonrelativistic limit) obtain the same "critical" coupling constants λ_i^+ as in the Schroedinger case. Using Lemma 3 it is simple to investigate the c dependence of λ_i^+. Inspecting the c dependence of $L^+_{\varkappa=0}$ we find

$$\nu_i^+(c) = \nu_i^+(\infty) + \sigma(1/c^2) .$$

Thus

$$\lambda_i^+(c) = \lambda_i^+(\infty) + \sigma(1/c^2),$$

with $\lambda_i^+(\infty)$ being the Schroedinger "critical" coupling constant. So the first relativistic correction is of the order $1/c^2$, (cf. /45/).

Decomposing $L^+_\varkappa$ we find for small $\varkappa$

$$L^+_\varkappa = L_0^+ + \varkappa L_1^+ + \varkappa^2 L_2^+ + \dots \tag{12}$$

with

$$L_0^+ = (1/4\pi c^2)|V|^{1/2}(H_0 + mc^2)\, T_0 |V|^{1/2} \tag{13a}$$

$$L_1^+ = -(m/2\pi)\, |v^+\rangle\langle v^+| \tag{13b}$$

$$L_2^+ = (1/8\pi c^2)|V|^{1/2}[(H_0 + mc^2)T_1 - (1/m)T_0]\,|V|^{1/2}, \tag{13c}$$

where T_0, T_1 are integral operators with kernels

$$T_0(x,y) = 1/|x-y|$$

$$T_1(x,y) = |x-y|$$

respectively, and $|v^+\rangle\langle v^+|$ denotes a rank-one operator acting as

$$|v^+\rangle\langle v^+| : f \longrightarrow v^+ (f,v^+)$$

with v^+ given by $v^+ = (|V|^{1/2}, |V|^{1/2}, 0, 0)$.

The standard perturbation theory implies now

<u>Lemma 4</u>: $\nu_i^+(\varkappa) = \nu_i^+ + a_i^+\varkappa + b_i^+\varkappa^2 + \sigma(\varkappa^3)$, where

i/ $$a_i^+ = -(m/2\pi)\left[\left(\int_{R^3}\varphi_{i,1}(x)\, V(x)\, d^3x\right)^2 + \left(\int_{R^3}\varphi_{i,2}(x)V(x)d^3x\right)^2\right] / \| \, |V|^{1/2}\varphi_i^+ \|^2 \tag{14}$$

with $\varphi_i^+ = (\varphi_{i,1}^+, \varphi_{i,2}^+, \varphi_{i,3}^+, \varphi_{i,4}^+)$ being the solution of the "critical" Dirac equation

$$\left(-ic\alpha \frac{\partial}{i\partial x_i} + \beta mc^2 + \lambda_i^+ V(x)\right)\varphi_i^+ = mc^2 \varphi_i^+$$

($\|\cdot\|$ is the ordinary $L^2(R^3) \otimes C^4$ norm)

ii/ If $a_i^+ = 0$, then $b_i^+ = \left(|V|^{1/2}\varphi_i^+, L_2^+|V|^{1/2}\varphi_i^+\right) / \| \, |V|^{1/2}\varphi_i^+\|^2$

iii/ If ν_i^+ is degenerate of multiplicity n then for $\varkappa$ near 0 $L^+_\varkappa$ has exactly n eigenvalues $\nu_{i,1}^+, \dots, \nu_{i,n}^+$ near ν_i^+ which are analytic in $\varkappa$. Moreover

$$\nu_{i,j}^+(\varkappa) = \nu_i^+ + a_{i,j}^+\varkappa + b_{i,j}^+\varkappa^2 + O(\varkappa^3),$$

where $a_{i,j}^+$ are zero for all j but one.

<u>Proof</u>: conclusion i/ follows from (12) , (13) and from the first order perturbation theory. Conclusion ii/ is just the second order perturbation theory, where one uses the fact that L_1^+ is of rank one. Conclusion iii/ follows from the degenerate perturbation theory. The numbers $a_{i,j}^+$ are eigenvalues of $P_i L_1^+ P_i$, where P_i is the eigenprojector which corresponds to ν_i^+. The operator L_1^+ is of rank one and this implies the result. ∎

The Lemma 4 represents a starting point for the further investigation of $E_i(\lambda)$ near λ_i^+. Knowing the $\varkappa$ dependence of $\nu_i^+(\varkappa)$ we can solve the equation

$$\nu_i^+(\varkappa) = 1/\lambda \tag{15}$$

obtaining in such a way

$$\varkappa_i(\lambda) = (\nu_i^+)^{-1}(1/\lambda). \tag{15a}$$

The λ dependence of $E_i(\lambda)$ is then given by

$$E_i(\lambda) = c\sqrt{m^2c^2 - \varkappa_i^2(\lambda)} \tag{16}$$

<u>Theorem 1</u>: Let λ_i^+ is the coupling constant at which a unique and nondegenerate eigenvalue $E_i(\lambda)$ approaches the upper continuum

$$\lim E_i(\lambda) = mc^2.$$

Let further φ_i^+ represents the corresponding solution of the "critical" Dirac equation

$$\left(-ic\alpha \frac{\partial}{i\partial x_i} + \beta mc^2 + \lambda_i^+ V(x)\right)\varphi_i^+ = mc^2 \varphi_i^+$$

and a_i^+ is given by the expression (14). Then either

<u>case A/</u> $a_i^+ \neq 0$ in which case $\varphi_i^+ \notin L^2(R^3) \otimes C^4$, $E_i(\lambda)$ is analytic at λ_i^+ and

$$E_i(\lambda) = mc^2 - \frac{(\lambda - \lambda_i^+)^2}{2m\, a_i^+ \lambda^2 \lambda_i^{+2}} + \sigma\left((\lambda - \lambda_i^+)^3\right) \tag{17}$$

or

case B $a_i^+ = 0$, in which case $\varphi_i^+ \in L^2(R^3) \otimes C^4$, $E_i(\lambda)$ is not analytic at λ_i^+ and

$$E_i(\lambda) = mc^2 + \frac{\lambda - \lambda_i^+}{2mb_i^+ \cdot \lambda \cdot \lambda_i^+} + o\left((\lambda - \lambda_i^+)^{3/2}\right), \qquad (18)$$

where $b_i^+ < 0$ is given in conclusion ii/ of Lemma 4 .

If n eigenvalues (counting multiplicity) approach simultaneously the upper continuum for $\lambda \to \lambda_i^+$ then at most one of them is in the case A.

Proof: Case A/ Since $\frac{d}{d\varkappa} \nu_i^+(\varkappa)\Big|_{\varkappa=0} = a_i^+ \neq 0$ (Lemma 4) we find that $\varkappa_i(\lambda)$ given by (15a) is analytic at λ_i^+. Hence $E_i(\lambda)$ is analytic at λ_i^+ and the relation (17) follows dirrectly from (15) and (16). Thus it remains to prove that $\varphi_i^+ \notin L^2(R^3) \otimes C^4$. Using the beautiful result of Klaus and Simon (see Lemma 5 below) we have only to show that

$$\lim_{\lambda \to \lambda_i^+} \frac{E_i(\lambda) - mc^2}{\lambda - \lambda_i^+} = 0$$

which is trivial.

Case B/ Lemma 4 yields now

$$\lim_{\lambda \to \lambda_i^+} \frac{E_i - mc^2}{\lambda - \lambda_i^+} = (1/2mb_i^+ \lambda_i^+) \neq 0 .$$

So (Lemma 5) $\varphi_i^+ \in L^2(R^3) \otimes C^4$ and the relation (18) follows again from (15) and (16) . Thus we have to show that $b_i^+ < 0$ and that $E_i(\lambda)$ is not analytic at λ_i^+ . b_i^+ cannot be zero simultaneously with a_i^+. If $a_i^+ = b_i^+ = 0$ then $\nu_i^+(\varkappa) = \nu_i^+ + c_i^+ \varkappa^3 + ..$ which implies

$$E_i(\lambda) = mc^2 - 1/2m \left(\frac{\lambda_i^+ - \lambda}{\lambda c_i^+ \cdot \lambda_i^+}\right)^{3/2} + \ldots$$

But this is not possible, since $E_i(\lambda)$ rises from the upper continuum with a finite slope (Consequence of Lemma 1) So $b_i^+ \neq 0$. Solving the equation (15) we find

$$\varkappa_i(\lambda) = \pm (1/\lambda_i^+)\left(\frac{\lambda_i^+ - \lambda}{b_i^+}\right)^{1/2} + \ldots \qquad (19)$$

Since $H(\lambda)$ has an eigenvalue for $\lambda > \lambda_i^+$, $\varkappa_i(\lambda)$ must obey /46/

$\varkappa_i(\lambda) > 0$ for $\lambda > \lambda_i^+$, which implies the negativity of b_i^+ . It remains to show that $E_i(\lambda)$ is not analytic at λ_i^+. Since $E_i(\lambda)$ is given by (15) we have to show that $\varkappa_i^2(\lambda)$ is not analytic at λ_i^+ . Due to (19) we have: $\varkappa_i^2(\lambda)$ is analytic at $\lambda_i^+ \iff \varkappa_i(\lambda) = (\lambda - \lambda_i^+)^{1/2} f(\lambda)$ where $f(\lambda)$ is a function analytic at λ_i^+. Since $\varkappa_i$ must be real for $\lambda > \lambda_i^+$ ($H(\lambda)$ has an eigenvalue for these λ) $f(\lambda)$ is an analytic function with real coefficients. Hence $\varkappa_i(\lambda)$ is pure imaginary for $\lambda < \lambda_i^+$ and corresponds in such a way to an eigenvalue imbeded in the upper continuum. But it is known that $H(\lambda)$ has no imbeded eigenvalues for $V \in C_0^\infty(R^3)$ /47/.

The degenerate case follows from the conclusion iii/ of Lemma 4.

Lemma 5: (Klaus and Simon) Let A and B are two self-adjoint operators and B is relatively A-compact, $B \leq 0$. Suppose that A has a gap in the spectrum, i.e., there are numbers a and b , $a < b$, such that $a,b \in \sigma(A)$; $(a,b) \notin \sigma(A)$. Let further $E(\lambda)$ be an eigenvalue of the operator $A + \lambda B$ lieing in the spectral gap (a,b) , $E(\lambda) \in (a,b)$ and suppose that for $\lambda = \lambda_0$ the eigenvalue $E(\lambda)$ approaches the gap endpoint b

$$\lim_{\lambda \searrow \lambda_0} E(\lambda) = b .$$

Then b is not an eigenvalue of $A + \lambda_0 B \iff \lim_{\lambda \to \lambda_0} \frac{E(\lambda) - b}{\lambda - \lambda_0} = 0.$

Proof: The Lemma was first proven by Simon /48/ for semibounded operators A. In the general case the proof was done by Klaus /42/ .

Of course there is nothing special about the point b and we could have taken a as well.

Remarks:

1/ Taking the nonrelativistic limit we get for b_i^+

$$\tilde{b}_i^+ = \lim_{c \to \infty} b_i^+ = (m/4\pi)\Big[\int_{R^3}\int_{R^3} V(x)\,\tilde{\varphi}_{i,1}^+(x)\,|x-y|\,\tilde{\varphi}_{i,1}^+(y)\,V(y)\,d^3x\,d^3y + \int_{R^3}\int_{R^3} V(x)\,\tilde{\varphi}_{i,2}^+(x)\,|x-y|\,\tilde{\varphi}_{i,2}^+(y)\,V(y)\,d^2x\,d^3y\Big]$$

with $\tilde{\varphi}_i^+ = n\lim_{c \to \infty} \varphi_i^+ = (\tilde{\varphi}_{i,1}^+, \tilde{\varphi}_{i,2}^+, 0, 0)$, which is just the result obtained by Klaus and Simon for the coupling constant thres - hold in the nonrelativistic case.

2/ The Theorem 1 can be reformulated in the terminology usually used in the scattering theory /49/. In the case A a single nondegenerate eigenvalue approaches the upper continuum as $\lambda \searrow \lambda_i^+$ and turns into an antibound (virtual) state for $\lambda < \lambda_i^+$, while in the case B an bound and antibound state collide for $\lambda \searrow \lambda_i^+$ producing a resonance pair for $\lambda < \lambda_i^+$.

Things become more simple for a spherically symmetric potential. Using the spherical coordinates one gets for φ_i^+ /51/

$$\varphi_i^+ = \begin{pmatrix} f(r)\,\Omega_{j,l,m} \\ (-1)^{(1+l-\tilde{l})/2}\, g(r)\,\Omega_{j,\tilde{l},m} \end{pmatrix}; \quad \tilde{l} = 2j - l, \qquad (20)$$

where $\Omega_{j,l,m}$ are the two component spherical harmonics

$$\Omega_{j,l,m} = \begin{pmatrix} \sqrt{\frac{j+m}{2j}}\; Y_{l,m-1/2} \\ \sqrt{\frac{j-m}{2j}}\; Y_{l,m+1/2} \end{pmatrix}; \qquad j = l + 1/2$$

$$\Omega_{j,l,m} = \begin{pmatrix} -\sqrt{\frac{j-m+1}{2j+2}}\; Y_{l,m-1/2} \\ \sqrt{\frac{j+m+1}{2j+2}}\; Y_{l,m+1/2} \end{pmatrix}; \qquad j = l - 1/2.$$

Inserting (20) into (14) we find

$$a_i^+ = \begin{cases} 0, & l = 0 \\ -\dfrac{m\left[\int_0^\infty V(r)\, f(r)\, r^2\, dr\right]^2}{2\pi \int_0^\infty V(r)\left(|f(r)|^2 + |g(r)|^2\right) r^2\, dr}, & l = 0. \end{cases}$$

So s-waves are in the case A while $l \geq 1$ waves are in the case B. As we will see in the next section, the situation changes for the lower continuum.

3. Dirac operator in three dimensions - the lower continuum

Let us now investigate the situation when an eigenvalue $E_i(\lambda)$ reaches the lower continuum as $\lambda \nearrow \lambda_i^-$

$$\lim_{\lambda \to \lambda_i^-} E_i = -mc^2.$$

As already mentioned in the introduction, once such an eigenvalue reaches the lower continuum, the system becomes unstable and a spontaneous positron creation occurs. The lifetime of the supercritical system is closely related to the behaviour of $E_i(\lambda)$ near $-mc^2$.

Klaus /28/ has shown, that the eigenvalue $E_i(\lambda)$ moves through the spectral gap $(-mc^2, mc^2)$ in an orderly fashion with increasing λ and do not stuck. In fact he showed the following:

" Let $E_i(\lambda)$ be an eigenvalue of $H(\lambda)$. Then there is a coupling constant λ_i^-, $\lambda_i^- < \infty$ such that $E_i(\lambda) \to -mc^2$ as $\lambda \to \lambda_i^-$ (i.e., the case $\lim_{\lambda\to\infty} E_i(\lambda) = a \in (-mc^2, mc^2)$ is prohibited). "

So any eigenvalue of $H(\lambda)$ will be (sooner or later) absorbed by the lower continuum.

Following the upper continuum case we introduce

$$a_i^- = (m/2\pi)\left[\left(\int_{\mathbb{R}^3} \varphi_{i,3}^-(x)\, V(x)\, d^3x\right)^2 + \left(\int_{\mathbb{R}^3} \varphi_{i,4}^-(x)\, V(x)\, d^3x\right)^2\right] / \| |V|^{1/2} \varphi_i^- \|^2 \qquad (21)$$

$$b_i^- = \left(|V|^{1/2}\varphi_i^-,\; L_2^- |V|^{1/2}\varphi_i^-\right) / \| |V|^{1/2}\varphi_i^- \|^2, \qquad (22)$$

where $\varphi_i^- = (\varphi_{i,1}^-, \varphi_{i,2}^-, \varphi_{i,3}^-, \varphi_{i,4}^-)$ solves the "critical" Dirac equation

$$\left(-ic\alpha \frac{\partial}{i\partial x_i} + \beta mc^2 + \lambda_i^- V(x)\right)\varphi_i^-(x) = -mc^2\, \varphi_i^-(x)$$

and

$$L_2^- = (1/8\pi c^2)\, |V|^{1/2}\left[(H_0 - mc^2)T_1 + (1/m)T_0\right]|V|^{1/2}. \qquad (23)$$

The Birman-Schwinger principle yields

<u>Theorem 2</u>: Let a unique nondegenerate eigenvalue $E_i(\lambda)$ approaches the lower continuum as $\lambda \nearrow \lambda_i^-$

$$\lim_{\lambda \to \lambda_i^-} E_i(\lambda) = -mc^2.$$

Then either

case A/ $a_i^- \neq 0$, in which case $E_i(\lambda)$ is analytic at λ_i^-, $\varphi_i^- \notin L^2(R^3) \otimes C^4$ and

$$E_i = -mc^2 + \frac{(\lambda - \lambda_i^-)^2}{2m(a_i^-)^2 (\lambda_i^-)^2 \lambda^2} + o\left((\lambda - \lambda_i^-)^3\right).$$

or

case B/ $a_i^- = 0$, in which case $b_i^- > 0$, $E_i(\lambda)$ is not analytic at λ_i^-, $\varphi_i^- \in L^2(R^3) \otimes C^4$ and

$$E_i = -mc^2 - \frac{\lambda - \lambda_i^-}{2mb_i^- \lambda_i^- \cdot \lambda} + o\left((\lambda - \lambda_i^-)^{3/2}\right).$$

If n eigenvalues (counting multiplicity) approach the lower continuum for $\lambda \to \lambda_i^-$, than at most one of them is in the case A.

Before proving the theorem 2 we derive an analogue of the Lemma 4. Starting with the Birman-Schwinger kernel K_E we define a new class of operators

$$L^-_\varkappa = K_{-c(m^2c^2-\varkappa^2)^{1/2}} .$$

The same argument as in the upper continuum case imply that $L^-_\varkappa$ form a holomorphic family of operators for $|\varkappa| < mc$. Consequently the eigenvalues $\nu^-_i(\varkappa)$ of $L^-_\varkappa$ are analytic functions of $\varkappa$. Decomposing $L^-_\varkappa$ we find for $\varkappa$ small

$$L^-_\varkappa = L^-_0 + \varkappa L^-_1 + \varkappa^2 L^-_2 + \dots ,$$

where L^-_0, L^-_1, L^-_2 are Hilbert-Schmidt operators defined as follows

$$L^-_0 = (1/4\pi c^2)|V|^{1/2}(H_0 - mc^2)T_0|V|^{1/2}$$

$$L^-_1 = (m/2\pi)|v^-\rangle\langle v^-|$$

with $v^- = (0,0,|V|^{1/2},|V|^{1/2})$. ($L^-_2$ is defined by (23)) The perturbation theory yields

<u>Lemma 6</u>: $\nu^-_i(\varkappa) = \nu^-_i + a^-_i\varkappa + b^-_i\varkappa^2 + \dots ,$ where a^-_i is given by (21). If $a^-_i = 0$ then b^-_i is given by (22). If ν^-_i is a degenerate eigenvalue of $L^-_{\varkappa=0}$ with multiplicity n then $L^-_\varkappa$ has exactly n eigenvalues (counting multiplicity) $\nu^-_{i,j}(\varkappa)$, $j = 1,2,\dots,n$ near ν^-_i for $\varkappa$ small. All these eigenvalues are analytic at 0 and

$$\nu^-_{i,j}(\varkappa) = \nu^-_i + a^-_{i,j}\varkappa + \dots$$

with at most one coefficient a^-_{ij} nonzero.

<u>Proof</u>: See the proof of the Lemma 4. The only modification needed is to replace $L^+_\varkappa$ by $L^-_\varkappa$.

Using Lemma 5 and Lemma 6 the reader can prove the theorem 2 in the same manner as the Theorem 1. Again: there are no eigenvalues imbeded in the lower continuum.

Inserting the decomposition (20) into (21) we find for a spherically symmetric potential

$$a^-_i = \begin{cases} 0 & \text{for } l = 0,2,3,\dots \text{ and for } l = 1,\ j = 3/2 \\ -\dfrac{m\int_0^\infty V(r)\,g(r)\,r^2\,dr}{2\pi\int_0^\infty V(r)\left(|f(r)|^2 + |g(r)|^2\right)r^2\,dr} & \text{for } l=1,\ j=1/2 \end{cases}$$

So only the $p_{1/2}$ states are in the case A while all the other states belong to the case B. The fact that the $p_{1/2}$ states cease to be quadratic integrable when approaching the lower continuum was already asserted in /51/, but it was not really proven there. The behaviour of the eigenvalues in the spherically symmetric case is schematically shown on Fig. 1.

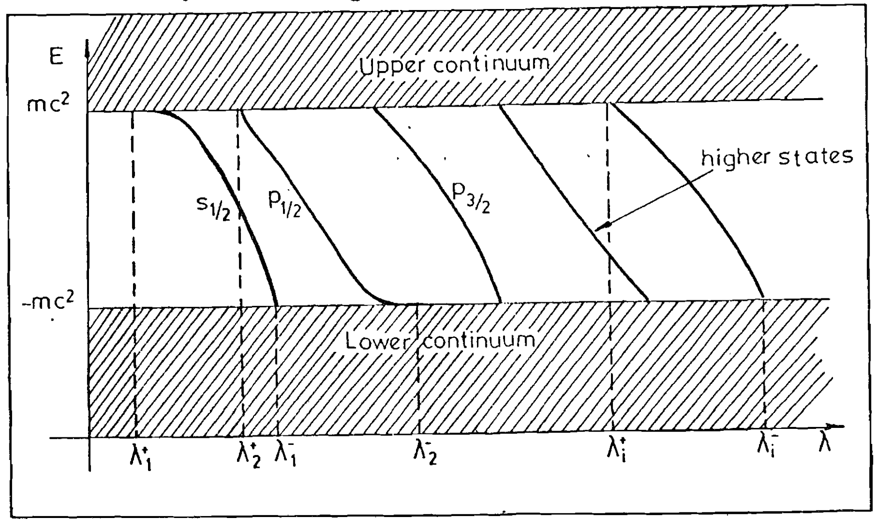

Fig.1

Paraphrazing the theorem 2 in physical terms we get the following image: For $a^-_i \neq 0$ case A the eigenvalue approaching the lower continuum turns into an anbound virtual state. For $a^-_i = 0$ case B the eigenvalue collides with an antibound state at $-mc^2$, producing in such a way a resonance pair or more resonance pairs in the degenerate case in the supercritical regime.

4. The one-dimensional case

Let us now briefly discuss the one-dimensional Dirac equation. We show that the only possible behaviour is this of the type A for the upper as well as for the lower continuum.

The one-dimensional Dirac Hamiltonian is defined on the Hilbert space $\mathcal{H} = L^2(R)\otimes C^2$ by

$$H(\lambda) = H_0 + \lambda V ,$$

where

$$H_0 = -ic\sigma_x\, d/dx + mc^2\sigma_z$$

(σ_x, σ_z are the standard Pauli matrices). We suppose again $V \in C_0^\infty(R)$ and $V \leq 0$ for simplicity.

The Birman-Schwinger kernel reads now

$$K_E = |V|^{1/2}(H_0 - E)^{-1}|V|^{1/2} = (1/c^2)|V|^{1/2}(H_0 + E)\cdot$$

$$\cdot\left(-d^2/dx^2 + m^2c^2 - E^2/c^2\right)^{-1}|V|^{1/2},$$

where $(-d^2/dx^2 + \varkappa^2)^{-1}$ is the well-known one-dimensional Schroedinger resolvent defined by the kernel

$$(-d^2/dx^2 + \varkappa^2)^{-1}(x,y) = (1/2\varkappa)\exp(-\varkappa|x-y|).$$

(The one-dimensional Dirac resolvent is calculated in full details in /52/,/53/.) Introducing the operators $L^\pm_\varkappa$

$$L^\pm_\varkappa = K_{\pm c(m^2c^2 - \varkappa^2)^{1/2}}$$

we get the following natural decomposition

$$L^\pm_\varkappa = L^\pm_{1,\varkappa} + L^\pm_{2,\varkappa},$$

where $L^\pm_{1,\varkappa}$ is a rank-one operator family with a pole at $\varkappa = 0$

$$L^\pm_{1,\varkappa} = (m/\varkappa)|v^\pm\rangle\langle v^\pm|$$

with $v^+ = (|V|^{1/2}, 0)$; $v^- = (0, |V|^{1/2})$
and

$$L^\pm_{2,\varkappa} = (1/c^2)|V|^{1/2}(H_0 \pm c(m^2c^2 - \varkappa^2)^{1/2})K(\varkappa)|V|^{1/2} \pm$$

$$\pm \frac{(m^2c^2 - \varkappa^2)^{1/2} - mc}{2c\varkappa}\, |v^+ + v^-\rangle\langle v^+ + v^-|.$$

Here $K(\varkappa)$ is an integral operator on $\mathcal{H}$ with kernel

$$K(\varkappa,x,y) = (\exp(-\varkappa|x-y|) - 1)/2\varkappa.$$

The operator $L^\pm_\varkappa$ decomposes in such a way into a part $L^\pm_{2,\varkappa}$ which is holomorphic for $\varkappa$ near 0 and a part $L^\pm_{1,\varkappa}$ which is singular at 0. As we will see later, this singular part is responsible for the coupling constant threshold at $\lambda = 0$.

Investigating the eigenvalues $\nu^\pm_i(\varkappa)$ of $L^\pm_\varkappa$ we get

Lemma 7: There is exactly one nondegenerate "great" eigenvalue $\nu^\pm_1(\varkappa)$ which behaves like

$$\nu^\pm_1(\varkappa) = \mp(m/\varkappa)\int_R V(y)\,dy + O(1)$$

for $\varkappa$ small. All the others eigenvalues are analytic at $\varkappa = 0$.

Proof: Since $L^\pm_{2,\varkappa}$ is bounded for $|\varkappa| < \delta$ $(L^\pm_{2,\varkappa} - z)^{-1}$ exist for $|z| > R = 2\max\|L^\pm_{2,\varkappa}\|$. So

$$(L^\pm_\varkappa - z)^{-1} = \left(1 + (L^\pm_{2,\varkappa} - z)^{-1}L^\pm_{1,\varkappa}\right)^{-1}(L^\pm_{2,\varkappa} - z)^{-1} \qquad (24)$$

and we find that all the poles of $(L^\pm_\varkappa - z)^{-1}$ (and hence the eigenvalues of $L^\pm_\varkappa$) are determined by the term

$$\left(1 + (L^\pm_{2,\varkappa} - z)^{-1}L^\pm_{1,\varkappa}\right)^{-1}$$

for those z. Since $L^\pm_{1,\varkappa}$ is of rank-one we have

$$\left(1 + (L^\pm_{2,\varkappa} - z)^{-1}L^\pm_{1,\varkappa}\right)^{-1} = 1 -$$

$$- \frac{(L^\pm_{2,\varkappa} - z)^{-1}L^\pm_{1,\varkappa}}{(m/4\varkappa)\left((\sigma_z \pm 1)|V|^{1/2}; (L^\pm_{2,\varkappa} - z)^{-1}(\sigma_z \pm 1)|V|^{1/2}\right) + 1}$$

and the eigenvalues of $L^\pm_\varkappa$ for which hold $|\nu^\pm_i(\varkappa)| > R$ are determined by

$$f^\pm(z) = \pm(m/4\varkappa)\left(|V|^{1/2}(\sigma_z \pm 1), (L^\pm_{2,\varkappa} - z)^{-1}(\sigma_z \pm 1)|V|^{1/2}\right) = -1. \quad (25)$$

Since $f^\pm(z)$ is a monotonic function of z

$$f^\pm(z)' = \pm(m/4\varkappa)\left(|V|^{1/2}(\sigma_z \pm 1), (L^\pm_{2,\varkappa} - z)^{-2}(\sigma_z \pm 1)|V|^{1/2}\right) \gtrless 0$$

we find that (25) has at most one solution. Hence there is at most one eigenvalue $\nu^\pm_1(\varkappa)$, $|\nu^\pm_1(\varkappa)| > R$ for $|\varkappa| < \delta$.

Decomposing $(L^\pm_{2,\varkappa} - z)^{-1}$ into a geometrical serie

$$(L^\pm_{2,\varkappa} - z)^{-1} = -(1/z)\left(1 - (1/z)L^\pm_{2,\varkappa}\right)^{-1} = -1/z + (1/z^2)L^\pm_{2,\varkappa} + \dots$$

we find

$$f^\pm(z) = \pm(m/\varkappa z)\int_R V(y)\,dy \pm (m/4\varkappa z^2)\left(|V|^{1/2}(\sigma_z \pm 1), L^\pm_{2,\varkappa}(\sigma_z \pm 1)|V|^{1/2}\right) + \dots$$

and from (25) finally

$$z = \pm(m/\varkappa)\int_R V(y)\,dy + O(1),$$

which proves the first part of the lemma.

It remains to show that the remaining eigenvalues are analytic at 0. From the above analysis it follows immediatelly, that they are smaller than R: $|\nu^\pm_i(\varkappa)| < R$, $i > 1$, $|\varkappa| < \delta$.
So we can introduce the operator projecting on the eigenspaces corresponding to $\nu^\pm_i(\varkappa)$, $i > 1$ by /34/,/38/

$$P^\pm(\varkappa) = (1/2\pi i)\oint(L^\pm_\varkappa - z)^{-1}dz.$$

The operator $\tilde{L}^\pm_\varkappa = P^\pm(\varkappa)L^\pm_\varkappa P^\pm(\varkappa)$ is analytic for $\varkappa$ in the punctured neighbourhood of 0 and is bounded for $\varkappa = 0$. So $\tilde{L}^\pm_\varkappa$ is analytic at $\varkappa = 0$. Since the eigenvalues $\nu^\pm_i(\varkappa)$, $i > 1$ of $L^\pm_\varkappa$ are exactly the same as those of $\tilde{L}^\pm_\varkappa$, their analyticity at 0 is proven.. ∎

Let now $E_i(\lambda)$ denotes the i-th eigenvalue of $H(\lambda)$. The Birman-Schwinger principle tells us, that the λ dependence of $E_i(\lambda)$ is managed by the $\varkappa$ dependence of $\nu_i^{\pm}(\varkappa)$ (cf. (15) and (16)). We obtain in such a way

Theorem 3: Let the eigenvalue $E_i(\lambda)$ approaches the upper (lower) continuum for $\lambda \searrow \lambda_i^+$ $(\lambda \nearrow \lambda_i^-)$, $\lambda_i^{\pm} \neq 0$

$$\lim_{\lambda\to\lambda_i^{\pm}} E_i(\lambda) = \pm mc^2 .$$

Then $E_i(\lambda)$ is analytic at $\lambda_i^{\pm}$, $E_i(\lambda) = \pm mc^2 \mp A_i^{\pm}(\lambda-\lambda_i^{\pm})^2 + \sigma(\lambda-\lambda_i^{\pm})^3$ with $A_i^{\pm} > 0$ and $\pm mc^2$ is not an eigenvalue of $H(\lambda_i^{\pm})$. Moreover there is exactly one nondegenerate eigenvalue $E_1(\lambda)$ approaching the upper continuum as $\lambda \to 0$

$$E_1(\lambda) = mc^2 - \frac{\lambda^2}{2}\left(\int_{\mathbb{R}} V(x)\,dx\right)^2 + \dots$$

$E_1(\lambda)$ is analytic at 0.

Expressing the Theorem 3 schematically we get the Fig. 2

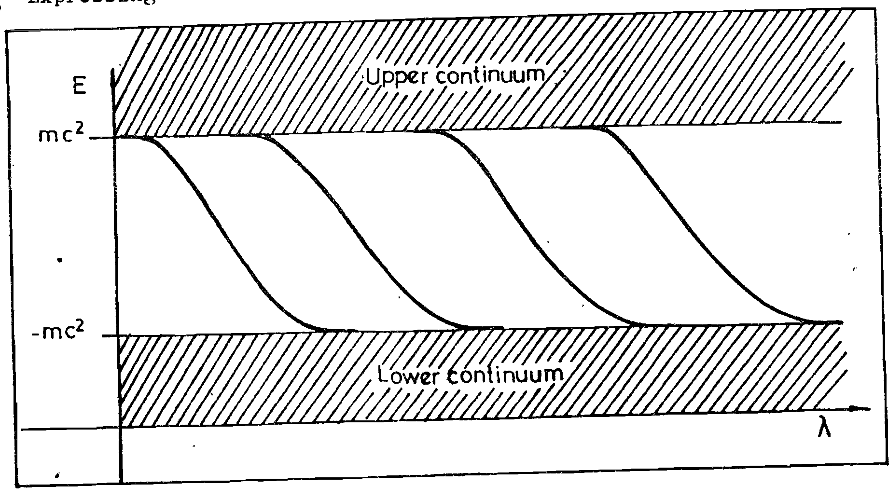

Fig.2

Proof: We prove the upper continuum case only. Solving the "critical" Dirac equation $H(\lambda_i^+)\varphi_i^+ = mc^2\varphi_i^+$ we find $\varphi_i^+ = (\varphi_{i,1}^+, \varphi_{i,2}^+)$ with

$$\varphi_{i,1}^+ = a + bx$$

$$\varphi_{i,2}^+ = c$$

for $|x| \to \infty$ (a,b,c are some constants). So φ_i^+ is never normalizable and the Klaus-Simon lemma imply

$$\lim_{\lambda\to\lambda_i^+} \frac{E_i(\lambda) - mc^2}{\lambda - \lambda_i^+} = 0 \qquad (26)$$

for all i. In order to prove the analycity of $E_i(\lambda)$ at λ_i^+ we apply the Birman-Schwinger principle. Using the relations (15) and (16) we see that

$$E_i(\lambda) = c\sqrt{m^2c^2 - \varkappa_i^2(\lambda)},$$

where $\varkappa_i$ solves

$$\lambda \cdot \nu_i^+(\varkappa) = 1 .$$

So $\varkappa_i(\lambda) = (\nu_i^+)^{-1}(1/\lambda)$ with ν_i^+ being an analytic function at 0 (Lemma 7)

$$\nu_i^+(\varkappa) = \nu_i^+ + a_i^+\varkappa + b_i^+\varkappa^2 + \dots$$

Moreover regarding (26) we have $\varkappa_i(\lambda) \approx C(\lambda - \lambda_i^+)$ with $C \neq 0$, which imply $a_i^+ \neq 0$. Thus $\frac{d}{d\varkappa}\nu_i^+(\varkappa)/_{\varkappa=0} = a_i^+ \neq 0$ which yields the analyticity of $\varkappa_i(\lambda)$ at λ_i^+ and hence the analyticity of $E_i(\lambda)$ The second part of the lemma is an easy consequence of the first part of the Lemma 7

5. Conclusions

As we can see from Theorems 1,2 and 3 the behaviour of the type A is generic. It is the only possible "threshold" behaviour in the one-dimensional case and it occurs also in the three-dimensional case unless a certain integral involving the solution of the "critical" Dirac equation vanishes. This has some dirrect consequences for the spontaneous positron creation in supercritical fields. It means that in the generic case the spontaneous positron production occurs already for the critical coupling $\lambda = \lambda_i^-$ (The corresponding wavefunction delocalizes for $\lambda = \lambda_i^-$ It was sometimes conjectured in the literature /17/, /54/ that such a delocalization takes place only for $\lambda > \lambda_i^-$, while the solution of the "critical" Dirac equation remains quadratic integrable. As we can see, this conjecture is true only for spherically symmertic potentials (but not for $p_{1/2}$ states), where a non-generic behaviour of the type B occurs. Moreover in the generic case the bound state turns into a virtual state for $\lambda > \lambda_i^-$ ($E_i(\lambda)$ is analytic at λ_i^-) . So there is not a non-zero resonant peak for spontaneous positron production in the case A for $\lambda > \lambda_i^-$ and $\lambda - \lambda_i^-$ small enough. On the other hand such a peak exists in the case B as observed for instance for the $s_{1/2}$ absorption in a spherically symmetric square well potential /17,p.259/.

Unfortunately, the Birman-Schwinger principle is not applicable for Coulomb potentials. The reason is that the continuous spectrum of K_E is not empty in this case see /55/ and so the above analysis breaks down.(The quantum electrodynamics is also not applicable in this case since the Bogolubov transform does not exist in a rigorous mathematical sense /55/,/56/.) Similar difficulties are present also for general long-range potentials, where some new phenomena ,like for instance the accumulation of eigenvalues near the upper continuum , appear. The difficulties connected with the long-range forces (which are cumbersome for the rigorous mathematical treatment) were recently reduced by replacing these potentials by a sum of separable potentials /57/,/58/. It was shown that this approach is able to reproduce many aspects of heavy ion collisions. Therefore it seems that the restriction to short-range potentials, which is crucial for the Birman-Schwinger approach, is not very essential from the physical point of view. Using these potentials one can describe the spontaneous positron production as well as by long-range ones.(The Birman-Schwinger principle works, of course, also for separable potentials.)

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Шеба П. E2-86-808

Дираковские собственные значения вблизи верхнего и нижнего континуумов

Современное исследование задач во внешних полях привело к изучению поведения собственных значений оператора Дирака вблизи верхнего и нижнего континуума. Обсуждается также связь этого поведения с проблемой рождения позитронов в интенсивных внешних полях.

Работа выполнена в Лаборатории теоретической физики, ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1986

Šeba P. E2-86-808

The Dirac Eigenvalues near Upper and Lower Continuum

Recent interest in external field problem led us to investigate the behaviour of eigenvalues of the Dirac operator near the upper and lower continuum. The connection with the spontaneous positron production in superstrong external fields is also discussed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1986