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A NON-RELATIVISTIC MODEL OF TWO-PARTICLE DECAY. The Pole Approximation

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1. Introduction

This is the third part of the paper devoted to analysis of a Lee-type decay model. The previous parts[1,2] are referred to hereafter as I and II, respectively. The model is described in I, where its Galilean invariance has been proven. In II, we have separated the centre-of-mass motion; then we have shown that the reduced resolvent of the Hamiltonian has just one simple second-sheet pole $z_p = \lambda_p - i\delta_p$ for small enough values of the coupling constant g.

Here we continue the analysis, with the main attention concentrated on the reduced evolution operator and the decay law. It can be checked easily that these two functions are exponential if one replaces the (analytically continued) reduced resolvent by the pole term of its Laurent expansion. Our aim is to estimate how accurate such an approximation could be. In order to get an explicit bound on the non-exponential corrections, we must require the function v additionally to have some mild regularity properties. They serve us to prove some technical lemmas collected in Section 2, which are used afterwards to derive our main results, in particular, the estimate expressed by Theorem 3.2. As a consequence of this result, we obtain in Section 4 bounds on the deviations of the decay law from exponentiality, and on the difference between the actual decay rate and $2\delta_p$. They give no restriction for very small and very large times, but between these two extremes, they can be used in the weak-coupling limit, being proportional to g^4 and g^6, respectively. This fact in turn justifies validity of Fermi golden rule for our model. In conclusion, we comment on another estimate of the pole-approximation error which could be obtained by adapting the method used by Demuth[3] for the Friedrichs model; we compare the two cases.

The next part of the paper will deal with spectral concentration and with the scattering theory for our model. As earlier, we refer to formulae, theorems, etc. of the preceding parts by adding I or II to

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their number. However, a comment should be made. For the printed version, due to appear in Czech.J.Phys.B, the material has been reorganized ; in the original preprint version the part II was just a short addendum. We make references here to the definitely shaped text ; hence (II.2.9) means the formula (I.4.9) of the preprint version, Theorem II.3.6 corresponds to I.5.6 in the preprint version, etc.

2. A few lemmas

Before formulating and proving the main result, we need some technical preliminaries. Let us mention first the assumptions. As we have remarked, we must add new requirements to those used in II . In order to make the exposition self-contained, we present the full list :

Assumptions 2.1 : (a) v is rotationally invariant, $\hat{v}(\vec{p}) = \hat{v}_1(p)$ for some $\hat{v}_1 \in L^2(\mathbb{R}^+, p^2 dp)$ and all $\vec{p} \in \mathbb{R}^3$,

(b) for brevity, let us introduce the functions $v_2 : v_2(p) = |\hat{v}_1(p)|^2 p$ and $v_3 : v_3(\lambda) = v_2(\sqrt{2m\lambda})$. Then we assume that v_3 can be continued analytically to a region $\Omega \subset \mathbb{C}$ whose intersection with $\mathbb{R}$ contains the point E ,

(c) $\hat{v}_1(\sqrt{2mE}) \neq 0$,

(d) there is a positive C_1 such that $|\hat{v}_1(p)|^2 \leqslant C_1$, $v_2(p) \leqslant mC_1$ and $|v_2'(p)| \leqslant C_1$ for all $p \in (0,\infty)$,

(e) the region Ω in (b) is such that $\Omega \supset (0,\infty)$,

(f) in addition to (d), we have $m|v_2''(p)| \leqslant C_1$ for all $p \in (0,\infty)$.

Notice that if (e) is valid, then the assumptions (d) and (f) are in fact restrictions to the behaviour of $\hat{v}_1$ in a neighbourhood of the points 0 and ∞ only.

Our main concern is the function r_u defined by the relations (II.3.4) and (II.3.5) which characterizes the reduced resolvent,

$$r_u(z,H_g) = \left[E - z + g^2 G(z)\right]^{-1} \quad , \tag{2.1}$$

where

$$G(z) = 4\pi \int_0^\infty \frac{v_2(p)}{z - \frac{p^2}{2m}}\, p\, dp = 4\pi m \int_0^\infty \frac{v_3(\lambda)}{z-\lambda}\, d\lambda \quad , \tag{2.2}$$

and its analytic continuation $r_u^\Omega(\cdot,H_g)$ from the upper halfplane $\mathbb{C}^+ \equiv \{z : \mathrm{Im}\, z > 0\}$ to $\mathbb{C}^+ \cup \Omega$ referring to the continuation G_Ω of G – cf. (II.3.11) and (II.3.15). For simplicity, we shall drop mostly H_g writing $r_u(z)$ instead of $r_u(z,H_g)$. Now we are going to derive a series of four lemmas.

Lemma 2.2 : Assume (a) and (d), then the inequality

$$|G(\xi \pm i\eta)| < C_2 \tag{2.3a}$$

holds for all $\xi \in \mathbb{R}$ and $\eta > 0$, where

$$C_2 = 4\pi^2 m\, C_1^{2/3} \left[m^2 C_1^{2/3} + 9 \left(\frac{2e^2+1}{\sqrt{2}\, e^2 \pi^2} \right)^{4/3} \|v\|^{4/3} \right]^{1/2} \quad . \tag{2.3b}$$

Proof : For any $z \in \mathbb{C} \setminus \mathbb{R}$, we have

$$\overline{G(z)} = G(\bar{z}) \quad , \tag{2.4a}$$

so it is possible to restrict to the upper sign only. The relation (2.2) gives the expression

$$G(\xi + i\eta) = -4\pi m \int_{-\xi}^{\infty} \frac{(x+i\eta) v_3(x+\xi)}{x^2+\eta^2}\, dx$$

which makes it possible to estimate the imaginary part immediately,

$$|\mathrm{Im}\, G(\xi+i\eta)| \leqslant 4\pi m^2 C_1 \int_{-\xi}^{\infty} \frac{\eta\, dx}{x^2+\eta^2} < 4\pi^2 m^2 C_1 \quad . \tag{2.5}$$

The real part is more difficult to manage. We choose a number $a > 0$ and distinguish four cases :

(i) $\xi \geqslant a$: then we split the expression for $\mathrm{Re}\, G(\xi+i\eta)$ into three parts,

$$\mathrm{Re}\, G(\xi+i\eta) = -4\pi m \left\{ J(-\xi,-a) + J(-a,a) + J(a,\infty) \right\} \quad , \tag{2.6a}$$

where

$$J(b,c) = \int_b^c \frac{x v_3(x+\xi)}{x^2+\eta^2}\, dx \quad . \tag{2.6b}$$

The first and the third term can be then estimated as

$$|J(-\xi,-a) + J(a,\infty)| \leqslant \frac{1}{a} \int_{-\xi}^{\infty} v_3(x+\xi)\, dx = \frac{\|v\|^2}{4\pi m a} \quad .$$

Since the function v_2 is real-valued, we have $v_2(\sqrt{2m(x+\xi)}) = v_2(\sqrt{2m\xi}) + v_2'(\sqrt{2m(\xi+\vartheta)})(\sqrt{2m(x+\xi)} - \sqrt{2m\xi})$ with ϑ between 0 and x , so

$$|J(-a,a)| \leqslant \int_{-a}^{a} |v_2'(\sqrt{2m(\xi+\vartheta)})| \frac{x}{x^2+\eta^2} \frac{2mx}{\sqrt{2m(x+\xi)} + \sqrt{2m\xi}}\, dx < 2C_1\sqrt{2ma} \quad .$$

Together we get

$$|\mathrm{Re}\, G(\xi+i\eta)| < \frac{1}{a}\|v\|^2 + 8\pi m C_1 \sqrt{2ma} \quad . \tag{2.7a}$$

(ii) $0 \leq \xi < a$: in this case we replace the curly bracket in (2.6a) by $J(-\xi,\xi)+J(\xi,a)+J(a,\infty)$. The first and the last integral can be estimated as above,

$$|J(-\xi,\xi)| < 2C_1\sqrt{2m\xi} < 2C_1\sqrt{2ma}$$

and

$$|J(a,\infty)| \leq \frac{1}{a}\int_a^\infty v_3(x+\xi)\,dx < \frac{\|v\|^2}{4\pi ma} \quad .$$

Finally, for the middle term we get

$$0 \leq J(\xi,a) \leq C_1\int_\xi^a \frac{x}{x^2+\eta^2}\sqrt{2m(x+\xi)}\,dx \leq \sqrt{2m}\,C_1\int_\xi^a \frac{x(\sqrt{x}+\sqrt{\xi})}{x^2+\eta^2}\,dx <$$

$$< \sqrt{2m}\,C_1\int_\xi^a\left(\frac{1}{\sqrt{x}}+\frac{\sqrt{\xi}}{x}\right)dx \quad .$$

The rhs can be easily calculated and shown to be bounded by $2C_1\sqrt{2ma}\,(1+e^{-2})$ from above ; it yields

$$|\mathrm{Re}\ G(\xi+i\eta)| < \frac{1}{a}\|v\|^2 + 8\pi m C_1(2+e^{-2})\sqrt{2ma} \quad . \tag{2.7b}$$

(iii) if $-a \leq \xi < 0$, we have to estimate $J(-\xi,\infty)=J(-\xi,a)+J(a,\infty)$. Since

$$0 \leq J(-\xi,a) < \sqrt{2m}\,C_1\int_{-\xi}^a \frac{dx}{\sqrt{x}} < 2C_1\sqrt{2ma} \quad ,$$

we get

$$|\mathrm{Re}\ G(\xi+i\eta)| < \frac{1}{a}\|v\|^2 + 8\pi m C_1\sqrt{2ma} \quad . \tag{2.7c}$$

(iv) finally, if $\xi < -a$, we use $J(-\xi,\infty) \leq J(a,\infty)$ obtaining in this way

$$|\mathrm{Re}\ G(\xi+i\eta)| < \frac{1}{a}\|v\|^2 \quad . \tag{2.7d}$$

Putting now the estimates (2.7) together, we see that (2.7b) holds true for every $\xi \in \mathbb{R}$. Next one has to optimize its rhs with respect to a : it gives

$$|\mathrm{Re}\ G(\xi+i\eta)| < 3\left[4\pi m\sqrt{2m}\,C_1\,(2+e^{-2})\|v\|\right]^{2/3} \quad , \tag{2.8}$$

and combining this inequality with (2.5), we arrive at (2.3). ∎

Lemma 2.3 : Assume (a)-(d), then there are positive numbers $\varkappa_1$ and η_0 such that to each g , $0<|g|<\varkappa_1$, a function $u_g \in L^1(\mathbb{R})$ exists, which fulfils the inequality

$$|\mathrm{Im}\ r_u(\xi \pm i\eta, H_g)| \leq u_g(\xi) \tag{2.9}$$

for all $\xi \in \mathbb{R}$ and $\eta \in (0,\eta_0)$.

Proof : Since

$$\overline{r_u(z)} = r_u(\bar{z}) \tag{2.4b}$$

holds for all $z \in \mathbb{C}\setminus\mathbb{R}$, we may again consider the upper sign only. If $\eta \in (0,\eta_0)$, where η_0 has yet to be specified, the relation (2.1) together with the preceding lemma give

$$|\mathrm{Im}\ r_u(\xi+i\eta)| = \frac{|\eta - g^2\,\mathrm{Im}\,G(\xi+i\eta)|}{|E-\xi-i\eta+g^2G(\xi+i\eta)|^2} < \frac{\eta_0+g^2C_2}{[E-\xi+g^2\,\mathrm{Re}\,G(\xi+i\eta)]^2} \quad . \tag{2.10}$$

Now one employs the analytically continued function G_{Ω_2} ; we are interested in zeros of the function $f : f(g,z) = -E+z-g^2G_\Omega(z)$. According to Theorem II.3.6, there is a complex neighbourhood $\Omega_2 = (E-2\delta,E+2\delta)\times(-2\eta_0,2\eta_0)$ of the point E and a positive $\varkappa_2$ such that for each g , $0<|g|<\varkappa_2$, there is just one $z_p(g)$ fulfilling $f(g,z_p(g))=0$. Furthermore, the assumption (c) together with Theorem II.3.6 imply existence of a positive $\varkappa_3$ such that $\mathrm{Im}\ z_p(g)<0$ for $0<|g|<\varkappa_3$. We can always choose $\varkappa_3 \leq \varkappa_2$, and therefore $f(g,\xi+i\eta) \neq 0$ for $0<|g|<\varkappa_3$, $\xi \in (E-2\delta,E+2\delta)$ and $\eta \in [0,2\eta_0)$, because the only zero lies then in the lower part of the rectangle Ω_2. It further implies that

$$m_g \equiv \min\left\{|f(g,\xi+i\eta)| \ :\ |E-\xi|\leq\delta,\ 0\leq\eta\leq\eta_0\right\} > 0$$

holds if $0<|g|<\varkappa_3$; combining this with (2.10), we get

$$|\mathrm{Im}\ r_u(\xi+i\eta)| < \frac{\eta_0+g^2C_2}{m_g^2} \tag{2.11a}$$

for $\xi \in (E-\delta,E+\delta)$ and $\eta \in (0,\eta_0)$. Finally, we set $\varkappa_1 = \min\{\varkappa_3, (\delta/2C_2)^{1/2}\}$. This number is positive, and $|\xi-E| - g^2C_2 > \delta - \varkappa_1^2C_2 > 0$ holds for $0<|g|<\varkappa_1$ and ξ which lies outside the interval $(E-\delta,E+\delta)$. Under these conditions, the inequality (2.10) gives

$$|\mathrm{Im}\, r_u(\xi+i\eta)| < \frac{\eta_0 + g^2C_2}{(|\xi-E| - g^2C_2)^2} \tag{2.11b}$$

and one can define the function u_g by the rhs of (2.11a) and (2.11b) in the respective intervals. The integrability of u_g is then obvious. ∎

Lemma 2.4 : Under the assumptions (a)-(d), there exists a finite $r_u^{(\pm)}(\xi,H_g) \equiv \lim_{\eta\to 0+} r_u(\xi\pm i\eta,H_g)$ for each $\xi\in\mathbb{R}$ and $0<|g|<\varkappa_1$. The functions $r_u^{(\pm)}$ are C^∞ in $(-\infty,0)\cup(\mathbb{R}^+\cap\Omega)$ and bounded in $\mathbb{R}$. If, in addition, the assumption (e) holds, then $r_u^{(\pm)}$ are C^∞ in $\mathbb{R}\setminus\{0\}$.

Proof : In view of (2.4b), we consider $r_u^{(+)}$ only. We have $r_u(\xi+i\eta,H_g) = -f(g,\xi+i\eta)^{-1}$; the preceding proof shows that $|f(g,.)|$ is bounded from below by a positive constant in $\mathbb{R}\times(0,\eta_0)$. Hence $|r_u^{(+)}(\xi)|$ is bounded by a constant independent of ξ, if only the limit exists. It is clear that the limit exists and defines a C^∞-function if the same is true for $\lim_{\eta\to 0+} G(\xi+i\eta)$. Therefore if $\xi\in\Omega\cap\mathbb{R}^+$, the assertion follows from Lemma II.3.4 . If $\xi<0$, we have

$$\left|\frac{v_3(\lambda)}{(\xi+i\eta-\lambda)^n}\right| \le \frac{v_3(\lambda)}{|\xi|^n}$$

and $4\pi m\int_0^\infty v_3(\lambda)\,d\lambda = \|v\|^2<\infty$ so existence of the limit and its differentiability is implied by the dominated-convergence theorem.

Let us turn now to checking existence of $\lim_{\eta\to 0+} G(\xi+i\eta)$ for the remaining ξ. If $\xi=0$, we have

$$|G(i\eta)| \le 4\pi m\int_0^\infty \frac{v_3(\lambda)}{|i\eta-\lambda|}\,d\lambda \le 4\pi m\int_0^\infty \frac{v_3(\lambda)}{\lambda}\,d\lambda \le$$

$$\le 8\pi m\int_0^m |\hat v_1(p)|^2\,dp + 8\pi m^{-1}\int_m^\infty |\hat v_1(p)|^2 p^2\,dp \le$$

$$\le 2m^{-1}(4\pi m^3C_1 + \|v\|^2) < \infty$$

and the limit exists due to the dominated-convergence theorem. If there are some positive ξ left, i.e., the assumption (e) is not valid, it is sufficient to verify that the function v_3 has a bounded derivative in a neighbourhood of ξ - cf. proofs of Lemmas II.3.3 and II.3.4 . It follows from the assumption (d), which yields

$$|v_3'(\lambda)| = \left|v_2'(\sqrt{2m\lambda})\sqrt{\frac{m}{2\lambda}}\right| \le C_1\sqrt{\frac{m}{2\lambda}}\,. \qquad ∎$$

Lemma 2.5 : Assume (a)-(f), then the continued function G_Ω fulfils

$$|G_\Omega'(\xi)| \le b_1 + \frac{b_2}{\sqrt{\xi}} + \frac{b_3}{\xi}\,, \tag{2.12a}$$

$$|\mathrm{Im}\, G'(\xi)| \le \frac{b_4}{\sqrt{\xi}} \tag{2.12b}$$

for all $\xi>0$, where

$$\begin{aligned} b_1 &= 8\pi mC_1\,,\\ b_2 &= 2\pi(\pi+2)mC_1\sqrt{2m}\,,\\ b_3 &= 12\pi m^2C_1\,,\\ b_4 &= 2\pi^2 mC_1\sqrt{2m}\,. \end{aligned} \tag{2.12c}$$

Proof : According to Lemma II.3.4 and the assumption (e), the function G_Ω is defined on some complex neighbourhood of the halfline $(0,\infty)$. The relation (2.2) gives

$$G'(z) = -4\pi m\int_0^\infty \frac{v_3(\lambda)}{(z-\lambda)^2}\,d\lambda$$

for $z\in\mathbb{C}\setminus\mathbb{R}$, and therefore

$$G_\Omega'(\xi) = -4\pi m\lim_{\eta\to 0+}\int_{-\xi}^\infty \frac{\lambda^2-\eta^2+2i\eta\lambda}{(\lambda^2+\eta^2)^2}\,v_3(\xi+\lambda)\,d\lambda\,. \tag{2.13a}$$

In order to estimate the rhs, we express $v_3(\xi+\lambda)$ by Taylor expansion. Since v_2 is twice differentiable and real-valued by assumption, we have

$$v_3(\xi+\lambda) = v_3(\xi) + v_2'(\sqrt{2m\xi})(\sqrt{2m(\xi+\lambda)}-\sqrt{2m\xi}) +$$

$$+\frac{1}{2}v_2''(\sqrt{2m(\xi+\vartheta)})(\sqrt{2m(\xi+\lambda)}-\sqrt{2m\xi})^2$$

for some ϑ between 0 and λ. After a rearrangement, the last expression becomes

$$v_3(\xi+\lambda) = v_3(\xi) + v_2'(\sqrt{2m\xi})\frac{m\lambda}{\sqrt{2m\xi}} +$$

$$+\left[v_2''(\sqrt{2m(\xi+\vartheta)}) - \frac{v_2'(\sqrt{2m\xi})}{\sqrt{2m\xi}}\right]\frac{2m^2\lambda^2}{(\sqrt{2m(\xi+\lambda)}+\sqrt{2m\xi})^2}\,; \tag{2.13b}$$

substituting it to (2.13a), we obtain expressions for the real and imaginary parts of $G_\Omega'(\xi)$ .Estimating them, we make use of the following relations which can be proved easily

$$\lim_{\eta\to 0+}\int_{\xi}^{\infty}\frac{\lambda^2-\eta^2}{(\lambda^2+\eta^2)^2}\,d\lambda=\frac{1}{\xi}\ ,$$

$$\lim_{\eta\to 0+}\int_{-\xi}^{\xi}\frac{\lambda^2-\eta^2}{(\lambda^2+\eta^2)^2}\,d\lambda=-\frac{2}{\xi}\ ,$$

$$\lim_{\eta\to 0+}\int_{-\xi}^{\xi}\frac{\lambda^2-\eta^2}{(\lambda^2+\eta^2)^2}\,\lambda\,d\lambda=0\ ,$$

$$\lim_{\eta\to 0+}\int_{-\xi}^{\xi}\frac{\lambda^2\pm\eta^2}{(\lambda^2+\eta^2)^2}\,\lambda^2 d\lambda=2\xi\ ,$$

$$\lim_{\eta\to 0+}\int_{\xi}^{\infty}\frac{\eta\lambda}{(\lambda^2+\eta^2)^2}\,d\lambda=\lim_{\eta\to 0+}\int_{-\xi}^{\xi}\frac{\eta\lambda}{(\lambda^2+\eta^2)^2}\,d\lambda=0\ ,$$

$$\lim_{\eta\to 0+}\int_{-\xi}^{\xi}\frac{\eta\lambda^2}{(\lambda^2+\eta^2)^2}\,d\lambda=\frac{\pi}{2}\ ,$$

$$\lim_{\eta\to 0+}\int_{-\xi}^{\xi}\frac{\eta|\lambda|^3}{(\lambda^2+\eta^2)^2}\,d\lambda=0\ .$$

In this way, we get

$$|\mathrm{Re}\ G'_{\Omega}(\xi)|\le\frac{4\pi m^2C_1}{\xi}+4\pi m\,\frac{2|v_3(\xi)|}{\xi}+8\pi m^2\left[\frac{C_1}{m}+\frac{|v_2'(\sqrt{2m\xi})|}{\sqrt{2m\xi}}\right]\qquad(2.14a)$$

and

$$|\mathrm{Im}\ G'_{\Omega}(\xi)|\le\frac{4\pi^2m^2}{\sqrt{2m\xi}}\,|v_2'(\sqrt{2m\xi})|\ ;\qquad(2.14b)$$

combining these inequalities with the assumption (d) and $|G'_{\Omega}(\xi)|\le$ $\le|\mathrm{Re}\ G'_{\Omega}(\xi)|+|\mathrm{Im}\ G'_{\Omega}(\xi)|$, we arrive at (2.12). ∎

Remark 2.6 : There are alternative ways how to estimate the rhs of (2.14a) from the assumption (d). Instead of (2.12c), we may choose, e.g., $b_1=8\pi mC_1$, $b_2=2\pi(\pi+6)mC_1\sqrt{2m}$ and $b_3=4\pi m^2C_1$. However, as far as we are not interested in actual values of the constants involved in the estimates, this ambiguity is irrelevant.

3. The pole approximation

Now we are ready to formulate and prove the main results. In accordance with (II.2.5), the undecayed state is represented by the vector $\Psi_0=\binom{1}{0}$. We have

$$E_uU(t)\Psi_0\equiv\begin{pmatrix}u(t)\\0\end{pmatrix}=u(t)\Psi_0\ ,\qquad(3.1)$$

where obviously u is a continuous function such that $|u(t)|\le 1$. We are interested in the quantity $u(t)$, which represents the non-decay probability amplitude at an instant t . One can check easily the relation

$$\frac{1}{2\pi i}\int_{\mathbb{R}}\left(\frac{1}{\lambda-\xi-i\eta}-\frac{1}{\lambda-\xi+i\eta}\right)e^{-i\xi t}\,d\xi=e^{-i\lambda t-\eta|t|}$$

and furthermore, $e^{-i\lambda t-\eta|t|}\to e^{-i\lambda t}$ pointwise with the lhs bounded independently of η . The functional-calculus rules (cf., e.g., Ref. I.27, Theorem VIII.5) then imply

$$U(t)\equiv e^{-iH_gt}=\text{s-}\lim_{\eta\to 0+}\frac{1}{2\pi i}\int_{\mathbb{R}}\left[R(\xi+i\eta,H_g)-R(\xi-i\eta,H_g)\right]e^{-i\xi t}\,d\xi\ .\qquad(3.2a)$$

From this relation, one can calculate $E_uU(t)\Psi_0$ interchanging the projection E_u with the integral (cf. Ref.5, Theorem III.2.19) ; it gives

$$u(t)=\lim_{\eta\to 0+}\frac{1}{2\pi i}\int_{\mathbb{R}}\left[r_u(\xi+i\eta,H_g)-r_u(\xi-i\eta,H_g)\right]e^{-i\xi t}\,d\xi\ .\qquad(3.2b)$$

Theorem 3.1 : Assume (a)-(d), then there is a positive $\mathcal{H}_1$ such that for $0<|g|<\mathcal{H}_1$, the function ω defined by

$$\omega(\lambda):=\frac{1}{\pi}\,\mathrm{Im}\ r_u^{(+)}(\lambda,H_g)=\frac{g^2}{\pi}\,\frac{\gamma(\lambda)}{(E-\lambda+4\pi g^2I(\lambda))^2+g^4\gamma(\lambda)^2}\ ,\qquad(3.3a)$$

where

$$I(\lambda)\equiv I(\lambda,v)=\mathcal{P}\int_0^{\infty}\frac{|\hat{v}_1(p)|^2p}{\lambda-\frac{p^2}{2m}}\,p\,dp\qquad(3.3b)$$

and

$$\gamma(\lambda)=4\pi^2m|\hat{v}_1(\sqrt{2m\lambda})|^2\sqrt{2m\lambda}\ \Theta(\lambda)\ ,\qquad(3.3c)$$

belongs to $L^1(\mathbb{R})$ and fulfils

$$u(t)=\int_{\mathbb{R}}e^{-i\lambda t}\omega(\lambda)\,d\lambda\qquad(3.4)$$

for all $t\in\mathbb{R}$.

Proof : The existence of $r_u^{(+)}(\lambda)$ is established by Lemma 2.4 . Proving this lemma, we have shown that the limit $G(\lambda+i0)$ exists under the present assumptions for all $\lambda\in\mathbb{R}$, and

$$r_u^{(+)}(\lambda)=\left[E-\lambda+g^2G(\lambda+i0)\right]^{-1}\ .$$

Furthermore, an inspection of the proof of Lemma II.3.4 shows that

$G(\lambda+i0) = 4\pi I(\lambda) - i\gamma(\lambda)$ for $\lambda > 0$, so a simple calculation leads to (3.3a). On the other hand, $r_u^{(+)}$ is real-valued in $(-\infty,0)$ so $\omega(\lambda) = 0$ there. In order to establish the relation (3.4), it is sufficient to interchange limit with the integral in (3.2b). If we choose $\varkappa_1$ as that of Lemma 2.3, the interchange is justified by this lemma and the dominated-convergence theorem. ■

Now we are going to use the obtained representation of the function u to prove the main result :

Theorem 3.2 : Assume (a)-(f), then there are positive numbers $\varkappa$, C (depending on v, but independent of g) such that the inequality

$$\left| u(t) - A e^{-iz_p t} \right| \leq \frac{C g^2}{t} , \tag{3.5}$$

where z_p is the pole specified in Theorem II.3.6 and

$$A = \left[1 - g^2 G_\Omega'(z_p) \right]^{-1} , \tag{3.6}$$

holds for all $t > 0$ and $|g| < \varkappa$.

Proof : The lhs of (3.5) is zero for $g = 0$, so we assume $g \neq 0$ in the following. The region Ω is open and contains E, hence we can choose $\rho \in (0, \frac{E}{2})$ so that $\bar{U}_{2\rho}(E) \equiv \{ z \in \mathbb{C} : |z-E| \leq 2\rho \} \subset \Omega$. Since G_Ω is continuous in $\bar{U}_{2\rho}(E)$, there is a positive C_3 such that $|G_\Omega(z)| \leq C_3$ within $\bar{U}_{2\rho}(E)$. In fact, we know more about this bound : Lemma 2.2 together with the relation (II.3.11) show that we can choose

$$C_3 = C_2 + 8\pi^2 m \sup\{ v_3(z) : |z-E| \leq 2\rho \} . \tag{3.7}$$

It is clear from Theorem II.3.6 that for small enough g the pole position fulfils

$$|z_p - E| < \rho , \quad |\lambda_p - E| < \tfrac{1}{2}\rho \tag{3.8a}$$

(cf. Fig.1). We can choose therefore $\varkappa \in (0, \varkappa_1)$ such that (3.8a) holds if $|g| < \varkappa$ and at the same time

$$12 \varkappa^2 C_3 < \rho . \tag{3.8b}$$

In such a case, we have also $12\varkappa^2 C_2 < \rho < \frac{1}{2}E$. Since G_Ω is holomorphic, the relation

$$G_\Omega'(z_p) = \frac{1}{2\pi i} \int_C \frac{G_\Omega(\zeta)}{(\zeta - z_p)^2} \, d\zeta$$

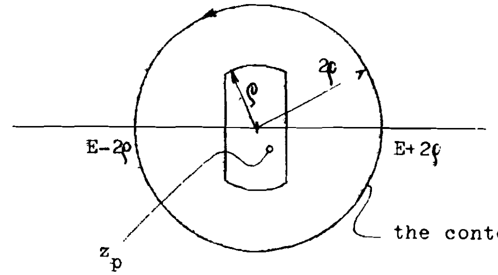

Fig.1. To the proof of Theorem 3.2.

holds, which yields the estimate*)

$$|G_\Omega'(z_p)| \leq \frac{4\pi\rho}{2\pi} \frac{C_3}{\rho^2} = \frac{2C_3}{\rho} . \tag{3.9}$$

It is easy to check by contour integration that to each $\varepsilon > 0$, there is some R_1 such that

$$\left| A e^{-iz_p t} - \frac{1}{\pi} \int_{-R}^{R} e^{-i\lambda t} \, \mathrm{Im} \frac{A}{z_p - \lambda} \, d\lambda \right| < \tfrac{1}{3}\varepsilon$$

holds for all $R > R_1$. Similarly Theorem 3.1 together with the dominated-convergence theorem show that

$$\left| u(t) - \int_{-R}^{R} e^{-i\lambda t} \omega(\lambda) \, d\lambda \right| < \tfrac{1}{3}\varepsilon$$

for all R larger than some R_2. Combining the last two inequalities, we get

$$\left| u(t) - A e^{-iz_p t} \right| < \tfrac{2}{3}\varepsilon + \frac{1}{\pi} \left| \int_{-R}^{R} e^{-i\lambda t} \, \mathrm{Im} \left[r_u^{(+)}(\lambda) - \frac{A}{z_p - \lambda} \right] d\lambda \right| \tag{3.10}$$

for $R > R_3 \equiv \max\{R_1, R_2\}$. In order to proceed further with the estimate, some manipulations with last integral are needed. We denote it for a moment as I_R and rewrite it as

$$I_R = \sum_{j=0}^{n-1} \frac{1}{\pi} \int_{\lambda_j}^{\lambda_{j+1}} e^{-i\lambda t} \, \mathrm{Im} \left[r_u^{(+)}(\lambda) - \frac{A}{z_p - \lambda} \right] d\lambda ,$$

where $\{\lambda_j\}_{j=0}^{n}$ is a finite sequence with $\lambda_0 = -R$ and $\lambda_n = R$, and

*) In combination with (3.8b), the inequality (3.9) shows that A is properly defined and $|A| < 6/5$. However, a little later we shall show that A is the residuum of $r_u^{\Omega}(.)$ at z_p, and as such it is defined independently of the assumption (d) and (3.8). We shall also mention that in a realistic physical situation, A is very close to 1 -cf. (4.4b) below.

perform the integration by parts. In view of Lemma 2.4, the only problem can arise at $\lambda = 0$, where the "resolvent part" of the integrated function might not be continuous. However, $\omega(\lambda) = 0$ for $\lambda < 0$, and at the same time, the relations (3.3a) and (3.3c) together with the assumption (d), Lemma 2.2 and the relation (3.8b) give

$$\omega(\lambda) \leqslant \frac{4\pi m g^2 C_1 \sqrt{2m\lambda}}{(E-\lambda-g^2C_2)^2}$$

for $\lambda \in (0, \frac{E}{2})$, i.e., $\lim_{\lambda \to 0+} \omega(\lambda) = 0$. Hence the integration can be performed ; it yields

$$I_R = \frac{1}{\pi}\left[\frac{i}{t}\, e^{-i\lambda t}\, \mathrm{Im}\left(r_u^{(+)}(\lambda) - \frac{A}{z_p-\lambda}\right)\right]_{\lambda=-R}^{R} -$$

$$- \frac{i}{\pi t}\sum_{j=0}^{n-1}\int_{\lambda_j}^{\lambda_{j+1}} e^{-i\lambda t}\, \frac{d}{d\lambda}\, \mathrm{Im}\left(r_u^{(+)}(\lambda) - \frac{A}{z_p-\lambda}\right) d\lambda \quad .$$

The functions $I(.)$ and $\gamma(.)$ are bounded in view of Lemma 2.2, and therefore $\lim_{\lambda\to\infty} \omega(\lambda) = 0$. Since $\lim_{\lambda\to-\infty} \omega(\lambda) = 0$ holds obviously, the first term on the rhs tends to zero as $R \to \infty$; we can therefore choose $R_4 \geqslant R_3$ such that its modulus will not exceed $\frac{1}{3}\varepsilon$ for any $R > R_4$. Combining this result with (3.10), we get

$$\left|u(t) - Ae^{-iz_p t}\right| < \varepsilon + \frac{1}{\pi t}\sum_{j=0}^{n-1}\left|\int_{\lambda_j}^{\lambda_{j+1}} e^{-i\lambda t}\, \frac{d}{d\lambda}\, \mathrm{Im}\left(r_u^{(+)}(\lambda) - \frac{A}{z_p-\lambda}\right) d\lambda\right| . \quad (3.11a)$$

Now we shall estimate the integrals on the rhs choosing

$$\lambda_1 = E - \varrho \quad \text{and} \quad \lambda_2 = E + \varrho \quad . \qquad (3.11b)$$

We start with

$$\left|\int_{-R}^{E-\varrho} e^{-i\lambda t}\, \frac{d}{d\lambda}\, \mathrm{Im}\, \frac{A}{z_p-\lambda}\, d\lambda\right| \leqslant \int_{-\infty}^{E-\varrho} \left|\mathrm{Im}\, \frac{A}{(z_p-\lambda)^2}\right| d\lambda =$$

$$= \int_{\lambda_p-E+\varrho}^{\infty} \left|\frac{(u^2-\delta_p^2)\,\mathrm{Im}\, A + 2\delta_p u\, \mathrm{Re}\, A}{(u^2+\delta_p^2)^2}\right| du \quad ,$$

where we have substituted $u = \lambda_p - \lambda$. It can be further estimated by the sum of two integrals ; for the one containing $|\mathrm{Re}\, A|$ we use

$$\int_{\lambda_p-E+\varrho}^{\infty} \frac{2\delta_p u}{(u^2+\delta_p^2)^2}\, du < 2\delta_p \int_{\frac{1}{2}\varrho}^{\infty} \frac{u\, du}{(u^2+\delta_p^2)^2} = \frac{4\delta_p}{\varrho^2 + 4\delta_p^2} < \frac{4\delta_p}{\varrho^2}$$

together with the above-mentioned fact that $|A| < \frac{6}{5}$. According to Theorem II.3.6 and the proof of Lemma 2.3, $\delta_p(.)$ is a C^∞-function

on a neighbourhood of $g = 0$ containing $[-\varkappa, \varkappa]$. Consequently, there is a positive C_4 such that

$$0 \leqslant \delta_p(g) \leqslant C_4 g^2 \qquad (3.12)$$

holds for all $g \in [-\varkappa, \varkappa]$. The first one of the above mentioned two integrals is estimated in the following way :

$$\int_{\lambda_p-E+\varrho}^{\infty} \left|\frac{u^2-\delta_p^2}{(u^2+\delta_p^2)^2}\right| du < \int_{\frac{1}{2}\varrho}^{\infty} \frac{du}{u^2+\delta_p^2} < \frac{2}{\varrho}$$

and

$$|\mathrm{Im}\, A| = \frac{g^2\,|\mathrm{Im}\, G'_\Omega(z_p)|}{|1-g^2G'_\Omega(z_p)|^2} \leqslant g^2 \frac{\frac{2C_3}{\varrho}}{(1-\varkappa^2\frac{2C_3}{\varrho})^2} < \frac{72\, C_3}{25\varrho}\, g^2 \quad ,$$

where we have used (3.9). Together we have

$$\left|\int_{-R}^{E-\varrho} e^{-i\lambda t}\, \frac{d}{d\lambda}\, \mathrm{Im}\, \frac{A}{z_p-\lambda}\, d\lambda\right| < \left(\frac{144}{25}C_3 + \frac{24}{5}C_4\right)\varrho^{-2} g^2 \quad . \qquad (3.13a)$$

Next one has to estimate the analogous integral for the function $r_u^{(+)}$. Notice that the functions I and γ are differentiable being multiples of the real and imaginary parts of the holomorphic function G_Ω. In fact, we are interested in their behaviour for $\lambda \geqslant 0$ only, because $\mathrm{Im}\, r_u^{(+)}(\lambda) = 0$ for $\lambda < 0$ and the integral can be taken therefore over $(0, E-\varrho)$. For an arbitrary $\lambda > 0$, an easy calculation yields

$$\mathrm{Im}\, \frac{d}{d\lambda}\, r_u^{(+)}(\lambda) = |E-\lambda+g^2G_\Omega(\lambda)|^{-4}\Big\{2g^2\gamma(\lambda)(1-4\pi g^2 I'(\lambda)) \times$$
$$\times (E-\lambda+4\pi g^2 I(\lambda)) + g^2\gamma'(\lambda)\left[(E-\lambda+4\pi g^2 I(\lambda))^2 - g^4\gamma(\lambda)^2\right]\Big\}. \qquad (3.14)$$

This quantity can be estimated using Lemma 2.2 and Lemma 2.5 as

$$\left|\mathrm{Im}\, \frac{d}{d\lambda}\, r_u^{(+)}(\lambda)\right| \leqslant$$
$$\leqslant \frac{g^2}{(\varrho - g^2C_2)^4}\left\{8\pi^2 m C_1 (E+g^2C_2)(1 + b_1 + \frac{b_2}{\sqrt{\lambda}} + \frac{b_3}{\lambda})\sqrt{2m\lambda} + \frac{b_4}{\sqrt{\lambda}}(E+g^2C_2)^2\right\} \quad ;$$

according to (3.7) and (3.8b), the term before the curly bracket is smaller than $g^2(\frac{11}{12}\varrho)^{-4}$. The integral of the rhs can be easily calculated ; we get

$$\int_0^{E-\varrho} \left|\mathrm{Im}\, \frac{d}{d\lambda}\, r_u^{(+)}(\lambda)\right| d\lambda < \left(\frac{144\, g}{121\varrho^2}\right)^2 (E + \frac{\varrho}{12})\Big\{2\Big[b_4(E + \frac{\varrho}{12}) +$$
$$+ 8\pi^2 m b_3 C_1\sqrt{2m}\Big](E-\varrho)^{1/2} + 8\pi^2 m b_2 C_1 \sqrt{2m}(E-\varrho) + \frac{16}{3}\pi^2 m(1+b_1)C_1\sqrt{2m}(E-\varrho)^{3/2}\Big\} . \qquad (3.13b)$$

The relations (3.13) together show that there is a positive C_5, which is depending on E, ρ and the function v but independent of t and R, such that

$$\left| \int_{-R}^{E-\rho} e^{-i\lambda t} \frac{d}{d\lambda} \mathrm{Im}\left(r_u^{(+)}(\lambda) - \frac{A}{z_p - \lambda} \right) d\lambda \right| < C_5 g^2 \quad . \tag{3.15}$$

Let us turn now to the integral over $(E-\rho, E+\rho)$. For $|z-E|<\rho$, we employ the following expansion /4/

$$G_\Omega(z) = G_\Omega(z_p) + G'_\Omega(z_p)(z-z_p) + F(z)(z-z_p)^2 \quad , \tag{3.16a}$$

where

$$F(z) = \frac{1}{2\pi i} \int_C \frac{G_\Omega(\xi)}{(\xi - z)(\xi - z_p)^2} d\xi \quad ; \tag{3.16b}$$

recall that the circle C has been chosen to be contained in the region Ω, where G_Ω is holomorphic. We use also the fact that z_p is a pole of r_u^Ω, i.e.,

$$E - z_p + g^2 G_\Omega(z_p) = 0 \quad . \tag{3.16c}$$

Since $r_u^{(+)} = r_u^\Omega$ on $(E-\rho, E+\rho)$, the relations (3.16) give

$$r_u^{(+)}(\lambda) - \frac{A}{z_p - \lambda} =$$

$$= \frac{1}{E - \lambda + g^2 G_\Omega(z_p) + g^2 G'_\Omega(z_p)(\lambda - z_p) + g^2 F(\lambda)(\lambda - z_p)^2} - \frac{A}{z_p - \lambda} =$$

$$= \frac{1}{(\lambda - z_p)(-1 + g^2 G'_\Omega(z_p)) + g^2 F(\lambda)(\lambda - z_p)^2} - \frac{A}{z_p - \lambda} = -g^2 A^2 \frac{F(\lambda)}{1 - g^2 A F(\lambda)(\lambda - z_p)}$$

so

$$\frac{d}{d\lambda}\left(r_u^{(+)}(\lambda) - \frac{A}{z_p - \lambda} \right) = -g^2 A^2 \frac{F'(\lambda) + g^2 A F(\lambda)^2}{\left[1 - g^2 A F(\lambda)(\lambda - z_p)\right]^2} \quad .$$

For $|z-E|<\rho$, we have

$$|F(z)| \leqslant \frac{1}{2\pi} \int_C \frac{G_\Omega(\xi)}{|\xi - z||\xi - z_p|^2} d\xi < \frac{1}{2\pi} \frac{C_3}{\rho\rho^2} 4\pi\rho = \frac{2C_3}{\rho^2}$$

and similarly

$$|F'(z)| < \frac{2C_3}{\rho^3} \quad .$$

These inequalities together with (3.8b) and $|A| < \frac{6}{5}$ give the relations

$$\left| \frac{d}{d\lambda} \mathrm{Im}\left(r_u^{(+)}(\lambda) - \frac{A}{z_p - \lambda} \right) \right| \leqslant \frac{36}{25} g^2 \frac{2C_3}{\rho^3} \frac{1 + \frac{6}{5}\varkappa^2 \frac{2C_3}{\rho}}{\left(1 - \frac{6}{5}\varkappa^2 \frac{2C_3}{\rho^2} 2\rho\right)^2} < \frac{48\,C_3}{5\rho^3} g^2$$

from which the sought estimate follows easily,

$$\left| \int_{E-\rho}^{E+\rho} e^{-i\lambda t} \frac{d}{d\lambda} \mathrm{Im}\left(r_u^{(+)}(\lambda) - \frac{A}{z_p - \lambda} \right) d\lambda \right| < C_6 g^2 \quad , \tag{3.17}$$

where

$$C_6 = \frac{96\,C_3}{5\rho^2} \quad .$$

Finally, let us estimate the third integral. According to Lemma 2.5, we have

$$|G'_\Omega(\lambda)| \leqslant b_1 + \frac{b_2}{\sqrt{E+\rho}} + \frac{b_3}{E+\rho} \equiv C_7$$

for $\lambda \geqslant E+\rho$. The relation (3.14) then gives

$$\left| \mathrm{Im} \frac{d}{d\lambda} r_u^{(+)}(\lambda) \right| < g^2 \left\{ \frac{2C_2(1+g^2 C_7)}{(\lambda - E - g^2 C_2)^3} + \frac{C_7}{(\lambda - E - g^2 C_2)^2} \right\} \quad .$$

Integral of the rhs can be easily calculated. Using (3.8b) again, we get

$$\begin{aligned} \int_{E+\rho}^{R} \left| \mathrm{Im} \frac{d}{d\lambda} r_u^{(+)}(\lambda) \right| d\lambda &< g^2 \left\{ \frac{C_2(1+g^2 C_7)}{(\rho - g^2 C_2)^2} + \frac{C_7}{\rho - g^2 C_2} \right\} \leqslant \\ &\leqslant g^2 \left\{ \frac{144\,C_2}{121\,\rho^2}(1 + \varkappa^2 C_7) + \frac{12\,C_7}{11\rho} \right\} \quad . \end{aligned} \tag{3.18a}$$

On the other hand, for the "pole part" of the third integral we have

$$\left| \int_{E+\rho}^{R} e^{-i\lambda t} \frac{d}{d\lambda} \mathrm{Im} \frac{A}{z_p - \lambda} d\lambda \right| < \int_{E+\rho-\lambda_p}^{\infty} \left| \mathrm{Im} \frac{A}{(u + i\delta_p)^2} \right| du \quad .$$

Estimating the first integral, we have calculated an analogous expression, with the only difference consisting in the interchanges $E \leftrightarrow \lambda_p$ and $\delta_p \leftrightarrow -\delta_p$. Since $E+\rho-\lambda_p > \frac{1}{2}\rho$, the same argument can be used, and we obtain

$$\left| \int_{E+\rho}^{R} e^{-i\lambda t} \frac{d}{d\lambda} \mathrm{Im} \frac{A}{z_p - \lambda} d\lambda \right| < \left(\frac{144\,C_3}{25\rho^2} + \frac{24\,C_4}{5\rho^2} \right) g^2 \quad . \tag{3.18b}$$

The relations (3.18) show that there is a positive C_8 such that

$$\left| \int_{E+\rho}^{R} e^{-i\lambda t} \frac{d}{d\lambda} \mathrm{Im}\left(r_u^{(+)}(\lambda) - \frac{A}{z_p - \lambda} \right) d\lambda \right| < C_8 g^2 \tag{3.19}$$

holds for $0 < |g| < \varkappa$ and all t and $R > E+\rho$.

Now one has to collect all the obtained estimates. The relations (3.11), (3.15), (3.17) and (3.19) together give

$$\left|u(t) - A\, e^{-iz_p t}\right| < \varepsilon + \frac{g^2}{\pi t}(C_5+C_6+C_8)$$

for $0<|g|<\varkappa$, any $t>0$ and $R>R_4$. However, ε has been an arbitrary positive number, and therefore the inequality (3.5) holds if we choose $C = (C_5+C_6+C_8)/\pi$. ∎

4. The decay law

The results of the preceding section can be now used to justify approximative validity of the exponential decay law (within a certain time interval) and to demonstrate accuracy of this approximation. The decay law of the state Ψ_0 is according to (3.1) given by

$$P(t) = |u(t)|^2 \quad ; \tag{4.1}$$

its meaning is the probability to find the heavy particle still undecayed at an instant t.

First we notice that the bound given by Theorem 3.2 is useless for very small and very large times. For small times, it is obvious from (3.5). On the other hand, the decay of $|u(.)|$ for large t is slower than exponential. This is a consequence of Paley-Wiener theorem, though not immediate (cf. Ref.I.3, Sec.1.5). Since the spectrum of H_g is semibounded, we have either $u \notin L^1(\mathbb{R}^+)$ or

$$\int_0^\infty \frac{\ln |u(t)|^2}{1+t^2}\, dt > -\infty$$

(notice that $\dim \mathcal{H}_u = 1$ so we can put $u(-t) = \bar{u}(t)$ if necessary). However, u is a continuous function such that $0 \leqslant |u(t)| \leqslant 1$ and $\lim_{t\to\infty} u(t) = 0$; the last assertion follows directly from (3.5), or alternatively from Theorem 3.1 and Riemann-Lebesgue lemma. Either of the above possibilities then requires $u(.)$ to decay slowly enough at infinity. Hence Theorem 3.2 gives nothing more than the bound

$$P(t) < \frac{g^2 C'}{t}$$

on the decay law for large t.

At the same time, the estimate (3.5) can be useful for a wide interval of intermediate times. Let us first comment on the meaning of the term "intermediate". A natural time scale for the problem under

consideration is given by the lifetime

$$T = \int_0^\infty P(t)\, dt \quad ; \tag{4.2a}$$

this quantity is closely related to the mean life of the heavy particle, but in general the two concepts are not fully identical - see Ref. I.3, Sec.1.2 for more details. If the decay is purely exponential, the lifetime is found easily. Our aim is now to show in which sense the actual lifetime can be approximated by such a simple expression. We remark that T can be affected substantially by the large-time behaviour of $P(t)$ if the latter decreases slowly enough at infinity. This is not disastrous, however. One must realize that the decay law itself and not T is a measurable quantity. The measurements are actually performed within a (possibly large, but finite) time interval and convergence of the integral is thus a matter of our extrapolation (a closely related problem is discussed in Ref.I.3, Secs.1.3 and 1.6). From the practical point of view, T can be therefore replaced by $T_1 = T(\tau_1)$ defined by

$$T_1 = \int_0^{\tau_1} P(t)\, dt \tag{4.2b}$$

for a sufficiently large τ_1. Now we have

<u>Proposition 4.1</u> : Assume (a)-(f). Let $0<\tau_0<\tau_1$ and $|g|<\varkappa$, then the following estimate is valid

$$\left|2\delta_p T_1 - 1\right| < 5\delta_p\tau_0 + 5Cg^2\delta_p \ln\frac{\tau_1}{\tau_0} + \frac{7C_3}{\varphi}g^2 + e^{-2\delta_p\tau_1} \quad , \tag{4.3}$$

where $C_3, \varphi, \varkappa$ are the constants used in the proof of Theorem 3.2 .

<u>Proof</u> : We have

$$\left|T_1 - \frac{1}{2\delta_p}\right| \leqslant \left|\int_0^{\tau_1}\left[P(t) - |A|^2 e^{-2\delta_p t}\right]dt\right| + \left|\,|A|^2 - 1\right|\int_0^{\tau_1} e^{-2\delta_p t}\, dt + \int_{\tau_1}^{\infty} e^{-2\delta_p t}\, dt \; .$$

The first term can be estimated using the relations (3.5) and (4.1) ; we get

$$\left|P(t) - |A|^2 e^{-2\delta_p t}\right| = \left|\left(|u(t)| + |A\, e^{-iz_p t}|\right)\left(|u(t)| - |A\, e^{-iz_p t}|\right)\right| \leqslant$$
$$\leqslant (1+|A|)\left|u(t) - A\, e^{-iz_p t}\right| \quad ,$$

i.e.,

$$\left| P(t) - |A|^2 e^{-2\delta_p t} \right| < \frac{11\,C}{5\,t}\, g^2 \quad , \tag{4.4a}$$

which can be used in $[\tau_0, \tau_1]$. On the other hand, in $[0, \tau_0]$, we can use the inequality

$$\left| P(t) - |A|^2 e^{-2\delta_p t} \right| \leq 1 + |A|^2 < \frac{61}{25} < \frac{5}{2} \quad .$$

Using further (3.6), (3.8b) and (3.9), we obtain

$$\left| |A|^2 - 1 \right| \leq \frac{\left(1 + |1 - g^2 G'_\Omega(z_p)|\right) \left| 1 - |1 - g^2 G'_\Omega(z_p)| \right|}{|1 - g^2 G'_\Omega(z_p)|^2} <$$

$$< \frac{\left(2 + \frac{2C_3}{\rho}\varkappa^2\right) g^2 |G'_\Omega(z_p)|}{\left(1 - \frac{2C_3}{\rho}\varkappa^2\right)^2} < \frac{78}{25}\, g^2\, \frac{2C_3}{\rho} \quad ,$$

i.e.,

$$\left| |A|^2 - 1 \right| < \frac{7\,C_3}{\rho}\, g^2 \quad . \tag{4.4b}$$

Putting now these estimates together, we get the inequality

$$\left| T_1 - \frac{1}{2\delta_p} \right| < \frac{5}{2}\tau_0 + \frac{11}{5} C g^2 \ln\frac{\tau_1}{\tau_0} + \frac{7\,C_3}{2\delta_p \rho} g^2 + \frac{e^{-2\delta_p \tau_1}}{2\delta_p}$$

from which (4.3) follows. ∎

The relation (4.3) shows that in the case of a sufficiently weak coupling the lifetime may be approximated by $(2\delta_p)^{-1}$. It is clear, that τ_1 cannot be chosen independently of g . We can put, e.g., $\tau_1 = N/2\delta_p$ and $\tau_0 = 1/2\delta_p N$ for some $N \gg 1$; then

$$\left| 2\delta_p T\!\left(\frac{N}{2\delta_p}\right) - 1 \right| < \frac{5}{2N} + 10\,C g^2 \delta_p \ln N + \frac{7\,C_3}{\rho} g^2 + e^{-N}$$

so the rhs can be made arbitrarily small by choosing a suitable N and $|g|$ small enough. These considerations show that an appropriate interval of intermediate times could be*)

$$\frac{1}{2\delta_p N} \leq t \leq \frac{N}{2\delta_p} \quad . \tag{4.5}$$

Within this restriction, the following estimates hold :

Theorem 4.2 : Suppose (a)-(f) are valid together with (4.5) and

*) It does not mean, of course, that the whole interval (4.5) is experimentally attainable ; remember that the real unstable systems (particles, nuclei) can have lifetime as short that it is impossible to measure the time plot of the decay law, or longer than duration of the Universe itself.

$$|g| < \min\left\{ \varkappa, \left(12\, C C_4 N e^N\right)^{-1/4} \right\}, \tag{4.6a}$$

then

$$\left| P(t) - |A|^2 e^{-2\delta_p t} \right| < 5 C C_4 N g^4 \quad . \tag{4.6b}$$

Furthermore, the function $\tilde{\Gamma}$ defined by $P(t) = |A|^2 e^{-\tilde{\Gamma}(t)}$ obeys in the considered interval the restriction

$$\left| \frac{\tilde{\Gamma}(t)}{t} - 2\delta_p \right| < 24 (\ln 2) C C_4^2 N^2 e^N g^6 \quad . \tag{4.6c}$$

Proof : The inequality (4.6b) follows immediately from (4.4a) and (3.12). Similarly.

$$\left| \frac{P(t) - |A|^2 e^{-2\delta_p t}}{|A|^2 e^{-2\delta_p t}} \right| < \frac{11}{5}\,\frac{49}{36}\, 2 C C_4 N e^N g^4 < 6 C C_4 N e^N g^4 \quad ,$$

where we have used the inequality $|A| > \frac{6}{7}$ following from (3.8b) and (3.9). Under the assumption (4.6a), we have $6 C C_4 N e^N g^4 < \frac{1}{2}$ so that

$$\left| \tilde{\Gamma}(t) - 2\delta_p t \right| = \left| \ln\left(1 + \frac{P(t) - |A|^2 e^{-2\delta_p t}}{|A|^2 e^{-2\delta_p t}} \right) \right| < 2 \ln 2 \,.\, 6 C C_4 N e^N g^4$$

and (4.6c) follows. ∎

Remark 4.3 : The procedure by which the additional powers of g are gained on the rhs of (4.6b) and (4.6c) can be formulated in a more mathematical way. Since the lifetime characterizing the natural time scale increases, in general, as $|g| \to 0$, it is useful to introduce the rescaled time $t' = g^2 t$ when dealing with the weak-coupling limit. One can write $z_p(g) = E + g^2 a_2(g)$, where $a_2 = \alpha_2 - i\beta_2$ is a C^∞-function in some neighbourhood of $g = 0$ such that $a_2(g) = 4\pi I(E,v) - 4\pi^2 \mathrm{im}\, v_2(\sqrt{2mE}) + O(g^2)$. The estimate (3.5) now acquires the form

$$\left| e^{ig^{-2}Et'} u(g^{-2}t') - A\, e^{-ia_2(g)t'} \right| < \frac{C g^4}{t'} \quad , \tag{4.7a}$$

from which we get, e.g.,

$$\left| P(g^{-2}t') - |A|^2 e^{-2\beta_2(g)t'} \right| < \frac{11\,C}{5\,t'}\, g^4 \quad . \tag{4.7b}$$

5. Fermi golden rule

Now we would like to establish strict validity of the popular rule which claims formally that the decay rate, i.e., the decay probability per unit time equals

$$\Gamma_F(g) = 2\pi g^2 |\langle E|V\Psi_0\rangle|^2 \qquad (5.1a)$$

(recall that we use $\hbar = 1$). Since the "projection" $|E\rangle\langle E|$ may be understood as $\frac{d}{d\lambda} E_\lambda^{(0)} P_c(H_0)\big|_{\lambda=E}$, where $\{E_\lambda^{(0)}\}$ is the decomposition of unity of the operator H_0 and $P_c(H_0)$ is the projection to the continuous subspace of H_0 , we can replace the formal expression in (5.1a) by

$$\Gamma_F(g) = 2\pi g^2 \frac{d}{d\lambda} (V\Psi_0, E_\lambda^{(0)} P_c(H_0) V\Psi_0)\Big|_{\lambda=E} \quad . \qquad (5.1b)$$

Proposition 5.1 : Under the assumptions (a)-(c), the decay rate (5.1) equals

$$\Gamma_F(g) = 8\pi^2 m g^2 |\hat{v}_1(\sqrt{2mE})|^2 \sqrt{2mE} \quad . \qquad (5.2)$$

Proof : We denote $\Psi_v = \begin{pmatrix} 0 \\ \hat{v} \end{pmatrix}$, then a straightforward computation gives

$$\Gamma_F(g) = 2\pi g^2 \frac{d}{d\lambda} (\Psi_v, E_\lambda^{(0)} P_c(H_0)\Psi_v)\Big|_{\lambda=E} =$$

$$= 2\pi g^2 \frac{d}{d\lambda} \int_{\{\vec{p}\,:\,|\vec{p}|^2 \le 2m\lambda\}} |\hat{v}(\vec{p})|^2 d\vec{p}\,\Big|_{\lambda=E} =$$

$$= 8\pi^2 m g^2 \frac{d}{d\lambda} \int_0^\lambda |\hat{v}_1(\sqrt{2m\xi})|^2 \sqrt{2m\xi}\, d\xi\,\Big|_{\lambda=E} \quad ,$$

i.e., the desired result. ∎

Comparing to Theorem II.3.6, we see that $\Gamma_F(g)$ given by (5.2) is nothing else than the lowest-order term in Taylor expansion of

$$\Gamma(g) \equiv 2\delta_p(g) \quad . \qquad (5.3a)$$

In order to justify the Fermi rule, one has to know therefore that (5.3a) represents the true decay rate (defined in a reasonable way, with an accuracy $\sim O(|g|^{2+\varepsilon})$). If we adopt the additional assumptions (d)-(f) beside those of Proposition 5.1, this is guaranteed by Theorem 4.2, even up to the sixth order in the coupling constant. One can write therefore

$$\Gamma(g) = \Gamma_F(g)\,(1 + O(g^2)) \quad . \qquad (5.3b)$$

6. Conclusions

We have not drawn out explicit connection between the constants in our final estimates such as (3.5) and those related directly to the function v . If required, these relations can be extracted easily from the proofs, but they would not be probably of much use. In order to get some quantitative information about the error tied with the pole approximation, it is more illustrative to treat a suitable example ; we are going to return to this problem later.

There are other ways how to estimate the pole-approximation error. One of them has been elaborated by Demuth /3/ for the Friedrichs model, and it can be adapted easily to the present case*). One obtains in this way a bound which is essentially time independent instead of (3.5) ; its rhs contains powers of the coupling constant g and of $\delta(g)$ which characterizes a family of intervals centred at E which leads to spectral concentration ; we shall see in the next part of this paper that one can choose $\delta(g) = O(g^\beta)$ with $\beta \in [0,2)$. Demuth's estimate can be optimized when we choose $\beta = \frac{2}{5}$; then it gives an error of the order of $g^{4/5}$, and the same type of estimate for the decay laws instead of (4.6b) . The decay rate is now constant within the interval (4.5) up to $g^{14/5}$ comparing to our estimate $\sim g^6$. Nevertheless, this is still sufficient to justify the Fermi rule. Generally speaking, Demuth's method provides us with weaker error estimates, but his restrictions imposed on the function v are also weaker than ours corresponding to the assumptions (a)-(c) only.

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*) There is a slight incorrectness in Demuth's paper, namely λ_+ in his eq.(26) should be replaced by z_p . However, this fact affects only the numerical values of the constants appearing in the obtained bound.

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Диттрих Я., Экснер П. E2-86-750
Нерелятивистская модель двухчастичного распада.
Полюсное приближение

Работа посвящена, в основном, обоснованию приближения, в котором приведенная резольвента заменяется одним полюсным членом. Налагая дополнительные условия регулярности на функцию v, описывающую взаимодействие, мы способны оценить разность соответствующих приведенных пропагаторов. Этот результат дальше используется для вывода оценки отклонений от экспоненциального закона распада, следующего из полюсного приближения. За исключением очень малых и очень больших времен, полученная оценка пропорциональна четвертой степени константы связи. Мы также показываем золотое правило Ферми для рассматриваемой модели и сравниваем предложенный метод с раньше применяемым методом Демута.

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Dittrich J., Exner P. E2-86-750
A Non-Relativistic Model of Two-Particle Decay.
The Pole Approximation

In this paper, we are concerned mostly with the problem of justifying the approximation in which the reduced resolvent is replaced by the pole term alone. Imposing additional regularity assumptions on the function v, which specifies the interaction, we are able to estimate the difference of the corresponding reduced propagators. This result is used further to derive an estimate of the deviations from the exponential decay law which results from the pole approximation. With exception of very small and very large times, the obtained bound is proportional to fourth power of the coupling constant. We prove also Fermi golden rule for the model under consideration, and compare the present method to the one previously used by Demuth.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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