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AN APPLICATION OF THE ABELIAN ANTISYMMETRIC-TENSOR GAUGE THEORIES TO CLASSICAL PHYSICS

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In the present short note we discuss the Abelian antisymmetric tensor gauge fields in somewhat different physical setting than that meant in the previous work on the subject.

In one of the pioneering studies [1] (one could also quote the work of Kemmer [2,3]) V.I.Ogievetsky and I.V.Polubarinov investigated the case of an antisymmetrical tensor-potential (and the corresponding four-vector field strength) and its massless zero-helicity excitation which they called notoph, to acknowledge its mirror properties and imitate relation to the photon.

After the work of Ogievetsky and Polubarinov [1] and Hayashi [4] the study of the Abelian antisymmetric tensor gauge fields was renewed [5,6,7] in relation to the string theories and was connected through the mediation of the latter with the problem of quark confinement [8]. Non-Abelian generalizations [9,11] and geometric interpretation [10,11] were also discussed. Much work was devoted to the study of the covariant quantization, ghost spectrum, BRST invariance and renormalization of the antisymmetric tensor gauge models [12–16]. Their discussion is usually related in the recent literature to the supersymmetry and supergravity [12–14,17–21]. They have also played certain role in the dynamical theory of currents [22] (in the days of current algebra) and in treating dual transformations [23] and defects [24] in solid-state physics. They have had their appearance in mathematics quite some time ago [25]. Most of the recent applications of the antisymmetric-tensor gauge fields are in the context of the (super) string theories.

Without entering into details we only resume the simplest case of a second-rank antisymmetric-tensor gauge field $\beta_2 = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu$ and its third-rank totally antisymmetric field strength

$$\gamma_3 = \frac{1}{3!} G_{\mu\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda = d\beta_2 = \frac{1}{3!}(\partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}) dx^\mu \wedge dx^\nu \wedge dx^\lambda . \quad (1)$$

One of the properties of such a field with Lagrangian (cf. the papers quoted above)

$$\mathcal{L} = \tilde{G}_\mu \tilde{G}^\mu = \frac{1}{6} G^{\nu\rho\sigma} G_{\nu\rho\sigma} = \frac{1}{4} \partial_\lambda B_{\mu\nu} \epsilon^{\lambda\mu\nu a} \partial^\rho B^{\sigma\tau} \epsilon_{\rho\sigma\tau a} \quad (2)$$

is that this free Lagrangian describes in four-dimensional Minkowski space-time a free massless scalar field. Indeed, it follows from (2), or the corresponding action

$$S = (d\beta_2, d\beta_2), \tag{3}$$

that the equation of motion is

$$\delta d\beta_2 = 0, \quad \text{i.e.,} \quad \delta\gamma_3 = 0 \tag{4}$$

or, in terms of the dual field strength ${}^*\gamma_3 = \frac{1}{3!} G^{\lambda\mu\nu} \epsilon_{\lambda\mu\nu\alpha} dx^\alpha = \tilde{G}_\alpha dx^\alpha$,

$$d^*\gamma_3 = 0 \tag{5}$$

which reads

$$\partial_\mu \tilde{G}_\alpha - \partial_\alpha \tilde{G}_\mu = 0 \tag{6}$$

in component form. On the other hand, eq.(1) and $d^2 = 0$ imply

$$d\gamma_3 = 0 \tag{7}$$

or $\partial_\mu \tilde{G}^\mu = 0$, in components. It follows from eq.(4) that there exists a scalar field (a 4-form χ_4) such that

$$\gamma_3 = \delta\chi_4 \tag{8}$$

which obeys, as is seen upon inserting in (7), the massless free field equation

$$0 = d\delta\chi_4 = (d\delta + \delta d)\chi_4 = \Delta\chi_4 . \tag{9}$$

Here the antisymmetric tensor gauge fields appear once again, in the simple classical context related to the unconventional variational formulation proposed in [26], and what we find curious enough to note, in this way it also turns out to be naturally linked to the theory of strings.

The conventional formulation via minimal coupling of the electromagnetic field to the current requires one to introduce a potential α for the field strength Φ,

$$\Phi = d\alpha \tag{10}$$

and to write then down an action essentially (up to gauge-fixing terms and kinetic terms for the fields of the current) of the form

$$S_{conv} = (d\alpha, d\alpha) + (\zeta, \alpha). \tag{11}$$

We denote by (α, β) the scalar product

$$(\alpha, \beta) = \sum_{p=0}^{n} \int_M \alpha_p \, {}^*\beta_p \tag{12}$$

in the relevant space of forms over space-time M and regard α, Φ, and ζ as potential-, field-strength-, and current-forms which can be represented as sums of homogeneous components, $\alpha = \sum_{p=0}^{n} \alpha_p$, etc.

In order to formulate the action principle not by means of gauge potentials but directly in terms of the gauge invariant field strength, we proposed to use the stream potential form κ,

$$\zeta = \delta\kappa \tag{13}$$

(for more details about this quantity we refer, e.g., to [27, 28]) whose existence follows from the charge conservation equation

$$\delta\zeta = 0 \tag{14}$$

and the converse of the Poincare lemma (we recall that the co-derivative δ is related, $\delta\omega_p = (-1)^{p} {}^{*-1} d {}^* \omega_p$, by means of the Hodge $*$ operator and its inverse ${}^{*-1}$ to the exterior derivative d). Then the (generalized) Maxwell equations

$$d\Phi = 0 \qquad \delta\Phi = \zeta \tag{15}$$

follow not only from (11) but also from the action

$$S = (\Phi, d\Phi) - (\kappa, d\Phi) \tag{16}$$

in which Φ and κ are the variational variables and κ is related only after the variation to the external current ζ by means of eq.(13).

In order to construct a model in which the gauge field is a true dynamical field one should add a kinetic term for κ to the action (16),

$$S = (\Phi, d\Phi) - (\kappa, d\Phi) + S(\kappa) = \int [\mathcal{L}(\Phi) + \mathcal{L}_{int}(\Phi, \kappa) + \mathcal{L}(\kappa)] d^4x. \tag{17}$$

The action (16) is invariant under the gauge transformations

$$\kappa \longrightarrow \kappa' = \kappa + \delta y \tag{18}$$

and a simple choice by which this invariance is not broken and which treats the gauge field κ as fundamental is

$$S(\kappa) = (\delta\kappa, \delta\kappa). \tag{19}$$

We note that the free second-rank antisymmetric-tensor field theory with the action $S(\kappa_2) = (\delta\kappa_2, \delta\kappa_2)$ is essentially the same as that discussed in eqs.(1)-(9) and describes, in particular, once again a massless scalar free field, as a consequence of the equation of the motion

$$d\delta\kappa_2 = 0, \quad \text{i.e.,} \quad d\zeta_1 = 0 \tag{20}$$

in terms of the current

$$\zeta_1 = \delta\kappa_2 \tag{21}$$

and the fact that eq.(21) and $\delta^2 = 0$ imply the charge conservation

$$\delta\zeta_1 = 0. \tag{22}$$

It follows from eq.(20) that there exists a scalar field (a 0-form λ_0) such that

$$\zeta_1 = d\lambda_0 \tag{23}$$

which obeys, as is seen upon inserting eq.(23) into eq.(22), the massless free field equation

$$0 = \delta d\lambda_0 = (\delta d + d\delta)\lambda_0 = \Delta\lambda_0 . \tag{24}$$

In the general case the equations of motion following from the action (17), (19) are

$$d\Phi = 2d\delta\kappa \qquad \delta\Phi = \delta\kappa - 2d\delta\kappa. \tag{25}$$

The Lagrangian densities appearing in (17) are

$$\mathfrak{L}(\kappa) = (\partial_\mu K^\mu)^2 + (\partial_\mu K^{\mu\alpha})(\partial^\nu K_{\nu\alpha}) - (\partial_\mu \tilde{K}_\nu - \partial_\nu \tilde{K}_\mu)\partial^\mu \tilde{K}^\nu - (\partial_\mu \tilde{K})(\partial^\mu \tilde{K}) \tag{26}$$

$$\mathfrak{L}(\Phi) = F\partial_\mu F^\mu + F_\mu \partial_\sigma F^{\sigma\mu} - \frac{1}{2}\epsilon_{\mu\sigma\alpha\beta}\tilde{F}^\mu \partial^\sigma F^{\alpha\beta} + \tilde{F}\partial_\mu \tilde{F}^\alpha \tag{27}$$

and

$$\mathfrak{L}_{int}(\Phi,\kappa) = -F\partial_\mu K^\mu - F_\mu \partial_\sigma K^{\sigma\mu} + \frac{1}{2}\epsilon_{\mu\sigma\alpha\beta}\tilde{K}^\mu \partial^\sigma F^{\alpha\beta} - \tilde{K}\partial_\mu \tilde{F}^\mu \tag{28}$$

under the supposition (assumed in the work) that the fields tend sufficiently fast to zero at infinity so that the integration by parts is legitimate. The fields appearing in eqs.(17), (26)-(28) are (in 3+1-dimensional Minkowski space-time)

$$\Phi_0 = F(x) \qquad \Phi_2 = \frac{1}{2}F_{\mu\nu}(x)dx^\mu \wedge dx^\nu$$
$$\Phi_1 = F_\mu(x)dx^\mu \quad *\Phi_3 = \tilde{F}_\mu(x)dx^\mu \quad *\Phi_4 = \tilde{F}(x) \tag{29}$$

and analogously for the stream potential κ, e.g.,

$$\kappa_2 = \frac{1}{2}K_{\mu\nu}(x)dx^\mu \wedge dx^\nu \tag{30}$$

which has no κ_0-component, however.

The equations of motion for the Maxwell field Φ_2, following from (17), are

$$d\Phi_2 = 2d\delta\kappa_3 \qquad \delta\Phi_2 = \delta\kappa_2 - 2d\delta\kappa_1 \tag{31}$$

and those for the other field strength

$$\begin{aligned} d\Phi_0 &= 2d\delta\kappa_1 & d\Phi_3 &= 2d\delta\kappa_4 \\ d\Phi_1 &= 2d\delta\kappa_2 & \delta\Phi_3 &= \delta\kappa_3 - 2d\delta\kappa_2 \\ \delta\Phi_1 &= \delta\kappa_1 & \delta\Phi_4 &= \delta\kappa_4 - 2d\delta\kappa_3 . \end{aligned} \tag{32}$$

In order to ensure that there are no magnetic monopoles in eq.(31) we shall assume that κ_3 vanishes in M_4 and shall further confine ourselves for simplicity to the case where the only field source not identically zero is κ_2 while $\kappa_1 \equiv 0$ and $\kappa_4 \equiv$ $\equiv 0$ in addition to $\kappa_3 \equiv 0$. Then the Maxwell equations take their usual form

$$d\Phi_2 = 0 \qquad \delta\Phi_2 = \delta\kappa_2 = \zeta_1 \tag{33}$$

while in the rest of the field equations we can also assume $\Phi_1 \equiv 0$ and $\Phi_4 \equiv 0$ and reduce the system (32) to

$$\begin{aligned} d\Phi_1 &= 2d\delta\kappa_2 & d\Phi_3 &= 0 \\ \delta\Phi_1 &= 0 & \delta\Phi_3 &= -2d\delta\kappa_2 . \end{aligned} \tag{34}$$

The fields Φ_1 and Φ_3 are indispencable to the new variational formulation of the Maxwell electrodynamics. Hence the right-hand side of eq.(20) is no longer zero identically and it cannot be claimed, as in the previous case, that the second-rank antisymmetric-tensor gauge field κ_2 describes a scalar particle.

The Lagrangians (26)-(28) simplify considerably

$$\mathfrak{L}(\kappa) = (\partial_\mu K^{\mu\alpha})(\partial^\nu K_{\nu\alpha}), \qquad \mathfrak{L}_{int}(\Phi,\kappa) = -F_\mu \partial_\sigma K^{\sigma\mu}, \tag{35}$$

and

$$\mathfrak{L}(\Phi) = F_\mu \partial_\sigma F^{\sigma\mu} - \frac{1}{2}\epsilon_{\mu\sigma\alpha\beta}\tilde{F}^\mu \partial^\sigma F^{\alpha\beta} . \tag{36}$$

It follows from the singular Lagrangian (36) or (27) that the momenta P_ν and $\tilde{P}_\nu$ canonically conjugate to the fields F^ν and $\tilde{F}^\nu$ are zero, i.e., satisfy the simple constraints $P_\nu = 0$, $\tilde{P}_\nu =$ $= 0$. The momenta $P_{\rho\nu}$ conjugate to the Maxwell field $F^{\rho\nu}$ are $P_{\rho\nu} = P_{0\rho\nu}$, where

$$P_{\lambda\rho\nu} = \frac{\partial\mathfrak{L}(\Phi)}{\partial(\partial^\lambda F^{\rho\nu})} = \eta_{\lambda\rho}F_\nu - \eta_{\lambda\nu}F_\rho + \epsilon_{\lambda\rho\nu\mu}\tilde{F}^\mu , \tag{37}$$

so that

$$P_{\rho\nu} = \eta_{o\rho}F_\nu - \eta_{o\nu}F_\rho + \epsilon_{o\rho\nu\mu}\tilde{F}^\mu = -P_{\nu\rho}, \tag{38}$$

or, for the electric F^{on} and the magnetic F^{mn} parts separately the constraints are

$$P_{on} = F_n, \quad P_{mn} = \epsilon_{mn\ell}\tilde{F}^\ell . \tag{39}$$

The action (16) (or (17)) is not equivalent to the usual action (11) in the sense that the latter cannot be obtained from the former by simply transforming the independent variables Φ and κ into α and ζ. We also remark that the canonical energy-momentum tensor which follows from the singular Lagrangian (36),

$$T_{\mu\nu} = F^\nu(\partial_\mu F_{\lambda\nu} - \eta_{\mu\lambda}\partial^\sigma F_{\sigma\nu}) + \frac{1}{2}\tilde{F}^\tau(-\epsilon_{\tau\lambda\rho\nu}\partial_\mu F^{\rho\nu} + \eta_{\mu\lambda}\epsilon_{\tau\sigma\rho\nu}\partial^\sigma F^{\rho\nu}) \tag{40}$$

or, provided the Maxwell equations (33) are taken into account,

$$T_{\lambda\mu} = F^\nu(\partial_\mu F_{\lambda\nu} - \eta_{\mu\lambda}j_\nu) - \frac{1}{2}\tilde{F}^\tau\epsilon_{\tau\lambda\rho\nu}\partial_\mu F^{\rho\nu}, \tag{41}$$

and the corresponding energy-

$$\mathcal{H} = T_{oo} = -F^\nu\partial^s F_{s\nu} + \frac{1}{2}\tilde{F}^\tau\epsilon_{\tau s\rho\nu}\partial^s F^{\rho\nu} \quad (s = 1,2,3) \tag{42}$$

and momentum-densities

$$\mathcal{P}_\ell = T_{o\ell} = F^\nu\partial_o F_{\ell\nu} - \frac{1}{2}\tilde{F}^\tau\epsilon_{\tau\ell\rho\nu}\partial_o F^{\rho\nu}, \tag{43}$$

look also differently and the rules of the first order formalism should be applied. We note, however, that one can introduce the field potential and current 1-forms α_1 and ζ_1 in the formalism we are considering here, after the Maxwell's equations (33) are derived from the action (17), and then one can also postulate the conventional action (11) which further implies the usual canonical energy-momentum tensor.

Although the physical content of the classical Faraday-Maxwell theory remains the same in this first-order variational framework, it provides us with an example of antisymmetric gauge field. It describes the electric charge source of the electromagnetic field and is not directly related to the electromagnetic field itself in the manner the notoph field is. The kinetic term (19) for the stream potential antisymmetric gauge field discussed here should not be regarded as something more than an example.

The fact that the gauge field in our alternative variational formulation is the matter field of the source current (while the electromagnetic field Φ is manifestly gauge invariant) may

have some interesting consequences, in spite the physical content of the theory remains intact by the reformulation, in the sense mentioned above.

To illustrate the latter point we recall that it is customary to consider point current sources $\zeta_1 = j_\mu(x)dx^\mu$ of the kind

$$j^\mu(y) = e\int_{-\infty}^{\infty}\dot{\xi}^\mu(\tau)\delta^4(y - \xi(\tau))d\tau, \tag{44}$$

supported on the world line

$$y^\mu = \xi^\mu(\tau), \tag{45}$$

of the point charge e. In order to find the corresponding stream potential κ_2 one should solve eq.(21) and, as we have noted in [26,29], the solution uses a homotopy

$$f: [0,1] \times M_4 \longrightarrow M_4 \qquad f(\sigma, x) = y, \tag{46}$$

and this yields for the example (44)

$$*\kappa_2 = \frac{1}{2}\tilde{K}_{\alpha\beta}(x)dx^\alpha \wedge dx^\beta$$

$$\tilde{K}_{\alpha\beta}(x) = e\int_0^1 d\sigma\int_{-\infty}^{\infty} d\tau\,\delta^4(f(\sigma,x) - \xi(\tau))\dot{\xi}^\mu(\tau)\epsilon_{\mu\nu\lambda\rho}\frac{\partial f^\nu}{\partial\sigma}\frac{\partial f^\lambda}{\partial x^\alpha}\frac{\partial f^\rho}{\partial x^\beta}. \tag{47}$$

In this way we see that to each point charge located at the point $\xi^\mu(\tau)$ there corresponds a string located at the support of (47), i.e., on the set of those x for which

$$f(\sigma, x) = \xi(\tau), \tag{48}$$

or in a form solved with respect to x

$$x^\mu = h^\mu(\xi(\tau), \sigma) = \phi^\mu(\tau, \sigma). \tag{49}$$

In other words, each point ξ^μ of M_4 gets naturally an internal string-like structure in such a good physical theory as is the Faraday-Maxwell theory of electromagnetism.

The appearance of string-like sources related to the antisymmetric tensor gauge field κ_2 allows one to look for other ways to introduce kinetic terms in the action (17) for the stream potential sources, for instance, of the kind of the Nambu-Goto string action [30,31].

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REFERENCES

1. Ogievetsky V.I., Polubarinov I.V. - Yad.Fiz., 1966, 4, p.216.
2. Kemmer M. - Proc.Roy.Soc. (London), 1938, A166, p.1127.
3. Kemmer M. - Helv.Phys.Acta, 1965, 33, p.829.
4. Hayashi K. - Phys.Lett., 1973, 44B, p.497.
5. Kalb M., Ramond P. - Phys.Rev., 1974, D9, p.2273.
6. Cremmer E., Scherk J. - Nucl.Phys., 1974, B72, p.117.
7. Marshall C., Ramond P. - Nucl.Phys., 1975, B85, p.375.
8. Nambu Y. - Phys.Reports, 1976, 23, p.250.
9. Freedman D.Z., Townsend P.K. - Nucl.Phys., 1981, B177, p.282.
10. Freund P.G.O., Nepomechie R.I. - Nucl.Phys., 1982, B199, p.482.
11. Nepomechie R.I. - Nucl.Phys., 1983, B212, p.301.
12. Townsend P.K. - Phys.Lett., 1979, 88B, p.97.
13. Siegel W. - Phys.Lett., 1980, 93B, p.170.
14. Sezgin E., van Nieuwenhuizen P. - Phys.Rev., 1980, D22,p.301.
15. Kumura T. - Prog.Theor.Phys., 1980, 64, p.357.
16. Thierry-Mieg J., Baulieu L. - Nucl.Phys., 1983, B228, p.259.
17. Gliozzi F., Scherk J., Olive D. - Nucl.Phys., 1977, B122, p.253.
18. Cremmer E., Julia B. - Nucl.Phys., 1979, B159, p.141.
19. Siegel W. - Phys.Lett., 1979, 85B, p.333.
20. Gates S.J., Jr., Grisaru M.T., Rocek M., Siegel W. Superspace, Benjamin, 1983.
21. Howe P., Karlhede A., Lindström U., Rocek M. - Phys.Lett., 1986, 168B, p.89.
22. Deser S. - Phys.Rev., 1969, 187, p.1931.
23. Sugamoto A. - Phys.Rev., 1979, D19, p.1820.
24. Julia B., Touless G. - J.de Phys., 1979, 16, p.395.
25. Cartan E. - Comp.Rend., 1926, 182, p.956.
26. Karloukovski V.I. JINR Communication E2-86-736, Dubna, 1986.
27. Nisbet A. - Proc.Roy.Soc.(London), 1955, A231, p.250.
28. Cohen J.M., Kegeles L.S. - Phys.Rev., 1974, D10, p.1070.
29. Karloukovski V.I. - Ann.de l'Univ.de Sofia, 1986, 77.
30. Nambu Y. Lectures in Copenhagen, 1970.
31. Goto T. - Prog.Theor.Phys., 1971, 46, p.1560.

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Применение калибровочной теории
с абелевским антисимметричным тензором
в классической физике

После короткого обсуждения калибровочных теорий с абелевским антисимметричным тензором обсуждается появление таких полей в классической электродинамике,как потенциалы потока. Они связаны с вариационной формулировкой абелевских калибровочных теорий, несколько отличающейся от обычной.Обсуждается возможность существования струноподобных объектов уже на уровне классической физики без специального предположения об их существовании.

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Karloukovski V.I. E2-86-744

An Application of the Abelian
Antisymmetric-Tensor Gauge Theories
to Classical Physics

After a brief review of the development of the Abelian antisymmetric-tensor gauge theories we comment on the appearance of such a kind of fields in classical electrodynamics as stream potentials. They are related to a, somewhat different from the conventional, variational formulation of the Abelian gauge theories. We speculate about the possibility that string-like objects have their natural existence even on a classical level in physics without the need to introduce them ad hoc.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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