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A NON-RELATIVISTIC MODEL OF TWO-PARTICLE DECAY: GALILEAN INVARIANCE REVISITED

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This is an addendum to our recently published paper [1], in which we have discussed a simple non-relativistic model of two-particle decay. In Section 3 of the paper, Galilean invariance of the model is treated ; we construct there the appropriate representation of the Galilei group $\mathcal{G}$. However, Theorem 3.1 of Ref.1 is in error as stated ; it requires an additional assumption, namely that the function v , which specifies the interaction Hamiltonian, is rotationally invariant. This corresponds just to the Galilean-invariant case, which is studied in the rest of the paper, and will be studied in the sequels ; so the conclusions are not affected. Nevertheless, we would like to correct the error, and to present at the same time a more detailed discussion which should enlight the role of Galilean transformations of the model also in the non-symmetric case*).

Throughout this note, we use the notation of Ref.1 . The corrected assertion reads as follows :

<u>Theorem 1</u> : Let $v(\vec{x}) = v_1(r)$ for some $v_1 \in L^2(\mathbb{R}^+, r^2 dr)$. Then there is a unitary projective representation of $\mathcal{G}$ on $\mathcal{H}$ defined by

$$(U(b,\vec{a},\vec{v},R)\Psi)(\vec{X},\vec{x}) = = e^{(i/2)M\vec{v}.(\vec{v}b+2\vec{X}-\vec{a})}\,(e^{iH_g b}\Psi)(R^{-1}(\vec{X}+\vec{v}b-\vec{a}),R^{-1}\vec{x}) \quad . \tag{1a}$$

It holds

$$U(\gamma')U(\gamma) = \omega(\gamma',\gamma)U(\gamma'\gamma) \tag{2a}$$

for all $\gamma,\gamma' \in \mathcal{G}$, where the multiplier ω is given by

$$\omega(\gamma',\gamma) = e^{(i/2)M(\vec{v}'.R'\vec{a}-\vec{a}'.R'\vec{v}-R'\vec{v}.\vec{v}'b)} \quad . \tag{2b}$$

It is useful to introduce the following Euclidean-transformations operators

*) In the relation (3.11a) of Ref.1, $\vec{a}$ should be replaced by $\vec{a}+\vec{v}t$ and b by zero (or we should set $t'=t=0$) . Other minor corrections are left to a printed version , since they probably cannot cause a misunderstanding.

$$S(R,\vec{a}) : \quad (S(R,\vec{a})\psi)(\vec{X},\vec{x}) = \psi(R^{-1}(\vec{X}-\vec{a}),R^{-1}\vec{x}) \tag{3a}$$

which obviously fulfil the relations

$$S(R',\vec{a}')S(R,\vec{a}) = S(R'R,\vec{a}'+R'\vec{a}) \quad . \tag{3b}$$

The definition relation (1a) can be then expressed for $\gamma \equiv (b,\vec{a},\vec{v},R)$ as

$$U(\gamma) = e^{(i/2)M\vec{v}.(\vec{v}b+2\vec{X}-\vec{a})}\, S(R,\vec{a}-\vec{v}b)\, e^{iH_g b} \quad . \tag{1b}$$

The proof relies primarily on Lemma 3.2 of Ref.1 . Beside it, however, other auxiliary assertions are needed.

<u>Lemma 2</u> : The operators $\hat{S}(R,\vec{a}) := FS(R,\vec{a})F^{-1}$ act as

$$(\hat{S}(R,\vec{a})\hat{\psi})(\vec{P},\vec{p}) = e^{-i\vec{P}.\vec{a}}\,\hat{\psi}(R^{-1}\vec{P},R^{-1}\vec{p}) \quad . \tag{4}$$

<u>Proof</u> : The relation (4) is verified directly for $\hat{\psi} = F\psi$ with $\psi \in L^2 \cap L^1$, and extended by continuity to $\mathcal{H}$. ∎

<u>Lemma 3</u> : The relation

$$e^{-iH_g b}\, S(R,\vec{a})\, e^{iH_g b} = S(R,\vec{a}) \tag{5a}$$

holds for all $R \in O(3)$, $\vec{a} \in \mathbb{R}^3$ and $b \in \mathbb{R}$ iff $v(\vec{x}) = v_1(r)$ for all $\vec{x} \in \mathbb{R}^3$ and some $v_1 \in L^2(\mathbb{R}^+,r^2dr)$.

<u>Proof</u> : According to Stone theorem, the relation (5a) is equivalent to $S(R,\vec{a})H_g \subset H_g S(R,\vec{a})$ or

$$\hat{S}(R,\vec{a})\hat{H}_g \subset \hat{H}_g \hat{S}(R,\vec{a}) \quad . \tag{5b}$$

For $\hat{\psi} \in D(\hat{H}_g)$, the functions $\vec{P}^2\hat{\psi}_u$ and $(\frac{\vec{P}^2}{2M}+\frac{\vec{p}^2}{2m})\hat{\psi}_d$ are square integrable so $\hat{S}(R,\vec{a})\hat{\psi} \in D(\hat{H}_g)$; it yields $\hat{S}(R,\vec{a})D(\hat{H}_g) = D(\hat{H}_g)$. Now one can take $\hat{\psi} \in D(\hat{H}_g)$ and calculate easily

$$([\hat{H}_g,\hat{S}(R,\vec{a})]\hat{\psi})(\vec{P},\vec{p}) = g\, e^{-i\vec{P}.\vec{a}} \begin{pmatrix} \int_{\mathbb{R}^3} \overline{[\hat{v}(R\vec{k})-\hat{v}(\vec{k})]}\, \hat{\psi}_d(R^{-1}\vec{P},\vec{k})\, d\vec{k} \\ [\hat{v}(\vec{p})-\hat{v}(R^{-1}\vec{p})]\,\hat{\psi}_u(R^{-1}\vec{P}) \end{pmatrix} . \tag{6}$$

Hence if v is rotationally invariant, the same is true for $\hat{v}$ and the relation (5b) holds (even as an equality). On the other hand, if the relations (5) are valid, then the rhs of (6) must be zero for all $\hat{\psi} \in D(\hat{H}_g)$; it is possible only if $\hat{v}(R\vec{p}) = \hat{v}(\vec{p})$ holds for all $\vec{p} \in \mathbb{R}^3$ and $R \in O(3)$. ∎

<u>Proof of Theorem 1</u> : It follows from (1b) that

$$e^{iH_g b'}\,U(\gamma) = e^{iH_g b'}\, e^{(i/2)M\vec{v}.(\vec{v}b+2\vec{X}-\vec{a})}\, e^{-iH_g b'}\, e^{iH_g b'}\, S(R,\vec{a}-\vec{v}b)\, e^{iH_g b}$$

so Lemma 3.2 of Ref.1 combined with Lemma 3 gives

$$e^{iH_g b'}\,U(\gamma) = e^{(i/2)M\vec{v}.[\vec{v}(b+b')+2\vec{X}-\vec{a}]}\, S(R,\vec{a}-\vec{v}(b+b'))\, e^{iH_g(b+b')} \quad .$$

Now one has to substitute from here to the relation

$$U(\gamma')U(\gamma) = e^{(i/2)M\vec{v}'.(\vec{v}'b'+2\vec{X}-\vec{a}')}\, S(R',\vec{a}'-\vec{v}'b')\, e^{iH_g b'}\, U(\gamma)$$

and a straightforward calculation leads to (2) . ∎

The assumption of rotational invariance in Theorem 1 is actually necessary as the following assertion shows :

<u>Proposition 4</u> : If the operators (1) fulfil the relations (2), then $v(\vec{x}) = v_1(r)$ for some $v_1 \in L^2(\mathbb{R}^+,r^2dr)$.

<u>Proof</u> : We set $\vec{v} = \vec{v}' = 0$, then (2a) acquires the form

$$S(R',\vec{a}')\, e^{iH_g b'}\, S(R,\vec{a})\, e^{iH_g b} = S(R'R,\vec{a}'+R'\vec{a})\, e^{iH_g(b+b')} \quad .$$

In particular, choosing $\gamma' = (-b,-R^{-1}\vec{a},0,R^{-1})$, we get the relation (5a) so the result follows from Lemma 3 . ∎

Let us turn now to the physical meaning of the representation U . Consider first the <u>isochronous subgroup</u> $\mathcal{G}' = \{\gamma \in \mathcal{G} : b = 0\}$ of $\mathcal{G}$. If two observers connected with the reference frames S,S′ describe the state of our system <u>at an instant</u> t by ψ_t and ψ_t' , respectively, then these vectors are related by

$$\psi_t' = U(0,\vec{a}+\vec{v}t,\vec{v},R)\psi_t = e^{(i/2)M\vec{v}.(2\vec{X}-\vec{a}-\vec{v}t)}\, S(R,\vec{a}+\vec{v}t)\psi_t \quad ; \tag{7a}$$

it can be written also as

$$\psi_t'(\vec{X}',\vec{x}') = e^{(i/2)M\vec{v}.(2R\vec{X}+\vec{a}+\vec{v}t)}\, \psi_t(\vec{X},\vec{x}) \quad . \tag{7b}$$

This is, of course, the passive interpretation. The active one, in which we have two decaying systems tied to the reference frames S and S′ , is obtained simply by replacing γ by γ^{-1} , or equivalently, by interchanging the primed and unprimed state vectors.

On the other hand, there is a substantial difference between the active and passive interpretations of the <u>time-translations subgroup</u> of $\mathcal{G}$. The active time translations are connected with the evolution, and therefore they are governed by the dynamics of the model. The state vector

$$\psi_t = e^{-iH_g t}\psi = U(-t,0,0,I)\psi \tag{8a}$$

corresponds to the initial condition $\psi_0 = \psi$. The same state is in the primed reference frame described by the vector ψ'_t which is in view of (7a) and (2a) given by

$$\psi'_t = U(-t,\vec{a},\vec{v},R)\psi \quad . \tag{8b}$$

This relation is consistent in the following sense. The initial condition in the primed reference frame is $\psi'_0 = \psi' = U(0,\vec{a},\vec{v},R)\psi$; expressing then $\psi'_t = U(-t,0,0,I)\psi'$, we arrive again at (8b).

In contrast with this, definition of time translations in the passive interpretation is a matter of convention. The simplest possibility is the following : the state is not changed, when an observer refixes his clock, $t' = t+b$, so

$$\psi'_{t'} = U(0,\vec{a}+\vec{v}t,\vec{v},R)\psi_t$$

for any $b \in \mathbb{R}$. The corresponding representation of $\mathcal{G}$ (for a fixed t) is trivial in the part of time translations. There are, however, other possibilities (similar as in Ref.2). Using the equation (8a) in the two frames, the last relation gives

$$\psi'_{t'=0} = U(b,\vec{a},\vec{v},R)\psi_{t=0} \quad ,$$

which is the definition of passive Galilei transformation used in (1a).

In conclusion, let us say a few words about the general case when v is not rotationally invariant. A brief inspection of the proof of Theorem 1 shows that (1) defines a unitary projective representation of the isochronous subgroup $\mathcal{G}'$ of $\mathcal{G}$ (in fact, the relations (2) remain to hold if only $b'=0$). Hence the conclusions concerning instantaneous Galilei transformations do not depend on the rotational invariance of v . Furthermore, the (active) time translations in the reference frame S are given by the operators $e^{-iH_g t}$ and the relations (8) hold again.

The only difference is that, according to Proposition 4, $U(.)$ is no longer a representation of the full Galilei group $\mathcal{G}$. The physical meaning of this fact can be easily illustrated. The relations (2) and (8) give

$$\psi'_t = U'(-t,0,0,I)\psi' \quad , \tag{9a}$$

where

$$U'(-t,0,0,I) = \tag{9b}$$
$$= U(0,\vec{a}+\vec{v}t,\vec{v},R)U(-t,0,0,I)U(0,-R^{-1}\vec{a}\,,\,-R^{-1}\vec{v}\,,\,R^{-1}) \quad .$$

If v is not rotationally invariant, then (9b) is not equal to $e^{-iH_g t}$. In particular, for a pure rotation, $\gamma = (0,0,0,R)$, one has/3/

$$U'(-t,0,0,I) = e^{-iH'_g t} \quad ,$$

where $H'_g = U(\gamma)H_g U(\gamma^{-1})$ is the rotated Hamiltonian.

References

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Нерелятивистская модель двухчастичного распада:
снова о галилеевой инвариантности

В этой заметке приведено подробное обсуждение проблемы галилеевой инвариантности для нерелятивистской модели двухчастичного распада рассмотренной в нашей недавней работе. В частности, мы исправляем здесь ошибку, допущенную при формулировке одной теоремы из этой работы.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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A Non-Relativistic Model of Two-Particle Decay:
Galilean Invariance Revisited

In this note, we discuss in detail the problem of Galilean invariance for a non-relativistic model of two-particle decay considered in our recent paper. Corrected version of a theorem deduced there is presented.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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