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институт
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исследований
дубна

E2-86-553

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SUPERSPACES FOR N = 3 SUPERSYMMETRY

Submitted to "Ядерная физика"

1986

I. Introduction

Invention of the harmonic superspace (SS) approach opened a way to unconstrained superfield (SF) formulations of all N=2 theories [1-3] and of the N=3 Yang-Mills theory [4]. An urgent problem ahead is to construct an unconstrained off-shell formulation of N=3 Einstein supergravity. Hitherto it was known only on shell[5,6]. We are led by reasonings [7] that follow a general compensating strategy (see [8] and references therein).
According to these reasonings an off-shell interaction of N=3 conformal supergravity[9] with three Maxwell multiplets produces the off-shell Einstein supergravity. To perform this program one has to find out N=3 conformal supergravity prepotentials and to establish how a local superconformal group acts in the N=3 real analytic superspace where the N=3 Yang-Mills action is written down. We shall see that this procedure is rather analogous to that in N=2 case[10].

We shall establish in the present paper the rigid superconformal properties of the real N=3 analytic SS and check the superconformal invariance of the off-shell N=3 Yang-Mills theory.

Moreover, in the present paper we reveal existence of an essentially complex analytic SS having only three Weyl spinor coordinates (instead of four in the real analytic SS). Possibilities are also indicated to impose additional analyticity conditions with respect to harmonic variables.

The paper is planned as follows: In section 2 we remind the reader of basics of the N=3 harmonic SS introduced by S.Kalitzin, E.Sokatchev and the present authors[4]. This is made for the reader's convenience and because we have improved some conventions and notation. We use a modernized combined conjugation definition that is easier to deal with and that relates more directly to the combined conjugation in the N=2 case[1].

The main section is the third one. Here we find such a realization of the superconformal group in the harmonic SS that leaves invariant its analytic sub SS. In what follows we shall often use the theorem of this section that the Berezianin (superdeterminant) of superconformal transformations in real analytic SS is unity. These transformations for harmonics and harmonic derivatives are presented in a compact form. In subsections 3.6 and 3.7 we digress temporarily from the basic line of attack and make some intriguing observations. The latter concerns the existence of complex analytic N=3 SS with a smaller number of Grassmann or/and harmonic coordinates and realization of a superconformal group in these SS's .

Finally, section 4 treats superconformal invariance of the N=3 SYM theory. This becomes rather evident after establishing $SU(2,2|3)$ transformation properties of the SYM prepotentials and field strengths.

Appendices contain the explicit form and algebra of harmonic derivatives, the N=3 superconformal transformations of analytic coordinates and some details connected with the complex analytic SS . In particular, we demonstrate that the latter contains the real analytic SS as a hypersurface.

2. The ABC of N=3 harmonic SS

In this section we give a brief review of basic conventions and concepts concerning N=3 harmonic SS /4/ . We adopt here a modernized combined conjugation operation. Being equivalent in essence to the original one /4/ the new operation is more convenient to keep it in mind and is in a direct correspondence with that for the N=2 case /1/ .

2.1. Central basis of harmonic N=3 SS contains the coordinates of the usual N=3 SS

$$R^{4|12} = \{ x^{\dot\alpha\alpha},\ \theta^\alpha_i,\ \bar\theta^{\dot\alpha i} \equiv \overline{(\theta^\alpha_i)} \} \qquad (2.1)$$

and, <u>in addition</u>, the harmonics $u^a_i, u^i_a \equiv \overline{(u^a_i)}$. The latter are coordinates of the N=3 supersymmetry automorphism's group manifold SU(3) . These harmonics obey the unitarity and unimodularity conditions

$$u^a_i u^j_a = \delta^j_i\ ,\quad u^a_i u^i_b = \delta^a_b\ ,\quad i,j,a,b = 1,2,3$$
$$\det u = 1 \Rightarrow \varepsilon^{ijk} u^a_i u^b_j u^c_k = \varepsilon^{abc}\ . \qquad (2.2)$$

Differentiation with respect to harmonics is performed by harmonic derivatives

$$D^a_b = u^a_i \frac{\partial}{\partial u^b_i} - u^i_b \frac{\partial}{\partial u^i_a} - \frac{1}{3}\delta^a_b \left(u^c_i \frac{\partial}{\partial u^c_i} - u^i_c \frac{\partial}{\partial u^i_c} \right)\ ,\qquad D^a_a = 0\ . \qquad (2.3)$$

The reader can easily check that D^a_b agree with the defining properties (2.2), that they form the $SU(3)_D$ algebra

$$[D^a{}_b, D^c{}_d] = \delta^c_b D^a{}_d - \delta^a_d D^c{}_b \qquad (2.4)$$

and that this $SU(3)_D$ commutes with $SU(3)_A$ that rotates indices i,j,k (but not a,b,c !). These $SU(3)_A$ and $SU(3)_D$ groups are realized on harmonics u^a_i by left and right multiplications, respectively.

The Cartan algebra of $SU(3)_D$ is given by harmonic derivatives

$$D^0_{I} \equiv D^1_1 - D^2_2\ ,\qquad D^0_{II} = 3\,(D^1_1 + D^2_2) \qquad (2.5)$$

that define two U(I)-charges. One can consider harmonics as eigenvectors of these charges (see (2.3))

$$D^0_I u^a_i = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}^a{}_b u^b_i\ ,\qquad D^0_{II} u^a_i = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}^a{}_b u^b_i\ , \qquad (2.6)$$

$$D^0_I u^i_a = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 0 \end{pmatrix}_a{}^b u^i_b\ ,\qquad D^0_{II} u^i_a = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 2 \end{pmatrix}_a{}^b u^i_b\ .$$

Correspondingly index a is represented by a pair of U(I)-indices

$$u^a_i = (u^{1,1}_i, u^{-1,1}_i, u^{0,-2}_i)\ ,\qquad u^i_a = (u^{-1,-1\,i}, u^{1,-1\,i}, u^{0,2\,i})\ . \qquad (2.7)$$

U(I) x U(I) notation for six remaining derivatives (2.3) does not need special explanation

$$D^1_2 = D^{2,0} = u^{1,1}_i \frac{\partial}{\partial u^{-1,1}_i} - u^{1,-1\,i} \frac{\partial}{\partial u^{-1,-1\,i}} \equiv \partial^{2,0}$$

$$D^1_3 = D^{1,3} = u^{1,1}_i \frac{\partial}{\partial u^{0,-2}_i} - u^{0,2\,i} \frac{\partial}{\partial u^{-1,-1\,i}} \equiv \partial^{1,3}$$

$$D^3_2 = D^{1,-3} = u^{0,-2}_i \frac{\partial}{\partial u^{-1,1}_i} - u^{1,-1\,i} \frac{\partial}{\partial u^{0,2\,i}} \equiv \partial^{1,-3}. \qquad (2.8)$$

Appendix A contains an explicit form for the rest of derivatives and algebra (2.4) . The latter consists of commutators with the U(I)-charge operators

$$\left[D^{0}_{I}, D^{q_I,q_{II}}\right] = q_I D^{q_I,q_{II}} \ , \quad \left[D^{0}_{II}, D^{q_I,q_{II}}\right] = q_{II} D^{q_I,q_{II}} \qquad (2.9)$$

and of commutators of $D^{q_I,q_{II}}$ between themselves.

2.2. A comment. The significance of U(I) x U(I) charges is defined by our ultimate goal, i.e. by the SF N=3 Yang–Mills theory and the N=3 supergravity. As has been shown in /4/ the first requires SF's of definite U(I)-charges. The even part of the corresponding SS includes the space-time M^4 and , in fact, the homogeneous space $SU(3)/U(1)\otimes U(1)$ (but not SU(3) itself). Indeed, harmonics u . have (taking into account (2.2)) 8 independent real degrees of freedom describing the SU(3)-manifold. However, the condition that SF $f^{q_I,q_{II}}$ has two definite U(I)-charges

$$D^{0}_{I} f^{q_I,q_{II}} = q_I f^{q_I,q_{II}} \ , \quad D^{0}_{II} f^{q_I,q_{II}} = q_{II} f^{q_I,q_{II}} \qquad (2.10)$$

fixes dependence on two degrees of freedom. Evidently, in such a framework one works in a manifestly SU(3) invariant way.

2.3. We shall finish a review of harmonics properties by discussion of the combined conjugation operation.

It includes the complex conjugation and one of the Weyl reflections of SU(3) algebra. The latter acts on indices α of harmonics and can be chosen in several ways. Time has shown that the original choice /4/ was not the best to deal with. The following one is better:

$$\overset{*}{\overline{u^{1,1}_i}} = u^{1,-1\,i} \ , \quad \overset{*}{\overline{u^{1,-1\,i}}} = -u^{1,1}_i \ , \quad \overset{*}{\overline{u^{0,-2}_i}} = u^{0,2\,i}$$
$$\overset{*}{\overline{u^{-1,1}_i}} = -u^{-1,-1\,i} \ , \quad \overset{*}{\overline{u^{-1,-1\,i}}} = u^{-1,1}_i \ , \quad \overset{*}{\overline{u^{0,2\,i}}} = u^{0,-2}_i \ . \qquad (2.11)$$

The new rule is easier to memorize; the first U(I)-charge remains unchanged while the second one changes its sign. The reader will easily see that the combined conjugation (2.11) is compatible with the conditions(2.2) and that the harmonic derivatives have the following reality properties:

$$\overset{*}{\overline{D^{0}_{I}}} = D^{0}_{I} \ , \quad \overset{*}{\overline{D^{0}_{II}}} = -D^{0}_{II} \ , \quad \overset{*}{\overline{D^{\pm2,0}}} = D^{\pm2,0}$$

$$\overset{*}{\overline{D^{1,\pm3}}} = \mp D^{1,\mp3} \ , \quad \overset{*}{\overline{D^{-1,\pm3}}} = \pm D^{-1,\mp3} . \qquad (2.12)$$

2.4. The introduction of harmonics enlarges the number of even dimensions. At the same time a wonderful possibility appears to single out in the harmonic SS its analytic sub SS /4/ . The latter has a smaller number of Grassmann coordinates and is closed with respect to the N=3 supersymmetry. Indeed, in the central basis $\{x^{\dot\alpha\alpha}, \theta^{\alpha}_{i}, \bar\theta^{\dot\alpha i}, u\}$ the N=3 supersymmetry transformations have a form

$$\delta_Q x^{\dot\alpha\alpha} = 2i\left(\theta^{\alpha}_{i}\bar\varepsilon^{\dot\alpha i} - \varepsilon^{\alpha}_{i}\bar\theta^{\dot\alpha i}\right) \ , \quad \delta_Q u = 0$$
$$\delta_Q \theta^{\alpha}_{i} = \varepsilon^{\alpha}_{i} \ , \quad \delta_Q \bar\theta^{\dot\alpha i} = \bar\varepsilon^{\dot\alpha i} \qquad (2.13)$$

Now let us pass from central basis to analytic one

$$\{x_A^{\dot\alpha\alpha}, \theta^{-1,-1}_{\alpha}, \theta^{1,-1}_{\alpha}, \theta^{0,2}_{\alpha}, \bar\theta^{1,1}_{\dot\alpha}, \bar\theta^{-1,1}_{\dot\alpha}, \bar\theta^{0,-2}_{\dot\alpha}, u\} , \qquad (2.14)$$

where

$$x_A^{\dot\alpha\alpha} = x^{\dot\alpha\alpha} + 2i\left(\theta^{1,-1\alpha}\bar\theta^{-1,1\dot\alpha} - \theta^{-1,-1\alpha}\bar\theta^{1,1\dot\alpha}\right)$$
$$\theta^{q_I,q_{II}}_{\alpha} = \theta_{\alpha i} u^{q_I,q_{II}\,i} \ , \quad \bar\theta^{q_I,q_{II}}_{\dot\alpha} = \bar\theta^{i}_{\dot\alpha} u^{q_I,q_{II}}_{i} . \qquad (2.15)$$

We see that the analytic sub SS

$$\{\zeta, u\} = \{x_A^{\dot\alpha\alpha}, \theta^{1,-1}_{\alpha}, \theta^{0,2}_{\alpha}, \bar\theta^{1,1}_{\dot\alpha}, \bar\theta^{0,-2}_{\dot\alpha}, u\} \qquad (2.16)$$

is closed under transformations (2.13):

$$\delta_Q x_A^{\dot\alpha\alpha} = 2i\left[2\theta^{1,-1\alpha}u^{-1,1}_{i} + \theta^{0,2\alpha}u^{0,-2}_{i}\right]\bar\varepsilon^{\dot\alpha i} -$$
$$-2i\varepsilon^{\alpha}_{i}\left[2\bar\theta^{1,1\dot\alpha}u^{-1,-1\,i} + \bar\theta^{0,-2\dot\alpha}u^{0,2\,i}\right] \qquad (2.17)$$
$$\delta_Q \theta^{q_I,q_{II}}_{\alpha} = \varepsilon_{\alpha i}u^{q_I,q_{II}\,i} \ , \quad \delta_Q \bar\theta^{q_I,q_{II}}_{\dot\alpha} = \bar\varepsilon^{i}_{\dot\alpha}u^{q_I,q_{II}}_{i} \ , \quad \delta_Q u = 0 .$$

The SS (2.16) is real with respect to the combined conjugation (2.11):

$$\overset{*}{\overline{x_A^{\dot\alpha\alpha}}} = x_A^{\dot\alpha\alpha} \ , \quad \overset{*}{\overline{\theta^{1,-1}_{\alpha}}} = -\bar\theta^{1,1}_{\dot\alpha} \ , \quad \overset{*}{\overline{\bar\theta^{1,1}_{\dot\alpha}}} = \theta^{1,-1}_{\alpha} \ ,$$
$$\overset{*}{\overline{\theta^{0,2}_{\alpha}}} = \bar\theta^{0,-2}_{\dot\alpha} \ , \quad \overline{\bar\theta^{0,-2}_{\dot\alpha}} = \theta^{0,2}_{\alpha} . \qquad (2.18)$$

We shall refer to SF's defined on (2.16) as to analytic SF's. The analytic SS and SF's on it are of great importance in N=3 theories. Indeed, (i) Prepotentials and gauge parameters of the N=3 Yang–Mills t theory are described by the analytic SF's and the action of this theory is given by an integral over analytic SS [4].

2.5. Analytic SF's obey automatically the analyticity conditions

$$D^{1,1}_{\alpha}\phi(\zeta,u) \equiv u^{1,1}_{i} D^{i}_{\alpha}\phi = 0, \quad \bar{D}^{1,-1}_{\dot\alpha}\phi(\zeta,u) \equiv u^{1,-1\,i}\bar{D}_{\dot\alpha i}\phi = 0 \tag{2.19}$$

because covariant spinor derivatives $D^{1,1}_{\alpha}$ and $\bar{D}^{1,-1}_{\dot\alpha}$ are reduced in the analytic basis to the partial derivatives $\partial/\partial\theta^{-1,-1\alpha}$, $\partial/\partial\bar\theta^{-1,1\dot\alpha}$.

The reader will easily perform one more exercise. Action of which harmonic derivatives preserves analyticity property of SF? Answer: $D^{0}_{\mathrm{I}}, D^{0}_{\mathrm{II}}, D^{1,\pm 3}, D^{2,0} = [D^{1,3}, D^{1,-3}]$. Their explicit form in the analytic basis is (in application to analytic SF)

$$(D^{0}_{\mathrm{I}})_{AB} = \partial^{0}_{\mathrm{I}} + \theta^{1,-1\alpha}\frac{\partial}{\partial\theta^{1,-1\alpha}} + \bar\theta^{1,1\dot\alpha}\frac{\partial}{\partial\bar\theta^{1,1\dot\alpha}}$$

$$(D^{0}_{\mathrm{II}})_{AB} = \partial^{0}_{\mathrm{II}} - \theta^{1,-1\alpha}\frac{\partial}{\partial\theta^{1,-1\alpha}} + 2\theta^{0,2\alpha}\frac{\partial}{\partial\theta^{0,2\alpha}} + \bar\theta^{1,1\dot\alpha}\frac{\partial}{\partial\bar\theta^{1,1\dot\alpha}} - 2\bar\theta^{0,-2\dot\alpha}\frac{\partial}{\partial\bar\theta^{0,-2\dot\alpha}}$$

$$(D^{2,0})_{AB} = \partial^{2,0} + 4i\theta^{1,-1\alpha}\bar\theta^{1,1\dot\alpha}\partial_{\alpha\dot\alpha}, \qquad \partial_{\alpha\dot\alpha} \equiv \tfrac{1}{2}\sigma^{m}_{\alpha\dot\alpha}\partial_m$$

$$(D^{1,3})_{AB} = \partial^{1,3} + 2i\theta^{0,2\alpha}\bar\theta^{1,1\dot\alpha}\partial_{\alpha\dot\alpha} + \bar\theta^{1,1\dot\alpha}\frac{\partial}{\partial\bar\theta^{0,-2\dot\alpha}},$$

$$(D^{1,-3})_{AB} = \partial^{1,-3} + 2i\theta^{1,-1\alpha}\bar\theta^{0,-2\dot\alpha}\partial_{\alpha\dot\alpha} - \theta^{1,-1\alpha}\frac{\partial}{\partial\theta^{0,2\alpha}}. \tag{2.20}$$

Additional terms with $\partial/\partial x$ and $\partial/\partial\theta$ in (2.20) arise upon the change of variables (2.15).

In this section we discussed the properties of the harmonic N=3 SS with respect to the N=3 Poincaré supersymmetry established in [4]. Now we turn to our main topic, to its superconformal properties.

3. Superconformal transformations and N=3 analytic superspace

This section is devoted to discussion of the realization of superconformal group SU(2, 2| 3) in the N=3 harmonic SS. As we shall show it can be defined from the requirement that the real analytic SS

is closed under these transformations. Berezinian of SU(2,2|3) transformations in the real analytic SS is proven to be unity. This fact will be of great importance for the proof of conformal invariance of the N=3 Yang-Mills theory (sect.4).

3.1. The SU(2, 2/3) realization in the standard N=3 SS (2.1) is known long ago (see, e.g., [11]) (we omit supertranslation given by (3.12) and the Poincaré transformations which are evident)

$$\begin{aligned}\delta x^{\dot\alpha\alpha} = {} & a x^{\dot\alpha\alpha} + k_{\beta\dot\beta}x^{\dot\alpha\beta}x^{\dot\beta\alpha} - 4k_{\beta\dot\beta}\theta^{\beta}_{i}\bar\theta^{\dot\alpha i}\theta^{\alpha}_{j}\bar\theta^{\dot\beta j} + \\ & + 2i(x^{\dot\alpha\beta} + 2i\theta^{\beta}_{i}\bar\theta^{\dot\alpha i})\theta^{\alpha}_{j}\eta^{j}_{\beta} + 2i(x^{\dot\beta\alpha} - 2i\theta^{\alpha}_{i}\bar\theta^{\dot\beta i})\bar\theta^{\dot\alpha j}\bar\eta_{\dot\beta j},\end{aligned} \tag{3.1}$$

$$\begin{aligned}\delta\theta^{\alpha}_{i} = {} & \left(\frac{a}{2} + ib\right)\theta^{\alpha}_{i} + \lambda_{i}^{\,j}\theta^{\alpha}_{j} + k_{\beta\dot\beta}(x^{\dot\beta\alpha} + 2i\theta^{\alpha}_{j}\bar\theta^{\dot\beta j})\theta^{\beta}_{i} - \\ & - 4i\theta^{\alpha}_{j}\theta^{\beta}_{i}\eta^{j}_{\beta} + (x^{\dot\beta\alpha} + 2i\theta^{\alpha}_{j}\bar\theta^{\dot\beta j})\bar\eta_{\dot\beta i}, \qquad \delta\bar\theta^{\dot\alpha i} = \overline{\delta\theta^{\alpha}_{i}},\end{aligned}$$

where a, b, $\lambda_{i}^{\,j}\ (\overline{\lambda_{i}^{\,j}} = -\lambda_{j}^{\,i},\ \lambda_{i}^{\,i} = 0)$, $k_{\beta\dot\beta}$ and $\eta^{\alpha i}$, $\bar\eta^{\dot\alpha}_{i} \equiv \overline{\eta^{\alpha i}}$ are the parameters of dilatations, γ_5 transformations, of SU(3) conformal boosts and special supersymmetry, respectively.

To find a SU(2, 2/3) realization in the harmonic SS, we have to add to (3.1) superconformal transformations of harmonics. We shall find the latter from the requirement of preserving the analytic SS (2.16). It suffices to find the conformal boosts $\delta_k u$ of harmonics because by commuting them with supertranslation we can recover all superconformal transformations.

3.2. We begin with a conformal boost of the coordinate $x^{\dot\alpha\alpha}_{A}$ (2.15) . We require that this boost does not contain the Grassmann coordinates $\theta^{-1,-1}_{\alpha}$, $\bar\theta^{-1,1}_{\dot\alpha}$ that do not enter into the real analytic SS (2.16). After a simple algebra we find

$$\delta_k x^{\dot\alpha\alpha}_{A} = k_{\beta\dot\beta}x^{\dot\alpha\beta}_{A}x^{\dot\beta\alpha}_{A} + 4(k_{\beta\dot\beta}\theta^{0,2\beta}\bar\theta^{0,-2\dot\beta})\theta^{0,2\alpha}\bar\theta^{0,-2\dot\alpha}. \tag{3.2}$$

provided

$$\delta_k u^{-1,1}_{i} = \delta_k u^{-1,-1\,i} = 0$$

$$\delta_k u^{1,1}_{i} = -4i(k_{\beta\dot\beta}\theta^{1,-1\beta}\bar\theta^{1,1\dot\beta})u^{-1,1}_{i} - 4i(k_{\beta\dot\beta}\theta^{0,2\beta}\bar\theta^{1,1\dot\beta})u^{0,-2}_{i} \tag{3.3}$$

$$\delta_K u^{1,-1\,i} = 4i\left(k_{\beta\dot\beta}\,\theta^{1,-1\beta}\bar\theta^{1,1\dot\beta}\right)u^{-1,-1\,i} + 4i\left(k_{\beta\dot\beta}\,\theta^{1,-1\beta}\bar\theta^{0,-2\dot\beta}\right)u^{0,2i}$$

$$\delta_K u^{0,2i} = 4i\left(k_{\beta\dot\beta}\,\theta^{0,2\beta}\,\bar\theta^{1,1\dot\beta}\right)u^{-1,-1\,i}$$

$$\delta_K u^{0,-2}_{\ i} = -4i\left(k_{\beta\dot\beta}\,\theta^{1,-1\beta}\,\bar\theta^{0,-2\dot\beta}\right)u^{-1,1}_{\ i}\,.$$

It follows from (3.1) and (3.3) (see also (2.15)) that analytic Grassmann coordinates are transformed according to

$$\delta_K\,\theta^{1,-1\,\alpha} = k_{\beta\dot\beta}\,X^{\dot\beta\alpha}_{A-}\,\theta^{1,-1\beta}\,,\qquad \delta\bar\theta^{1,1\dot\alpha} = k_{\beta\dot\beta}\,X^{\dot\alpha\beta}_{A+}\,\bar\theta^{1,1\dot\beta}$$
$$\delta_K\,\theta^{0,2\alpha} = k_{\beta\dot\beta}\,X^{\dot\beta\alpha}_{A+}\,\theta^{0,2\beta}\,,\qquad \delta\bar\theta^{0,-2\dot\alpha} = k_{\beta\dot\beta}\,X^{\dot\alpha\beta}_{A-}\,\bar\theta^{0,-2\dot\beta}, \tag{3.4}$$

where we have introduced the notation

$$X^{\dot\alpha\alpha}_{A\pm} = X^{\dot\alpha\alpha}_{A} \pm 2i\,\theta^{0,2\alpha}\,\bar\theta^{0,-2\dot\alpha}. \tag{3.5}$$

It is remarkable that conformal boosts act within the analytic SS; the right-hand parts of (3.2)-(3.4) involve only analytic coordinates (2.16). We leave for the reader to check compatibility of the transformations obtained with the defining conditions (2.2) and the combined conjugation (2.11), (2.18).

3.3. As was said above, the remaining transformations can be obtained by commuting (3.2) - (3.4) with (2.17) . For analytic coordinates X and θ they are given in Appendix B while for harmonics we prefer to give them here in the following compact form (that will be used often in what follows)

$$\delta u^{-1,1}_{\ i} = \delta u^{-1,-1\,i} = 0$$
$$\delta u^{1,1}_{\ i} = \lambda^{2,0}\,u^{-1,1}_{\ i} + \lambda^{1,3}\,u^{0,-2}_{\ i}$$
$$\delta u^{1,-1i} = -\lambda^{2,0}\,u^{-1,-1\,i} - \lambda^{1,-3}\,u^{0,2i} \tag{3.6}$$
$$\delta u^{0,2i} = -\lambda^{1,3}\,u^{-1,-1i}\,,\qquad \delta u^{0,-2}_{\ i} = \lambda^{1,-3}\,u^{-1,1}_{\ i},$$

where

$$\lambda^{2,0} = -4i\,k_{\beta\dot\beta}\,\theta^{1,-1\beta}\bar\theta^{1,1\dot\beta} - 4i\left(\bar\eta_{\dot\beta j}\,\bar\theta^{1,1\dot\beta}\,u^{1,-1j} + \theta^{1,-1\beta}\eta^{j}_{\beta}\,u^{1,1}_{\ j}\right) + \lambda^{i}_{\ j}\,u^{1,-1\,j}\,u^{1,1}_{\ i}\,,$$

$$\lambda^{1,3} = -4i\,k_{\beta\dot\beta}\,\theta^{0,2\beta}\bar\theta^{1,1\dot\beta} - 4i\left(\bar\eta_{\dot\beta j}\,\bar\theta^{1,1\dot\beta}\,u^{0,2j} + \theta^{0,2\beta}\eta^{j}_{\beta}\,u^{1,1}_{\ j}\right) + \lambda^{i}_{\ j}\,u^{0,2j}\,u^{1,1}_{\ i} \tag{3.7}$$

$$\lambda^{1,-3} = -4i\,k_{\beta\dot\beta}\,\theta^{1,-1\beta}\bar\theta^{0,-2\dot\beta} - 4i\left(\bar\eta_{\dot\beta j}\,\bar\theta^{0,-2\dot\beta}\,u^{1,-1j} + \theta^{1,-1\beta}\eta^{j}_{\beta}\,u^{0,-2}_{\ j}\right) + \lambda^{i}_{\ j}\,u^{1,-1\,j}\,u^{0,-2}_{\ i}\,.$$

Analytic parameters $\lambda^{2,0}$, $\lambda^{1,\pm3}$ have the following properties:

$$\overset{*}{\widetilde{\lambda^{2,0}}} = \lambda^{2,0}\,,\qquad \overset{*}{\widetilde{\lambda^{1,3}}} = -\lambda^{1,-3}\,,\qquad \overset{*}{\widetilde{\lambda^{1,-3}}} = \lambda^{1,3} \tag{3.8a}$$

$$\lambda^{2,0} = D^{1,3}\lambda^{1,-3} = -D^{1,-3}\lambda^{1,3}$$

$$D^{2,0}\lambda^{2,0} = D^{2,0}\lambda^{1,\pm3} = D^{1,-3}\lambda^{1,-3} = D^{1,3}\lambda^{1,3} = 0. \tag{3.8b}$$

3.4. We suggest the reader proving the theorem that will be intensively used in sect.4. The Berezinian of rigid N=3 superconformal transformations in the real analytic SS is unity,

$$\mathrm{Ber}\,\frac{\partial(\zeta+\delta\zeta,\,u+\delta u)}{\partial(\zeta,u)} = 1\,. \tag{3.9}$$

Hints. (i) The Berezinian of infinitesimal transformations has the form

$$\mathrm{Ber}\,\frac{\partial(\zeta+\delta\zeta,\,u+\delta u)}{\partial(\zeta,u)} = 1 + \frac{\partial}{\partial X_A}\delta X_A + \frac{\partial}{\partial u}\delta u - \frac{\partial}{\partial\theta}\delta\theta\,, \tag{3.10}$$

where sum is implied over all analytic coordinates.
(ii) Due to (3.6)

$$\frac{\partial}{\partial u}\delta u = \partial^{-2,0}\lambda^{2,0} + \partial^{-1,-3}\lambda^{1,3} + \partial^{-1,3}\lambda^{1,-3}. \tag{3.11}$$

The important equality (3.9) could be guessed on purely dimensional grounds. Indeed the analytic SS integration measure $d^4x\,d^8\theta\,du$ has zero dimensionality ($[dx]=m^{-1}$, $[d\theta]=m^{+1/2}$, $[du]=0$) and, consequently, zero Weyl weight.

3.5. To establish superconformal properties of the N=3 Yang-Mills prepotentials and to check superconformal invariance of the action, we have to know a transformation law of harmonic derivatives (2.20) with respect to SU(2, 2 3) . First of all, variations of the U(I)-charge operators (see (2.10) have to vanish because these charges are respected by superconformal transformations:

$$\delta D^{0}_{\mathrm{I}} = \delta D^{0}_{\mathrm{II}} = 0\,. \tag{3.12}$$

This can be checked using their explicit form. Also

$$\delta D^{1,3} = -\frac{1}{2}\lambda^{1,3}\left(D^{\circ}_{I}+D^{\circ}_{II}\right)$$
$$\delta D^{1,-3} = -\frac{1}{2}\lambda^{1,-3}\left(D^{\circ}_{I}-D^{\circ}_{II}\right) \qquad (3.13)$$
$$\delta D^{2,0} = \lambda^{1,3}D^{1,-3}-\lambda^{1,-3}D^{1,3}-\lambda^{2,0}D^{\circ}_{I}.$$

To check these laws it is convenient to use the central basis where harmonic derivatives do not contain $\partial/\partial x$, $\partial/\partial\theta$ (A.1–A.3),(2.8). Note also that (3.13c) follows in fact from (3.13a,b)

$$D^{2,0} = [D^{1,3}, D^{1,-3}].$$

3.6. Digression. When reading section 2 a careful reader could notice the following. The real analytic SS (2.16), being invariant sub SS of the harmonic SS (2.14), itself contains Sub SS

$$\left\{ X^{\dot\alpha\alpha}_{A+}=X^{\dot\alpha\alpha}_{A}+2i\theta^{0,2\alpha}\bar\theta^{0,-2\dot\alpha},\ \theta^{1,-1}_{\alpha},\theta^{0,2}_{\alpha},\bar\theta^{1,1}_{\dot\alpha},u\right\} \qquad (3.14)$$

which is closed under all N=3 Poincaré supersymmetry transformations. The sub SS (3.14) does not contain the Grassman coordinate $\bar\theta^{0,-2}_{\dot\alpha}$ (entering into (2.16)) and variations of coordinates (3.14) do not involve $\bar\theta^{0,-2}_{\dot\alpha}$ (see (2.17)), e.g.,

$$\delta_Q X^{\dot\alpha\alpha}_{A+} = 4i\left(\theta^{1,-1\alpha}u^{-1,1}_{\ \ i}+\theta^{0,2\alpha}u^{0,-2}_{\ \ i}\right)\bar\varepsilon^{\dot\alpha i}-4i\,\varepsilon^{\alpha}_{i}\bar\theta^{1,1\dot\alpha}u^{-1,-1 i}. \qquad (3.15)$$

In N=1,2 theories, SS's with the least number of spinor coordinates played the important role /12–15/ . From this standpoint it is rather interesting to investigate the superconformal properties of the SS (3.14) having in mind possible applications of the latter in N=3 theories. It is not difficult to note that (3.14) is not closed with respect to N=3 superconformal transformations in the form we have discussed above. For example, the conformal boost variations (3.3), (3.4)

$$\delta_K u^{0,-2}_{\ \ i} = -4i\left(k_{\beta\dot\beta}\,\theta^{1,-1\beta}\bar\theta^{0,-2\dot\beta}\right)u^{-1,1}_{\ \ i}$$
$$\delta_K\theta^{1,-1\alpha} = k_{\beta\dot\beta}\left(X^{\dot\beta\alpha}_{A+}-4i\theta^{0,2\alpha}\bar\theta^{0,-2\dot\beta}\right)\theta^{1,-1\beta}$$

involve $\bar\theta^{0,-2}$, i.e., they take us out of the SS (3.14). The reason is that (3.14) contains mutually conjugated coordinates $\theta^{1,-1}_{\alpha}$ and $\bar\theta^{1,1}_{\dot\alpha}$ and also conjugated pairs of harmonics. Therefore, if a transformation law includes $\theta^{0,2}$, then it has necessarily to include the conjugated coordinate $\bar\theta^{0,-2}$ as well.

One could conclude that a realization of SU(2, 2|3) group in SS of the type (3.14) is impossible. However, such a conclusion would be premature. We have postulated above that harmonics $u^{a,b}$ and $u^{a,-b}$ are mutually conjugated and so do also spinor variables $\theta^{a,b}$ and $\theta^{a,-b}$. This mutual conjugation property can be avoided if one redefines harmonics starting from harmonics of the complexification of SU(3), i.e., SL(3,C). Remarkably, one can realize the full SC group SU(2,2/3) on such a complexified SS (3.14). We discuss this realization in Appendix C.

There, we show that the real analytic SS (2.16) forms a real hypersurface in the complexified SS (3.14). Thus, an intriguing analogy arises with the interpretation of the real N=1 SS as a hypersurface in the chiral (or complex) N=1 SS /16/ . The latter property is known to be crucial in construction of the geometric minimal formulation of the N=1 supergravity. Therefore, the complex version of SS (3.14) deserves further study. At the same time it is rather difficult to connect it with the N=3 Yang–Mills and conformal supergravity theories (in contrast with the real version (2.16)).

3.7. Digression continued. A possibility of a shift $X^{\dot\alpha\alpha}_{A}$ on $\theta^{0,2\alpha}\bar\theta^{0,-2\dot\alpha}$ yields some interesting consequences also in the framework of the real analytic SS (2.16) itself. In the complex parametrization of this SS

$$\left\{X^{\dot\alpha\alpha}_{A+}=X^{\dot\alpha\alpha}_{A}+2i\,\theta^{0,2\alpha}\bar\theta^{0,-2\dot\alpha},\theta^{1,-1}_{\alpha},\theta^{0,2}_{\alpha},\bar\theta^{1,1}_{\dot\alpha},\bar\theta^{0,-2}_{\dot\alpha},u\right\} \qquad (3.16)$$

the harmonic derivative $D^{1,-3}$ does not involve $\partial/\partial X_{A+}$

$$\left(D^{1,-3}\right)_{compl.\ param} = \partial^{1,-3}-\theta^{1,-1\alpha}\frac{\partial}{\partial\theta^{0,2\alpha}}. \qquad (3.17)$$

Therefore, for complex analytic SF's, one can define an extra analyticity in harmonics by means of the condition

$$D^{1,-3}\,\phi^{q,p}(\zeta,u)=0. \qquad (3.18)$$

This analyticity is compatible with the superconformal group iff the first U(I)-charge equals the second one x), p=q (see (3.13b))

x) Notice an interesting analogy. External Lorentz indices of N=1 chiral superfields agree with N=1 conformal supersymmetry only for special representations of the Lorentz group /17/ .

$$\delta^{\lambda}(D^{1,-3}\varphi^{q,q}) = -\frac{1}{2}\lambda^{1,-3}(D_{I}^{o}-D_{II}^{o})\varphi^{q,q}=0. \quad (3.19)$$

Some possible consequences will be discussed elsewhere.

The following observation is also worth mentioning. There is one more harmonic derivative $D^{-1,3}$ that takes a (3.17)-like form when applied to an analytic superfield defined on (3.14)

$$D^{-1,3} = \partial^{-1,3} - \theta^{0,2\alpha}\frac{\partial}{\partial\theta^{1,-1\alpha}} \quad (3.20)$$

This derivative together with $D^{1,\pm 3}$, $D^{2,0}$ form the full set of harmonic derivatives that makes up a closed algebra together with the spinor derivatives $D_{\alpha}^{1,1}$, $\bar{D}_{\dot\alpha}^{1,-1}$, $\bar{D}_{\dot\alpha}^{0,2}$. (The latter single out the analytic SS (3.14)). Equations (2.9) and (A.4) say that $D^{-1,3}$, $D^{1,-3}$ and $-D_{I}^{o}+D_{II}^{o}$ form an SU(2) algebra. In particular

$$[D^{-1,3}, D^{1,-3}] = \frac{1}{2}(-D_{I}^{o}+D_{II}^{o}). \quad (3.21)$$

One can impose simultaneously condition (3.18) and the condition

$$D^{-1,3}\varphi^{q,q}=0 \quad (3.22)$$

on SF's defined on SS (3.14) (in agreement with (3.21) because $q_{I}=q_{II}=q$). These conditions have the following simple group-theoretic meaning. Among the functions defined on the homogeneous space $SU(3)/U(1)\otimes U(1)$, these conditions single out those functions which are defined on its subspace $SU(3)/SU(2)\otimes U(1)$. SU(2) contains harmonic derivatives (3.21), and the U(I)-generator is given by a combination $D_{I}^{o}+D_{II}^{o}$. Correspondingly, SS (3.14) for such functions can be considered as an analytic sub SS of the N=3 harmonic SS, having $M^{4}\otimes SU(3)/SU(2)\otimes U(1)$ as its even part. Then, harmonics $(u^{1,-1}{}_{i}, u^{0,2}{}_{i})$, $(u^{1,1}{}_{i}, u^{0,-2}{}_{i})$ and, consequently spinor coordinates $(\theta_{\alpha}^{1,-1}, \theta_{\alpha}^{0,2})$, $(\bar\theta_{\dot\alpha}^{1,-1}, \bar\theta_{\dot\alpha}^{0,-2})$ will be SU(2) doublets while all other coordinates (3.14) will be singlets of SU(2).

So much for digression. Now we return to the N=3 real analytic SS.

4. Superconformal invariance of N=3 Yang-Mills theory

This section is devoted to the N=3 Yang-Mills theory. After briefly reminding its basics we establish here superconformal properties of gauge prepotentials and prove the SU(2,2/3) invariance of its action.

4.1. As has been shown in /4/, N=3 gauge prepotentials are connections entering into three covariant harmonic derivatives. In new notation we have

$$\mathcal{D}^{1,3} = D^{1,3} + iV^{1,3}(\zeta,u),$$
$$\mathcal{D}^{1,-3} = D^{1,-3} + iV^{1,-3}(\zeta,u), \quad (4.1)$$
$$\mathcal{D}^{2,0} = D^{2,0} + iV^{2,0}(\zeta,u).$$

Prepotentials in (4.1) are the analytic Lie algebra valued SF's having definite U(I) charges

$$D_{I}^{o}V^{q_I,q_{II}} = q_{I}V^{q_I,q_{II}}, \quad D_{II}^{o}V^{q_I,q_{II}} = q_{II}V^{q_I,q_{II}} \quad (4.2)$$

and reality properties

$$\overset{*}{\overline{V^{1,3}}} = -V^{1,-3}, \quad \overset{*}{\overline{V^{1,-3}}} = V^{1,3}, \quad \overset{*}{\overline{V^{2,0}}} = V^{2,0}. \quad (4.3)$$

Their gauge transformation law is the standard one for connections

$$\delta V^{q_I,q_{II}} = \mathcal{D}^{q_I,q_{II}}\lambda(\zeta,u). \quad (4.4)$$

where $\lambda(\zeta,u)$ is a real analytic Lie algebra valued superparameter having zero U(I)- charges.

Commutators of harmonic derivatives (4.1) determine the corresponding field strengths

$$F^{3,3} = \frac{1}{i}[\mathcal{D}^{2,0},\mathcal{D}^{1,3}] = D^{2,0}V^{1,3} - D^{1,3}V^{2,0} + i[V^{2,0},V^{1,3}] \quad (4.5a)$$

$$F^{3,-3} = \frac{1}{i}[\mathcal{D}^{1,-3},\mathcal{D}^{2,0}] = D^{1,-3}V^{2,0} - D^{2,0}V^{1,-3} + i[V^{1,-3},V^{2,0}] \quad (4.5b)$$

$$F^{2,0} = \frac{1}{i}[\mathcal{D}^{1,3},\mathcal{D}^{1,-3}] - \frac{1}{i}\mathcal{D}^{2,0} = D^{1,3}V^{1,-3} - D^{1,-3}V^{1,3} + i[V^{1,3},V^{1,-3}] - V^{2,0}, \quad (4.5c)$$

obeying the evident Bianchi identity

$$\mathcal{D}^{2,0}F^{2,0}+\mathcal{D}^{1,3}F^{3,-3}+\mathcal{D}^{1,-3}F^{3,3}=0. \qquad (4.6)$$

Their reality properties follow from (2.12) and (4.3)

$$\overset{*}{\overline{F^{3,3}}} = F^{3,-3}, \quad \overset{*}{\overline{F^{3,-3}}} = -F^{3,3}, \quad \overset{*}{\overline{F^{2,0}}} = F^{2,0}. \qquad (4.7)$$

4.2. The equations of motion of the N=3 Yang–Mills theory are obtained by equating to zero all three field strengths (4.5) /4/. They follow from the action principle in the analytic SS.

$$S^{N=3}_{YM} = \frac{1}{g^2}\, tr \int d\zeta^{-4,0} du \,\{ V^{1,3}F^{3,-3} + V^{1,-3}F^{3,3} + V^{2,0}F^{2,0} - i V^{2,0}[V^{1,3}, V^{1,-3}]\}, \qquad (4.8)$$

where g is the coupling constant. The integration measure

$$d\zeta^{-4,0}du = d^4x_A (d^2\theta)^{-2,2}(d^2\theta)^{0,-4}(d^2\bar\theta)^{-2,-2}(d^2\bar\theta)^{0,4}du \qquad (4.9)$$

has U(I) charges (-4,0) because the Grassmann integration is equivalent to the Grassman differentiation, e.g.,

$$\int (d^2\theta)^{-2,2} f = \frac{\partial}{\partial\theta^{1,-1\alpha}}\frac{\partial}{\partial\theta^{1,-1}_{\alpha}} f.$$

Action (4.8) is gauge invariant up to full derivatives in the integrand and is as a whole the Chern–Simons-type action.

4.3. Now we shall demonstrate the superconformal invariance of the N=3 Yang–Mills theory. For covariantized Yang–Mills derivatives (4.1) we postulate the same SU(2,2/3) transformation law (3.13) as for simple ones

$$\delta\mathcal{D}^{1,\pm3} = -\frac{1}{2}\lambda^{1,\pm3}(D^0_I \pm D^0_{II})$$
$$\delta\mathcal{D}^{2,0} = \lambda^{1,3}\mathcal{D}^{1,-3} - \lambda^{1,-3}\mathcal{D}^{1,3} - \lambda^{2,0}D^0_I. \qquad (4.10)$$

Then, we find for prepotentials

$$\delta V^{1,3} = \delta V^{1,-3} = 0$$
$$\delta V^{2,0} = \lambda^{1,3}V^{1,-3} - \lambda^{1,-3}V^{1,3} \quad (\delta V = V'(\zeta',u') - V(\zeta,u)) \qquad (4.11)$$

and for field strengths (taking into account 3.8b)

$$\delta F^{3,3} = -\lambda^{1,3}F^{2,0}, \quad \delta F^{3,-3} = \lambda^{1,-3}F^{2,0}, \quad \delta F^{2,0} = 0. \qquad (4.12)$$

Due to the theorem (3.9) the integration measure in the analytic SS is superconformally invariant x)

$$(d\zeta^{-4,0}du)' = Ber\frac{\partial(\zeta',u')}{\partial(\zeta,u)}\, d\zeta^{-4,0}du = d\zeta^{-4,0}du. \qquad (4.13)$$

Using equations (4.11)-(4.13) the reader can easily demonstrate that action (4.8) is invariant

$$\delta_{SU(2,2|3)} S^{N=3}_{YM} = 0. \qquad (4.14)$$

5. Conclusion

Thus, we have shown that the real analytic N=3 SS is closed with respect to all rigid superconformal transformations. The Berezinian of the latter is unity. Transformation laws obtained imply that the N=3 Yang–Mills theory /4/ is superconformally invariant. Establishing of these simple statements is not an end in itself. Their generalization to the case of local transformations will bring us to construction of the N=3 off-shell supergravity theory.

Appendix A. Explicit form and algebra of harmonic derivatives

$$D^0_I = u^{1,1}_i\frac{\partial}{\partial u^{1,1}_i} - u^{-1,1}_i\frac{\partial}{\partial u^{-1,1}_i} + u^{-1,-1i}\frac{\partial}{\partial u^{-1,-1i}} + u^{1,-1i}\frac{\partial}{\partial u^{1,-1i}} \equiv \partial^0_I \qquad (A.1)$$

$$D^0_{II} = u^{1,1}_i\frac{\partial}{\partial u^{1,1}_i} + u^{-1,1}_i\frac{\partial}{\partial u^{-1,1}_i} - 2u^{0,-2}_i\frac{\partial}{\partial u^{0,-2}_i} - u^{-1,-1i}\frac{\partial}{\partial u^{-1,-1i}} - u^{1,-1i}\frac{\partial}{\partial u^{1,-1i}} + 2u^{0,2i}\frac{\partial}{\partial u^{0,2i}} \equiv \partial^0_{II}, \qquad (A.2)$$

$$D^2_1 = D^{-2,0} = u^{-1,1}_i\frac{\partial}{\partial u^{1,1}_i} - u^{-1,-1i}\frac{\partial}{\partial u^{1,-1i}} \equiv \partial^{-2,0},$$

$$D^2_3 = D^{-1,3} = u^{-1,1}_i\frac{\partial}{\partial u^{0,-2}_i} - u^{0,2i}\frac{\partial}{\partial u^{1,-1i}} \equiv \partial^{-1,3}$$

x) In contradistinction with the dimentionful integration measure in the full harmonic SS (2.14), $[d^4x\, d^{12}\theta\, du] = m^2$

$$\mathcal{D}_1^{\ 3} = \mathcal{D}^{-1,-3} = u_i^{0,-2}\frac{\partial}{\partial u_i^{1,1}} - u^{-1,-1i}\frac{\partial}{\partial u^{0,2i}} \equiv \partial^{-1,-3}.$$

Nonvanishing commutators of U(I)-charged harmonic derivatives are

$$[\mathcal{D}^{2,0}, \mathcal{D}^{-1,3}] = \mathcal{D}^{1,3}, \quad [\mathcal{D}^{2,0}, \mathcal{D}^{-2,0}] = \mathcal{D}_I^0, \quad [\mathcal{D}^{2,0}, \mathcal{D}^{-1,-3}] = -\mathcal{D}^{1,-3},$$
$$[\mathcal{D}^{1,3}, \mathcal{D}^{-2,0}] = -\mathcal{D}^{-1,3}, \quad [\mathcal{D}^{1,3}, \mathcal{D}^{-1,-3}] = \tfrac{1}{2}(\mathcal{D}_I^0 + \mathcal{D}_{II}^0), \quad [\mathcal{D}^{1,3}, \mathcal{D}^{1,-3}] = \mathcal{D}^{2,0} \qquad (A.4)$$
$$[\mathcal{D}^{-1,3}, \mathcal{D}^{-1,-3}] = \mathcal{D}^{-2,0}, \quad [\mathcal{D}^{-1,3}, \mathcal{D}^{1,-3}] = \tfrac{1}{2}(-\mathcal{D}_I^0 + \mathcal{D}_{II}^0), \quad [\mathcal{D}^{-2,0}, \mathcal{D}^{1,-3}] = \mathcal{D}^{-1,-3}.$$

Appendix B. N=3 superconformal transformations of analytic coordinates X, θ (besides (2.17), (3.2), (3.3))

$$\begin{aligned}
\delta X_A^{\dot\alpha\alpha} &= a X_A^{\dot\alpha\alpha} - 2i\bar\eta_{\dot\beta i} X_{A-}^{\dot\beta\alpha}\bar\theta^{0,-2\dot\alpha} u^{0,2i} - 4i\bar\eta_{\dot\beta i} X_A^{\dot\beta\alpha}\bar\theta^{-1,1\dot\alpha} u^{-1,-1i} \\
&- 2i\eta^i_\beta X_{A+}^{\dot\alpha\beta}\theta^{0,2\alpha} u_i^{0,-2} - 4i\eta^i_\beta X_A^{\dot\alpha\beta}\theta^{1,-1\alpha} u_i^{-1,1} - \\
&- 4i(\lambda_i^{\ j} u^{-1,-1i} u_j^{-1,1})\theta^{1,-1\alpha}\bar\theta^{1,1\dot\alpha} - 2i(\lambda_i^{\ j} u^{0,2i} u_j^{-1,1})\theta^{1,-1\alpha}\bar\theta^{0,2\dot\alpha} - \\
&- 2i(\lambda_i^{\ j} u^{-1,-1i} u_j^{0,-2})\theta^{0,2\alpha}\bar\theta^{1,1\dot\alpha},
\end{aligned} \qquad (B.1)$$

$$\begin{aligned}
\delta\theta^{0,2\alpha} &= \left(\tfrac{a}{2} + ib\right)\theta^{0,2\alpha} - X_A^{\dot\beta\alpha}\bar\eta_{\dot\beta j} u^{0,2j} - 4i[\theta^{1,-1\alpha} u_j^{-1,1} + \theta^{0,2\alpha} u_j^{0,-2}]\theta^{0,2\beta}\eta_\beta^{\ j} + \\
&+ (\lambda_i^{\ j} u^{0,2i} u_j^{0,-2})\theta^{0,2\alpha} + (\lambda_i^{\ j} u^{0,2i} u_j^{-1,1})\theta^{1,-1\alpha},
\end{aligned}$$

$$\begin{aligned}
\delta\theta^{1,-1\alpha} &= \left(\tfrac{a}{2} + ib\right)\theta^{1,-1\alpha} + X_{A-}^{\dot\beta\alpha}\bar\eta_{\dot\beta j} u^{1,-1j} - 4i\theta^{1,-1\alpha}\theta^{1,-1\beta}\eta_\beta^{\ j} u_j^{-1,1} + \\
&+ \lambda_i^{\ j} u^{1,-1i} u_j^{-1,1}\theta^{1,-1\alpha},
\end{aligned}$$

$X_A^{\dot\beta\alpha}$ was defined by eq. (3.5),

$$\delta\bar\theta^{1,1\dot\alpha} = -\overset{*}{\delta\theta^{1,-1\alpha}}, \quad \delta\bar\theta^{0,-2\dot\alpha} = \overset{*}{\delta\theta^{0,2\alpha}}.$$

Appendix C. Complexified analytic superspace (3.14)

This SS has coordinates

$$\{X_{A+}^{\dot\alpha\alpha},\ \theta_u^{1,-1\alpha},\ \theta_u^{0,2\alpha},\ \bar\theta_u^{-1,1\dot\alpha},\ u\} \qquad (C.1)$$

connected with coordinates of customary N=3 SS (2.1) by formulas (2.15) and (3.5). However, now harmonics $u_i^{\ a}$ are not self-conjugated in the sense of (2.11); they parametrize that time SL(3.C) group instead of SU(3) . So SS (C1) has a doubled set of harmonics in comparison with the real analytic SS (2.16). Under conjugation $\overset{*}{\ }$ we have

$$\begin{aligned}
&u_i^{1,1} \to v^{1,-1i} \to -u_i^{1,1}, \\
&u_i^{-1,1} \to -v^{-1,-1i} \to -u_i^{-1,1}, \\
&u_i^{0,-2} \to v^{0,2i} \to u_i^{0,-2}, \\
&u^{1,-1i} \to -v_i^{1,1} \to -u^{1,-1i}, \\
&u^{-1,-1i} \to v_i^{-1,1} \to -u^{-1,-1i}, \\
&u^{0,2i} \to v_i^{0,-2} \to u^{0,2i}.
\end{aligned} \qquad (C.2)$$

Here v are conjugated harmonics not connected generally to u by any algebraic relation . Respectively $\theta_\alpha^{1,1}$ and $\bar\theta^{1,1\dot\alpha}$ are not conjugated to each other. Spinor coordinates in (C.1) are furnished with index u to stress that they are related to a set of harmonics u. Their complex conjugated coordinates will be supplied by index v . One should have in mind that the whole complex nature of SS (C.1) is just due to the complex nature of harmonics. At the same time the central basis coordinates remain real.

To find a realization of SU(2; 3/3) in (C.1) we shall copy derivation in Sect. 3 of superconformal transformations of real analytic SS. Then, we easily find conformal boosts

$$\delta_K X_{A+}^{\dot\alpha\alpha} = k_{\beta\dot\beta} X_{A+}^{\dot\alpha\beta} X_{A+}^{\dot\beta\alpha}, \quad \delta_K\theta^{1,-1\alpha} = k_{\beta\dot\beta} X_{A+}^{\dot\beta\alpha}\theta^{\beta(1,-1)},$$
$$\delta_K\theta^{0,2\alpha} = k_{\beta\dot\beta} X_{A+}^{\dot\beta\alpha}\theta^{0,2\beta}, \quad \delta_K\bar\theta^{1,1\dot\alpha} = k_{\beta\dot\beta} X_{A+}^{\dot\alpha\beta}\bar\theta^{1,1\dot\beta} \qquad (C.3)$$

$$\delta u_i^{1,1} = -4i k_{\beta\dot\beta}\, \theta^{1,-1\beta} \bar\theta^{1,1\dot\beta} u_i^{-1,1} - 4i k_{\beta\dot\beta}\, \theta^{0,2\beta} \bar\theta^{1,1\dot\beta} u_i^{0,-2},$$

$$\begin{aligned}
\delta_K u^{1,-1i} &= 4i k_{\beta\dot\beta}\, \theta^{1,-1\beta} \bar\theta^{1,1\dot\beta} u^{-1,-1i}, \\
\delta_K u^{0,2i} &= 4i k_{\beta\dot\beta}\, \theta^{0,2\beta} \bar\theta^{1,1\dot\beta} u^{-1,-1i} \\
\delta_K u_i^{0,-2} &= \delta_K u_i^{-1,1} = \delta_K u^{-1,-1i} = 0
\end{aligned} \qquad (C.4)$$

Under conjugation ~~a~~ boosts (C.2) turn to conformal boosts of the conjugated $SS\ \{x_{A-}^{\dot\alpha\alpha}, \bar\theta_v^{1,1\dot\alpha}, \bar\theta_v^{0,-2\dot\alpha}, \theta_v^{1,-1\alpha}, v\}$. Now we can easily obtain the remaining superconformal transformations (like we did it in sect.3 for (2.16))

$$\begin{aligned}
\delta x_{A+}^{\dot\alpha\alpha} &= a x_{A+}^{\dot\alpha\alpha} - 4i \bar\eta_{\dot\beta i}\, \bar\theta^{1,1\dot\alpha} u^{-1,-1i} x_A^{\dot\beta\alpha} - \\
&\quad - 4i \eta_\beta^j [\theta^{1,-1\alpha} u_j^{-1,1} + \theta^{0,2\alpha} u_j^{0,-2}] x_{A+}^{\dot\alpha\beta} + 4i \lambda_e^k \bar\theta^{1,1\dot\alpha} \theta^{1,-1\alpha} u_k^{-1,1} u^{-1,-1e} + \\
&\quad + 4i \lambda_e^k \bar\theta^{1,1\dot\alpha} \theta^{0,2\alpha} u_k^{0,-2} u^{-1,-1e}, \\
\delta\theta^{1,-1\alpha} &= (\tfrac{a}{2} + ib)\theta^{1,-1\alpha} + \bar\eta_{\dot\beta i} x_{A+}^{\dot\beta\alpha} u^{1,-1i} - 4i \eta_\beta^j [\theta^{1,-1\alpha} u_j^{-1,1} + \theta^{0,2\alpha} u_j^{0,-2}] \theta^{\beta,1,-1} \\
&\quad - \lambda_i^j [\theta^{1,-1\alpha} u_j^{-1,1} + \theta^{0,2\alpha} u_j^{0,-2}] u^{1,-1i}, \\
\delta\theta^{0,2\alpha} &= (\tfrac{a}{2} + ib)\theta^{0,2\alpha} + \bar\eta_{\dot\beta i} x_{A+}^{\dot\beta\alpha} u^{0,2i} - 4i \eta_\beta^j [\theta^{1,-1\alpha} u_j^{-1,1} + \theta^{0,2\alpha} u_j^{0,-2}] \theta^{0,2\beta} - \\
&\quad - \lambda_i^j [\theta^{1,-1\alpha} u_j^{-1,1} + \theta^{0,2\alpha} u_j^{0,-2}] u^{0,2i}, \\
\delta\bar\theta^{1,1\dot\alpha} &= (\tfrac{a}{2} - ib)\bar\theta^{1,1\dot\alpha} + \eta_\beta^i x_{A+}^{\dot\alpha\beta} u_i^{1,1} - 4i \bar\eta_{\dot\beta j}\, \bar\theta^{1,1\dot\alpha} \bar\theta^{1,1\dot\beta} u^{-1,-1j} - \\
&\quad - \lambda_i^j \bar\theta^{1,1\dot\alpha} u^{-1,-1i} u_j^{1,1}.
\end{aligned} \qquad (C.5)$$

(We omit here indices of spinor coordinates). The full realization of SU(2,2/3) on harmonics is given by equations (3.6) where one should discard terms with $\lambda^{1,-3}$ (now $\lambda^{1,3}$ is not conjugated to $\lambda^{1,-3}$ and $\lambda^{2,0}$ is not self-conjugated).

Now we shall show that the real analytic SS (2.16) arises as a real hypersurface in SS (C.1). Let us impose the following constraints:

$$\begin{aligned}
u_i^{-1,1} &= -v_i^{-1,1}, && (a) \\
u_i^{0,-2} &= v_i^{0,-2} + (v^{1,-1k} u_k^{0,-2}) v_i^{-1,1}, && (b) \\
u^{1,-1i} &= -v^{1,-1i} + (v^{1,-1k} u_k^{0,-2}) u^{0,2i} && (c)
\end{aligned} \qquad (C.6)$$

and conjugated constraints.

It can be checked that constraints (C.6) are closed with respect to (C.2) and superconformal transformations (C.3), (C.4) . After imposing these constraints, the number of independent harmonics reduces just to half the original one [x]. Let us take now a coordinate set

$$\{x_A^{\dot\alpha\alpha}, \theta^{0,2\alpha}, \theta^{1,-1\alpha}, \bar\theta^{1,1\dot\alpha}, \bar\theta^{0,-2\dot\alpha}, \tilde u\}, \qquad (C.7)$$

where

$$\theta^{0,2\alpha} \equiv -\theta_u^{0,2\alpha}, \quad \theta^{1,-1\alpha} \equiv \theta_v^{1,-1\alpha}, \qquad (C.8)$$

$$\bar\theta^{1,1\dot\alpha} \equiv \bar\theta_u^{1,1\dot\alpha}, \quad \bar\theta^{0,-2\dot\alpha} \equiv -\bar\theta_v^{0,-2\dot\alpha}, \qquad (C.9)$$

$$\{\tilde u\} = \{u_i^{1,1}, -u^{0,2i}, u^{-1,-1i}, v^{1,-1i}, -v_i^{0,-2}, v_i^{-1,1}\}.$$

Using the rule (C.2), one can check that the set (C.7) is self-conjugated in the sense (2.11) (2.18) and that $\tilde u$ -harmonics satisfy all defining relations (2,2), (2.11) due to constraints (C.6), e.g.,

$$\tilde u^{0,2i} \tilde u_i^{0,2} = 1, \quad \varepsilon^{ijk} \tilde u_i^{11} \tilde u_j^{-1,1} \tilde u_k^{0,-2} = 1, \ etc.$$

Starting with (C.6) after some straightforward calculations one can see that the SU(2, 2/3) transformations in SS (C.1) induce for coordinates (C.7) just the transformations (3.2), (3.3), (3.4), (3.6) and (B.1) of the real analytic SS. (C.7) can be identified with the real analytic SS (2.16).

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[x] When solving these constraints explicitly, one has to fix parametrization of SL(3,C).

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Received by Publishing Department
on August 27, 1986.

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Суперпространства для суперсимметрий (N = 3)

Найдена реализация суперконформной группы в вещественном аналитическом N = 3 суперпространстве. Установлено, что березиниан ее преобразований равен единице. Наличие такой реализации делает очевидной конформную инвариантность суперполевого действия N = 3 теории Янга - Миллса. Полученные результаты являются предварительным этапом в построении N = 3 супергравитации. Указаны также комплексно аналитические суперпространства с меньшим числом спинорных переменных и получена реализация в нем N = 3 суперконформной группы.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1986

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Superspaces for N = 3 Supersymmetry

N = 3 superconformal group (SU(2,2/3) is realized in the real analytic subspace of harmonic N = 3 superspace. Berezinian of its transformations is shown to be unity. Conformal invariance of the N = 3 Yang - Mills superfield action becomes evident within such a framework. A complex analytic superspace is also indicated, having a smaller number of spinor coordinates. A realization of SU(2,2/3) in this superspace is found.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1986