сообщения
объединенного
института
ядерных
исследований
дубна

E2-86-735

V.I.Karloukovski

ON THE VARIATIONAL FORMULATION OF CLASSICAL ABELIAN GAUGE FIELD THEORIES

1986

Let $j_\mu(x)$ be a given external electric current and $\zeta_1 = j_\mu(x)dx^\mu$ the corresponding one-form in n-dimensional flat space-time M with t plus signs and s minus signs in the diagonal metric $\eta^{\mu\nu} = \langle dx^\mu, dx^\nu \rangle$. The Maxwell equations

$$d\Phi_2 = 0, \quad \delta\Phi_2 = \mu_0\zeta_1 \qquad (1)$$

are to be solved in order to find the electromagnetic field $F_{\mu\nu}(x)$, or the corresponding two-form $\Phi_2 = \frac{1}{2} F_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu$ determined by the current. Here $\mu_0 = 4\pi \cdot 10^{-7}$ H/m is the magnetic constant of the vacuum (which we shall not write in the following) and the co-differentiation δ is the adjoint of the exterior derivative d with respect to the scalar product

$$(\lambda_p, \omega_p)_M = \int_M \lambda_p \wedge * \omega_p \qquad (2)$$

in the space of smooth p-forms on M (approaching zero sufficiently fast at the boundary if M is noncompact and with a boundary, or even with compact support), $\delta\omega_p = (-1)^p *^{-1} d * \omega_p$. We denote by $*^{-1}$ the inverse of the (unitary with respect to the same inner product (2)) Hodge * operator, $*^{-1}\omega_p = (-1)^{p(n-p)+s} * \omega_p$. The use of the differential form language in electrodynamics is too well-known and widely discussed to be reviewed in more detail here and we only quote /1-7/ as an instance.

There are two immediate concequences of the Maxwell equations (1). The one is the local charge-conservation law, or continuity equation

$$\delta\zeta_1 = 0 \qquad (3)$$

implied by $\delta^2 = 0$. The other is the existence of a potential one-form $a_1 = A_\mu(x)\, dx^\mu$ determined up to a gauge transformation $(a_1 \to a_1 + d\chi_0)$, such that

$$\Phi_2 = da_1 \qquad (4)$$

which is somewhat more indirect. It follows from the fact that Φ_2 is closed, the first equation in (1), and relies on the converse of the Poincare' lemma (the triviality of the de Rham cohomology for the degree considered).

The existence of vector potential gauge field $A_\mu(x)$ plays a crucial role for the variation and quantization of electrodynamics. In fact, another approach is not even known.

On the other hand, given the (global) current form ζ_1 on M, it seems more natural to apply the converse of the Poincaré lem-

© Объединенный институт ядерных исследований Дубна, 1986.

Объединенный институт ядерных исследований БИБЛИОТЕКА

ma to the charge-conservation equation (3) rather than to the Bianchi identity that the unknown two-form Φ_2 should obey. In this way we end up with the two-form $\kappa_2 = \frac{1}{2} K_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu$ of the stream potential [8]

$$\zeta_1 = \delta\kappa_2 \tag{5}$$

also determined modulo a gauge transformation

$$\kappa_2 \to \kappa_2' = \kappa_2 + \delta\gamma_3 \tag{6}$$

with an arbitrary three-form γ_3 and the possibility for an alternative variational formulation of the gauge theories.

We shall briefly discuss here this alternative variational formulation of the classical Abelian gauge theories, on a general level, considering the (generalized) Maxwell equations

$$d\Phi = 0 \tag{7}$$

and

$$\delta\Phi = \zeta \tag{8}$$

for arbitrary field-strength

$$\Phi = \sum_{p=0}^{n} \Phi_p \tag{9}$$

and current

$$\zeta = \sum_{p=0}^{n} \zeta_p \tag{10}$$

smooth differential forms on the flat $n = s + t$)- dimensional space-time manifold M, where Φ_p and ζ_p are their homogeneous components of degree p. We define the scalar product in the space of smooth forms on M to be

$$(\Phi, \Psi)_M = \sum_{p=0}^{n} (\Phi_p, \Psi_p)_M \tag{11}$$

with the scalar products $(\Phi_p, \Psi_p)_M$ of the p-forms defined in eq.(2).

The (generalized) Maxwell equations (7) and (8) can be obtained from the action

$$S = (\Phi, d\Phi)_M - (\kappa, d\Phi)_M \tag{12}$$

in which Φ is the field-strength form and the external current enters via its stream potential κ,

$$\zeta = \delta\kappa. \tag{13}$$

The variation of the action S with respect to κ yields the source-free equation

$$d\Phi = 0 \tag{14}$$

while the variation with respect to Φ results in

$$d\Phi + \delta\Phi - \delta\kappa = 0 \tag{15}$$

which is simplified to

$$\delta\Phi = \delta\kappa \tag{16}$$

in view of eq.(14) and coincides with the second equation, eq.(8), according to (13).

The stream potential can be constructed, given the current ζ, as a solution of eq.(13), i.e., of

$$d * \kappa = (-1)^{\deg\zeta - 1} * \zeta. \tag{17}$$

This can always be done, at least locally, in any contractable neighbourhood U of every point x of the space-time M. If K is a homotopy operator and f is some homotopy between U and x, one can write (of., e.g. [2,9]) the solution of equation (17) explicitly in U

$$* \kappa = K[f^*((-1)^{\deg\zeta - 1} * \zeta)], \tag{18}$$

where f^* is the pull-back of f. If the manifold is R^n (endowed with the metric $\eta^{\mu\nu}$ we are considering), for instance, and the current is defined in it everywhere, the solution (18) is defined in the whole space-time. More generally, if M_0 is the submanifold of M, where the current can be defined, the triviality of the de Rham cohomology $H^p(M_0)$ is the condition that the homogeneous component $(*\kappa)_p$ of the stream potential can be defined by solving equation (17). For example, if one deals with currents ζ with compact support in R^n and looks for compactly supported stream-potential forms as solution of (17) one should beware of the fact that $H^n_c(R^n) = R$.

The stream potential is determined up to a gauge transformation or

$$*\kappa' = *\kappa + d(-1)^{\deg\gamma} * \gamma, \tag{19}$$

or

$$\kappa' = \kappa + \delta\gamma \tag{20}$$

under which the action (12) is invariant. Of course, one can also solve Eq.(14) (provided there are no obstructions for the manifold where Φ is defined) to introduce a gauge potential a

$$\Phi = da \tag{21}$$

with its own gauge transformation

$$a' = a + d\chi \tag{22}$$

and the usual geometric interpretation. This can be done, however, (in the variational formulation discussed here) only after the field equations are written down. And the action functional (12) not only has the property that it equals zero for any field configuration which is a solution of the field equations (14) and (16) but it is also annulated by any solution of the form (21) of only the first of the equations, eq.(14). Hence we cannot get the conventional action by simply inserting $\Phi = d\alpha$ in the action (12). We slao mention that our action (12) is obviously invariant under the usual gauge transformations (22) by construction.

The stream potential κ was conceived above, and in the derivation of equation (14) in particular, as a dynamical variable which was fixed after that as an external field by its postulated relation (13) to the given external current. It can be treated as well as an unknown dynamical variable, on an equal footing with the field strength Φ, with its own dynamics coupled to that of Φ, and equation (13) can be used at the end to translate it in terms of the current. There are different possibilities to write more actions in conformity with such a point of view. The simplest one is perhaps to add to the previous action (12) a kinetic term of the form $(\delta\kappa, \delta\kappa)_M$ which respects the invariance under the gauge transformations (20) or (19). The variation of the action

$$S = (\Phi, d\Phi)_M - (\kappa, d\Phi)_M + (\delta\kappa, \delta\kappa)_M \tag{23}$$

so obtained, in the fields Φ and κ yields the equations

$$d\Phi = 2d\delta\kappa, \qquad \delta\Phi = \delta\kappa - 2d\delta\kappa. \tag{24}$$

There are, in general, sources now in both (generalized) Maxwell equations. One could argue if it is possible to include to conventional electrodynamics in this new framework, and if there always should be magnetic monopoles, in particular, implied by the action (23). To explain this latter point let us consider, e.g., the case of 3+1-dimensional Minkowski space-time M_4. The Maxwell equations for the homogeneous part of Φ of degree 2, $\Phi_2 = \frac{1}{2} F_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu$, now read (with μ_0 included again):

$$d\Phi_2 = \frac{2}{\mu_0} d\delta\kappa_3, \qquad \delta\Phi_2 = \mu_0 \delta\kappa_2 - \frac{2}{\mu_0} d\delta\kappa_1, \tag{25}$$

and it follows from them that there will not be magnetic monopoles, provided $d\delta\kappa_3 \equiv 0$ or $\kappa_3 \equiv 0$ in particular, and vice versa, one possibility to incorporate them in the present theory is by means of a 3-form component κ_3 of the stream potential in the action, such that $d\delta\kappa_3 \neq 0$.

A point to be noted is that classical strings naturally appear in this new formulation due to the fact that the support of elementary stream potential source κ_2 in degree 2 (corresponding to a point current source ζ_1) actually is a string. This follows from eq.(18) and is clarified in more detail in a following publication under preparation.

It is pleasure to thank A.T.Filippov and E.Kapuscik for valuable discussions helping the improvement of the work.

References

1. Misner C.W., Wheeler J.- Ann.Phys.(N.Y.), 1957, 2, p.525.
2. Flanders H. Differential Forms, Academic Press, N.Y. 1963.
3. Cohen J.M., Kegeles L.S.- Phys.Rev., 1974, D10, p.1070.
4. Sternberg S. In: Differential Geometrical Methods in Mathematical Physics, Lecture Notes in Mathematics, vol.676, Springer-Verlag, 1978.
   Thirring W., Wallner R., ibid. A.O.Barut, ibid.
5. Eguchi T., Gilkey P.B., Hanson A.J.- Phys.Rep., 1980, 66, p.213.
6. Curtis W.D., Miller F.R. Differentiable Manifolds and Theoretical Physics, Academic Press, N.Y., 1985.
7. Karloukovski V. Ann.de l'Univ. de Sofia, 1986, 77.
8. Nisbet A.- Proc.Roy.Soc.(London), 1955, A231, p.250.
9. Bott R., L.W.Tu. Differential Forms in Algebraic Topology, Springer-Verlag, 1982.

Received by Publishing Department
on November 11, 1986.

Карлуковски В.И. E2-86-735
О вариационной формулировке классических абелевых калибровочно-полевых теорий

Показано, как можно сформулировать принцип действия для классических абелевых калибровочных теорий не через калибровочные потенциалы и токи, а с помощью калибровочно-инвариантных напряженностей поля и зависящих от калибровки потенциалов потока. Обсуждение проводится для формально общего числа n = s+t пространственно-временных измерений и использует для краткости язык дифференциальных форм.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1986

Karloukovski V.I. E2-86-735
On the Variational Formulation of Classical Abelian Gauge Field Theories

It is shown how one can formulate an action principle for classical Abelian gauge theories not by means of gauge potentials and currents but in terms of the gauge invariant field strengths and gauge variant stream potentials. The discussion is on a general formal level in n = s+t space-time dimensions and uses, for brevity, the language of differential forms.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1986