сообщения
объединенного
института
ядерных
исследований
дубна

E2-86-460

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ON INTERRELATION OF DIFFERENT RENORMALIZATION GROUPS

1986

1. INTRODUCTION

The renormalization-group method widely used in various branches of physics has a more than thirty-year-long history. As early as 1953 Stueckelberg and Petermann[1] found the group of continuous transformations related with a finite arbitrariness arising in quantum field theory (QFT) while removing the ultraviolet divergences. A year after almost the same renormalization group (RG) was used by Gell-Mann and Low[2] for studying quantum electrodynamics at short distances. There the notion was introduced of a fixed point of the RG transformation. Functional equations for the Green functions and for new specific quantities, effective coupling constants, as well as the Lie differential group equations constructed later by Bogolubov and Shirkov[3] allowed then to develop a regular procedure for improving the results of perturbation theory in the ultraviolet and infrared regions known as the renormalization-group method (RGM)[4].

From the early seventies the RGM has been intensively used in the theory of critical phenomena the construction of which represents a most important problem of modern physics. The reason is that, on the one hand, the investigation of phase transitions and related critical phenomena promotes the development of technique, and on the other hand, that very different systems (quantum fields, condensed matter, biological or chemical objects) near singularities exhibit a general behaviour. Such a generality permits the elaboration of a universal approach and a fruitful exchange by ideas between various branches, first of all, between QFT and statistical mechanics.

In modern literature one may find statements according to which the RG in the theory of critical phenomena is something different in principle from the quantum-field group of multiplicative renormalizations (see, e.g., the Introduction in[5], and sects.1,2 of chapter V in[6]). As a matter of fact, there exists a rather simple formulation of specific symmetry underlying RG transformations, which allows one to interpret these "various" renormalization groups from a unique mathematical point of view. The symmetry is based on the so-called[7] property of functional self-similarity (FS) that is a generalization of the property of power self-similarity well-known in the problems of hydrodynamics and gas dynamics. However, though there is a formal identity of the mathematical apparatus, it is possible to establish the boundary [3] between applications of the RG in various fields based on physical arguments. It turns out that such applications may be divided into two classes. First of them incorporates the cases when the introduction of RG is connected with the properties of internal symmetry of a considered physical system formulated in the language of natural variables of this problem. Here as an example one may point to the formulation of the quantum-field RG given in [1,3,4] and based on concepts and functions belonging to one local renormalizable model. The other class contains the RG connected with the construction of a certain set of models having the same or almost the same properties and differing from each other by the scale of some variable. It is just this class to which the RGM in the theory of critical phenomena belongs.

The above-mentioned principle of FS is common for both the classes though it may appear in different forms, being either exact or approximate for a given problem.

In this work, we shall analyse the basic content of the principle of FS and using it consider the relationship between various RG.

2. FUNCTIONAL SELF-SIMILARITY

Let a continuous positive parameter t defines the transformation of quantities x and g of the form

$$x \to x' = x/t, \qquad g \to g' = \bar{g}(t, g)\,, \tag{1}$$

where $\bar{g}(\cdot)$ is a single-valued function of its arguments, such that $\bar{g}(1, g) = g$. The first of transformations (1) is obviously a scaling transformation and has a group nature. Transformations (1) for g also compose a group if together with transformations t and τ there also is their composition

$$\bar{g}(t\tau, g) = \bar{g}(\tau, \bar{g}(t, g))\,. \tag{2}$$

Setting $t\tau = x$ we hence find $\bar{g}(x, g) = \bar{g}(x/t, \bar{g}(t, g))$, therefore, it follows that $\bar{g}(x, g)$ is an invariant of transformation (1).

Relation (2) may be treated as a functional equation for $\bar{g}$ from which it can easily be derived the Lie differential equation

$$t\frac{\partial \bar{g}}{\partial t} = \beta(\bar{g})\,. \tag{3}$$

The function $f(\cdot)$ transforming by the law $f(x', g') = z(t, g)\, f(x, g)$ realizes a representation of the FS group. At $z \equiv 1$ $f(\cdot)$ is an invariant of the FS group. When the second transformation in (1) is linear in g, the general solution may be represented by the simplest power self-similarity: $\bar{g} = t^a g$ being thus a particular case of the FS.

Transformations (1) and equations (2), (3) in QFT correspond to a single-charged RG in the massless case [3,4]. The argument x is then an invariant momentum variable, $\bar{g}$ is the so-called invariant or effective coupling constant. A natural generalization of this case is the RG for a massless QFT with several coupling constants. In this case the argument g "multiplies": $g \to \{g\} = \{g_1 \dots g_k\}$, and equations (2), (3) turn into systems of coupled equations for the corresponding effective $\bar{g}_i$. For the first time an RG of that type for the case of two constants was considered in ref. [9]. The system of differential equations obtained there may be written in the form

$$\frac{\partial \bar{g}_i}{\partial \ln t} = \beta_i(\bar{g}_1(t, g_1, g_2),\ \bar{g}_2(t, g_1, g_2)) \qquad (i = 1, 2)\,. \tag{4}$$

Its important property consists in that there is no explicit dependence on the variable t in the right-hand sides of the nonlinear differential equations.

The second essential generalization is due to the homogeneity in variable x being broken. Let there be a fixed parameter y of the same physical nature as x. Then the transformation of FS is written in the form

$$x \to x/t, \quad y \to y/t, \quad g \to g' = \bar{g}(t, y, g)\,, \tag{1a}$$

while (2) in the form

$$\bar{g}(x, y, g) = \bar{g}(x/t,\ y/t,\ \bar{g}(t, y, g))\,. \tag{2a}$$

This case was first considered in ref. [3].

3. SOME ILLUSTRATIONS

We have said above that the FS property may appear either as a result of internal symmetry of a certain model, or as a reflection of some scale invariance on a set of related models. Simplest examples of the FS of the first of the mentioned classes may be classical systems (a flexible rod with a fixed end, flat transport problem, a hydrodynamic wave in homogeneous medium) considered in ref. [7]. In these examples, the FS follows from the property of invariance of the model itself formulated in the language of natural variables and functions with a direct physical meaning.

Let us more carefully analyse the problem of the transport theory in a one-dimensional medium [10]. It is assumed that onto the boundary of semi-infinite one-dimensional medium there falls a stationary monochromatic radiation of the intensity g. Considering x to be the depth of penetration of the radiation into the medium and performing transformation (1), for the radiation intensity $\bar{g}(x, g)$ at the depth x inside the medium one may obtain a functional equation of type (2) that results from the Ambartsumyan invariance principle. In this case the differetial equation of RG (3) differs from the integro-differential kinetic equation usually used for describing transport phenomena. The function $\beta(\bar{g})$ entering into its r.h.s. characterizes the damping of the radiation intensity in the semi-infinite medium when an infinitely thin layer is put in front of the medium. Explicit form of that function may be found from given optical characteristics of the medium [10].

Therefore, the formulation of a nonlinear problem on the radiation transport in a semi-infinite medium is made in full analogy with the single-chaged RG in the massless case in QFT.

In the case of radiation transport in a finite-depth medium the consideration leads to an analog [10] of the massive equation of FS (2a).

An illustration of the second type of FS is the analysis made in [2] in connection with the study of effects of vacuum polarization by virtual e^+e^--pairs at short distances from the electron. The model in this case associates the electron with a certain spatial dimension, the radius of smearing. The RG transformation changes one smearing radius by another with a simultaneous change. A transformation of that king may be interpreted as the transition from one nonlocal physical problem to another, each of them being at long distances equivalent to the local model [7,8]. In other words, here the RG takes place on a set of models.

Construction is analogous also when the RGM is used in the theory of critical phenomena which we shall consider below.

4. CRITICAL PHENOMENA

We shall not expound the modern theory of critical phenomena in detail, and shall consider only those basic assumptions that will be important for our consideration.

First of all we note that the RGM in the theory of critical phenomena are applied only to homophase (i.e. homogeneous in symmetry) states of macroscopic systems. Critical phenomena in such states are due to increasing contribution of elementary excitations as the phase-transition point is approached.

The RGM in the theory of critical phenomena was initially based on the Kadanoff transformations [11,12] as applied to the Ising model. These transformations consist in the following:

since below the critical point profitable are those configurations in which neighbouring spins are parallel to each other, it is possible to enlarge an initial lattice with the constant a by passing to a block lattice with sizes of blocks 2a, 3a.... With each block an effective spin μ is associated, and it is assumed that correlations between μ_i and μ_j in the englarged lattice should be the same as those in the initial lattice but with a respective change of the dimensionless coupling constant K. Near the critical point the correlation length $\xi \gg a$, and therefore the change of scale

$$a \to a' = at, \quad K \to K' = \bar{K}(t, K) \tag{5}$$

may be characterized by the almost continuous parameter t, $1 \ll t \ll \xi/a$. Transformations (5) are basically the same as (1), and Wilson in his first work [13] has postulated for them the validity of the Lie differential equation (2). The critical point β_c in this case is defined as a fixed point of that equation, and linearization of the equation near β_c and use of a special iteration procedure lead to the Kadanoff phenomenological results connected with scaling laws. Here the FS has the meaning of the simplest power self-similarity, and the RG transformations are defined, after Gell-mann and Low, on a set of models and represent transformations of the scale of microscopic distances with a simultaneous renormalization of the demensionless coupling constant. As far the Kadanoff construction is concerned, the following comments are to the point.

1. The parameter t is only approximately continuous, and the sequence of transformations $a \to a' = at \to a'' = a'/\tau$ may lead not to the return to one of the block lattice considered above but to the construction of a new lattice (model). In other words, for the Kadanoff approximately continuous scaling transformation no inverse transformation exists, and therefore, one may speak only about a renormalization semigroup (RSG).

2. Conservation of the form of spin correlations for a set of models should, by definition of the correlation function, imply conservation of the form of Hamiltonians: $H(K, a) \to H(K', a')$ which is not true for the Ising model. In the course of enlarging the lattice a new Hamiltonian acquires interaction terms of a higher order than the starting Hamiltonian, and their contribution to thermodynamics is negligible only when $\beta \to \beta_c$.

3. The validity of an RG equation of type (2) is postulated. Thus, the Kadanoff construction is to be considered only as approximate FS (in the simplest form PS) leading to the RSG.

5. THE WILSON APPROACH

Wilson has attempted to avoid difficulties in the Kadanoff construction, in the first place, to determine the class of

Hamiltonians invariant under the scaling transformation of natural variables for which the RG equations are not already postulated but follow immediately from the consideration [14,15].

The first step represents the transition from the Ising model having a simple and obvious physical meaning to a phenomenological model which contains, instead of a local spin a spin field $S(x)$ and, in addition to a pair interaction, an additional functional of spin fields $Q[S(x)]$ that essentially takes into account an infinite sequence of extra interactions. A natural variable in such a problem (an analog of g in (1)) is the spin field, while a renormalizable quantity (an analog of g in (1)) is the functional Q.

The most simple interpretation may be obtained in the momentum representation in which spatial contributions of the spin field may be characterized by index ℓ defining a sequence of the spherical shells of the radius $k = 2^{-(\ell-2)}$ in the momentum space (see [5,6]). In this case the expression for the partition function involves integration over S_ℓ and product of all ℓ. Integration over S_ℓ with fixed $\ell = 0, 1 \ldots$ at each step lowers the number of variables by unity, whereas the statistical sum conserves its form up to renormalization of the functional Q. At an ℓ-th step of integration it becomes possible to write a recurrence of the type

$$\bar{Q}(\ell + \ell', Q) = \bar{Q}(\ell', Q(\ell, Q)), \qquad (6)$$

which is consistent with the FS equation (2).

Let us make two comments. First, the integration over S_ℓ at each step, $\int_{-\infty}^{+\infty} dS_\ell$, prevents from the construction of inverse transformation since it is no longer possible to find an integrand corresponding to a definite integral (the number); consequently, the Wilson construction also possesses semigroup properties. Second, one cannot exactly integrate the partition function, and the Wilson approximations have no rigorous grounds.

So, the FS in the Wilson RG defined by relation (6) is only approximate, as in the Kadanoff construction given by transformation (5).

The next step in the Wilson approach consists in an approximate solution of the nonlinear RSG equation (6). A fixed point corresponds to an ℓ-independent solution of equation (6). For the approximate solution of those equations Wilson and Fisher [14] have used an old idea on the possibility to introduce a continuously changing dimensionality [16] d and have constructed the perturbation theory in small parameter ϵ characterizing the deviation of dimensionality from $d = 4$. The choice of dimensionality $d = 4$ is caused by the possibility of constructing perturbation theiry near the known (at $d = 4$) solution.

6. THE COLLECTIVE-VARIABLE METHOD

An interesting version of the RSG approach has been developed by Yukhnovsky and his followers [17-20]. Within this approach, the interaction parameter in the Ising model is treated as a function of site coordinates having the Fourier transform

$$J(\vec{r}) = \frac{1}{N} \sum_k \tilde{J}(\vec{k})\, e^{i\vec{k}\vec{r}},$$

where the vector $\vec{k}$ runs over values inside the first Brillouin zone. The spin variables are replaced by the Fourier transform ρ_k, and the partition function assumes the form

$$Z = \int (d\rho)^N I(\rho) \exp\{ \sum_k \beta \tilde{J}(\vec{k})\, \rho_k\, \rho_{-k} \},$$

where $I(\rho)$ is the Jacobian of transition from spin to collective variables ρ_k. Calculation of this function is based on the explicit determination of $I(\rho)$ and representation of the partition function as a sum of moments of the Gaussian distribution; the behaviour at the critical point is described by a specially constructed basis distribution of collective variables, an exponential of the polynomial in even powers of collective variables. A further procedure consists in integration over the layers of phase space of collective variables, which, after Wilson, leads to an RSG. However, in the Yukhnovsky approach critical indices can be calculated with the use of recurrence relations obtained from the consideration of partial partition functions within the real space with $d = 3$ without resorting to $d = 4 - \epsilon$.

7. CLASSES OF UNIVERSALITY OF MODELS

Universality of critical phenomena as defined by Kadanoff is understood as equivalence of the critical behaviour of various physical systems, i.e. it is assumed that the critical exponents characterizing the thermodynamics of a system near the critical point are the same for different models within a certain class.

In some cases, the class of universality may be determined by using analogy in the formulation of problems of statistical mechanics and quantum field theory. Consider, as an example, problems of statistical mechanics with the following density of the potential energy

$$U(x) = \frac{1}{2} \int \psi^+(x)\, \psi^+(x')\, \Phi(x, x')\, \psi(x')\, \psi(x)\, dx', \qquad (7)$$

where $\psi(x)$ is a field operator. Form (7) corresponds to a number of physical models applied for the description of liquids, crystals, magnetics, etc. (in particular, to the three-dimensional Ising model).

The theory of critical phenomena assumes that the quantum properties of systems do not influence their critical behaviour usually observed at sufficiently high temperatures [5,12,26]. Therefore the quantum representation of the Hamiltonian can be replaced by its classical analog through changing field operators $\psi(x)$ by scalar functions $\phi(x)$. As a result, we have

$$U(x) = \frac{1}{2} \int \Phi(x, x') \phi^2(x) \phi^2(x') dx'. \quad (8)$$

For the local interaction

$$\Phi(x, x') = C\delta(x - x'), \quad (9)$$

the potential-energy density (8) reduced to the expression

$$U(x) = g\phi^4(x), \quad g \equiv C/2, \quad (10)$$

corresponding to the potential part of the Lagrangian density in field theory ϕ^4. Consequently, statistical systems with a finite radius of two-particle interaction should belong to the same class of universality as the ϕ^4 field-theory model in three dimensions.

A different situation occurs for interaction of an infinite redius when, for instance,

$$\Phi(x, x') = \gamma/V, \quad \gamma = const, \quad (11)$$

where V is the volume of a system. In this case the mean-field approximation for $V \to \infty$ gives thermodynamical solutions asymptotically coincident with exact one, i.e. the potential-energy density (8) for interaction (11) and $V \to \infty$ asymptotically equals

$$U(x) \simeq U_0 \phi^2(x), \quad U_0 \equiv \frac{\gamma}{2V} \int n(x) dx, \quad (12)$$

where $n(x) = <\phi^2(x)>$. It is then seen that the statistical model with long-range interaction corresponds to the ϕ^2 field theory in an effective external field generally dependent on thermodynamic variables.

The equivalence between statistical and field models allows one to calculate the critical exponents by methods of quantum field theory [21,22].

Among other interesting statistical problems that may be reformulated in the language of field theory we mention the problem of isotropic turbulence of an incompressible viscous fluid [23-25].

The above transition from (7) to (8) was made for the simplest case of a one-component system consisting of particles of one sort without internal degrees of freedom. Just for this reason $\phi(x)$ was understood as a scalar function. In the general

case the class of universality in the language of field theory is defined by the following characteristics: the number of different fields, the number of components of each field, leading powers of the fields, space dimensionality, and the nature of interaction (short- or long-range interaction, the presence or absence of anisotropy).

The introduction of classes of universality essentialy simplifies the description of critical phenomena since a large number of models with a different physical content may be substituted by a small number of their mathematical equivalents. It is to be stressed that belonging of particular models to a certain class of universality is usually postulated a priory rather than proved. The assumption of classes of universality is verified by a universal behaviour experimentally observed for various substances when asymptotically approaching the critical point.

It is, however, to be recognized that the critical behaviour experimentally observed for many substances does not allow one to put them into a certain class of universality. For a number of model problems of statistical mechanics the concept of classes of universality is also meaningless; this, for instance, concerns the exactly solvable Baxter model.

8. CRITICAL PHENOMENA AND HETEROPHASE STATES

The renormalization-group method is usually used solely for considering homophase critical phenomena, when only a single thermodynamic phase is present on each side of a phase-transition point. However, in many physical systems the so-called heterophase states are experimentally observed, the latter being a mixture of two or several phases, having, as a rule, different macroscopic symmetries. For example, in ferro - and antiferromagnets paramagnetic fluctuations can occur, in ferroelectrics paraelectric fluctuational nuclei appear, in liquids crystalline clusters exist and so on. A distinguishing feature of heterophase phenomena is their occurrence in quite wide temperature intervals below as well as above of the phase-transition point that can be of first or second order. Note that the Wilson approach efficiently works exceptionally near second-order transitions, while there are some attempts to apply it to first-order transitions [26].

A microscopic description of heterophase states has been done in refs. [27-29] and exploited in the case of various systems [27-30]. An essential point is that the heterophase fluctuations require for their existence the presence of disordering interaction entering into the Hamiltonian side by side with an ordering interaction. According to the correlation of these competing forces the system can have either first- or second order transition, and even the critical indices can be changed.

In such a way the same physical model can belong to different universality classes depending on the values of interactions but not merely on their effective radii.

9. CONSCLUSION

To summarize, let us make the following conclusions.

1. The functional self-similarity, reflecting the property of transitivity of physical characteristics with respect to the way of giving their initial values and being an exact or approximate consequence of the internal symmetry of a system or realized on a set of models, makes the basis for constructing the renormalization group.
2. In the theory of homophase critical phenomena the property of functional self-similarity is usually realized on a set of models as an approximate one leading to renormalization semigroup. This is just the main difference from problems of QFT and classical physics for which the functional self-similarity and the corresponding renormalization group are exact.
3. The introduction of classes of universality simplifies the description of critical phenomena by allowing a simplest model to be chosen from a considered class.
4. There are exampes of the systems which cannot be related to a definite class of universality (heterophase systems) or even to any of the classes of universality (the Baxter model).
5. In many cases being based on equivalence between statistical and field models (in the sense of critical behaviour) it is convenient to employ renormalization group methods developed in field theory.

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Received by Publishing Department
on July 7, 1986.

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О взаимосвязи различных ренормгрупп

Рассмотрено свойство функциональной автомодельности ряда физических систем. Исследованы разные формулировки ренормгрупповых и полугрупповых преобразований. Обсуждены их различия и общие свойства. С математической точки зрения введение группы перенормировок можно производить двумя способами. Один способ основан на выявлении внутренней симметрии конкретной физической системы и описании данной симметрии на языке естественных переменных рассматриваемой задачи. Такой метод построения ренормгрупп использовался Штюкельбергом и Петерманом, Боголюбовым и Ширковым. Другой способ связан с выделением совокупности моделей, обладающих близкими свойствами и отличающихся друг от друга масштабом некоторой переменной. К этому методу относятся подходы Гелл-Манна и Лоу, Каданова и Вильсона, Юхновского. Оба указанных способа являются частными следствиями общего принципа функциональной автомодельности.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1986

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On Interrelation of Different Renormalization Groups

The functional self-similarity property of a number of physical systems is considered. Different formulations of renormalization group and semi-group transformation are analysed. Their general and particular properties are discussed. From the mathematical point of view a renormalization group can be introduced by two ways. One is based on the elucidation of an internal symmetry of a concrete physical system and on the description of the symmetry given using the language of variables that are natural for the problem considered. Such a method of constructing renormalization groups has been used by Stueckelberg and Petermann and by Bogolubov and Shirkov. The other way is connected with the separation of a class of models that have similar properties and differ one from another by the scale of some variable. This method is illustrated by the approaches due to Gell-Mann and Low, Kadanoff and Wilson, and Yukhnovsky. Both the ways mentionned are particular consequences of the general principle of the functional self-similarity.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1986