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# THE COUNTERTERM $\delta \mathcal{L} = -\frac{1}{2} M^2 A_\mu^2$ IN THE LAGRANGIAN OF EUCLIDEAN QUANTUM ELECTRODYNAMICS AND THE GAUGE INVARIANCE

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1. Euclidean quantum electrodynamics is determined by the Lagrangian

$$\mathcal{L} \equiv \mathcal{L}(\psi, \psi^*, A) = -\psi^*\left[\gamma_\alpha\left(\frac{\partial}{\partial x_\alpha} - ieA_\alpha\right) + m\right]\psi - \frac{1}{4}\left(\frac{\partial A_\alpha}{\partial x_\beta} - \frac{\partial A_\beta}{\partial x_\alpha}\right)^2 \qquad (1)$$

which is invariant under gauge transformations

$$\begin{aligned}
\psi^*(x) &\to \psi^{*\prime}(x) = \psi^*(x)\, e^{-ie\Lambda(x)} \\
\psi(x) &\to \psi'(x) = \psi(x)\, e^{+ie\Lambda(x)} \\
A_\alpha(x) &\to A'_\alpha(x) = A_\alpha(x) + \frac{\partial \Lambda}{\partial x_\alpha}.
\end{aligned} \qquad (2)$$

Such an Euclidean QED contains quadratic divergences of the inverse photon propagator. In order to get rid of these quadratic divergences, one introduces into the Lagrangian the counterterm

$$\delta_1 \mathcal{L} = -\frac{1}{2} M^2 A_\mu^2(x) \qquad (3)$$

(see, e.g., the textbook /1/); usually one also adds to the Lagrangian the counterterm

$$\delta_2 \mathcal{L} = -\frac{1}{2} d_l \left(\partial_\alpha A_\alpha\right)^2 \qquad (4)$$

so that the total Lagrangian becomes

$$\mathcal{L}_1(\psi, \psi^*, A) = \mathcal{L}(\psi, \psi^*, A) + \delta_1\mathcal{L} + \delta_2\mathcal{L}. \qquad (5)$$

The counterterms (3), (4) are not invariant under gauge transformations (2); but the QED, determined by the Lagrangian (5) is gauge invariant /1/.

2. In this work we give a simple proof of this statement for the Euclidean QED.

2.1. First of all, we shall introduce instead of the fields $\psi, \psi^*$ $A_\alpha$ the fields $\xi, \xi^*, B_\alpha, \varphi$ according to equations

$$A_\alpha(x) = B_\alpha(x) + \partial\varphi(x)/\partial x_\alpha$$

$$\xi(x) = \psi(x)\exp[-ie\varphi(x)]$$

$$\xi^*(x) = \psi^*(x)\exp[\,ie\varphi(x)]$$

$$\varphi(x) = \Box^{-1}\partial A_\alpha/\partial x_\alpha . \tag{6}$$

Here $\Box = \sum_{\alpha=1}^{4} \partial^2/\partial x_\alpha^2$ so that if

$$u(x) = \Omega^{-1}\sum_{k\neq 0} e^{ikx} u_k , \tag{7a}$$

then

$$\Box^{-1} u(x) = -\Omega^{-1}\sum_{k\neq 0} e^{ikx} u_k/k^2 \tag{7b}$$

(we have introduced four-dimensional periodicity cube $\Omega$ ). Last equation contains $k^2$ in the denominator. This is the point where we use our Euclidean metric: in pseudo-Euclidean metric the quantity $k^2$ may be zero even if $k \neq 0$ and the operator $\Box^{-1}$ is ill- defined.

2.2. One can easily prove that under gauge transformations (2) the quantities $B_\alpha(x)$ are invariant, and the quantities $\xi, \xi^*$ get the constant phase factor :

$$\xi(x) \to \xi'(x) = \xi(x)e^{ie\Lambda_0},$$

where $\Lambda_0$ is the coefficient of the expansion $\Lambda(x) = \sum_k \Lambda_k e^{ikx}$ with $k = 0$ .

2.3. In the continual integral method the photon, e.g., propagator is determined by the equation (see /2/, chap. 4)



$$D_{\alpha\beta}(x,y) = \int \delta A\,\delta\psi\,\delta\psi^*\, e^{-S} A_\alpha(x) A_\beta(y) \Big/ \int \delta A\,\delta\psi\,\delta\psi^*\, e^{-S}, \tag{8}$$

where

$$S = -\int_\Omega \mathcal{L}_1\, d^4x . \tag{9}$$

Let us express eq. (8) in terms of variables $\xi, \xi^*, B_\alpha, \varphi$ .

2.3.1. First of all note that

$$\delta A\,\delta\psi\,\delta\psi^* \equiv \Big(\prod_{(x)}\Big(\prod_{\alpha=1}^{4} dA_\alpha(x)\Big) d\psi(x)\, d\psi^*(x)\Big) =$$

$$= const\Big(\prod_{(x)} \delta(\partial_\beta B_\beta(x))\Big(\prod_{\alpha=1}^{4} dB_\alpha(x)\Big) d\varphi(x)\, d\xi(x)\, d\xi^*(x)\Big)$$

$$\equiv \delta B\,\delta\xi\,\delta\xi^*\,\delta\varphi \tag{10}$$

(remember that eq. (6) gives

$$\partial_\beta B_\beta(x) = 0 \quad ). \tag{11}$$

2.3.2. Let us prove the action (9) to be the sum of two terms, where the first term depends on $B, \xi, \xi^*$; and the second, on $\varphi$:

$$S = S_1(\xi, \xi^*, B) + S_2(\varphi). \tag{12}$$

Really, eq. (6) determines the gauge transformation (2) with $\Lambda = \varphi$ , so that

$$\mathcal{L}(\psi, \psi^*, A) = \mathcal{L}(\xi, \xi^*, B). \tag{13}$$

One has $\partial_\alpha A_\alpha = \Box\varphi$ and

$$\delta_2 \mathcal{L} = -(d_\ell/2)(\Box\varphi)^2. \tag{14}$$

Equation (11) implies

$$\int_\Omega B_\alpha(x)\,\partial\varphi(x)/\partial x_\alpha\, d^4x = -\int \varphi(x)\partial B_\alpha(x)/\partial x_\alpha\, d^4x = 0$$

so that

$$\int_\Omega \delta_1 \mathcal{L}\, d^4x = -(M^2/2)\int d^4x\,[B_\alpha(x)^2 + (\partial\varphi/\partial x_\alpha)^2]. \tag{15}$$

Thus, we have arrived at eq. (12) with

$$S_1(\zeta,\zeta^*,B) = \int_\Omega d^4x \Big\{ \zeta^*\Big[\gamma_\alpha\Big(\frac{\partial}{\partial x_\alpha} - ieB_\alpha\Big) + m\Big]\zeta + \frac{1}{4}\Big(\frac{\partial B_\alpha}{\partial x_\beta} - \frac{\partial B_\beta}{\partial x_\alpha}\Big)^2 + \frac{M^2}{2} B_\alpha(x)^2 \Big\} \tag{16}$$

$$S_2(\varphi) = \int_\Omega d^4x \left\{ (d_\ell/2)(\Box\varphi)^2 + (M^2/2)(\partial\varphi/\partial x_\alpha)^2 \right\}. \tag{17}$$

2.4. Equations (6), (8), (10), (12) give

$$\mathcal{D}_{\alpha\beta}(x,y) = \mathcal{D}^{tr}_{\alpha\beta}(x,y) + \mathcal{D}^{\ell}_{\alpha\beta}(x,y) \tag{18}$$

where

$$\mathcal{D}^{tr}_{\alpha\beta}(x,y) = \int \delta B\,\delta\zeta\,\delta\zeta^*\, e^{-S_1} B_\alpha(x) B_\beta(y) \Big/ \int \delta B\,\delta\zeta\,\delta\zeta^*\, e^{-S_1} \tag{19}$$

$$\mathcal{D}^{\ell}_{\alpha\beta}(x,y) = \int \delta\varphi\, e^{-S_2} \frac{\partial\varphi(x)}{\partial x_\alpha}\frac{\partial\varphi(y)}{\partial y_\beta} \Big/ \int \delta\varphi\, e^{-S_2}$$

$$= \int d^4k\, \frac{\exp[ik(x-y)]}{k^2 d_\ell + M^2}\, \frac{k_\alpha k_\beta}{k^2} \tag{20}$$

the integrations over variables $\zeta, \zeta^*, B$ and over $\varphi$ get separated, giving rise to the gauge invariant photon propagator $\mathcal{D}^{tr}$ and the propagator $\mathcal{D}^{\ell}$ of a noninteracting longitudinal particle. (The propagator $\mathcal{D}^{tr}$ is gauge-invariant as the variables $B_\alpha$ do not change under transformation (2) and the variables $\zeta, \zeta^*$ get only a constant phase factor under this transformation).



2.4.1. Let us define the gauge invariant electron propagator $G_0(x,y)$ by the substitution $\zeta(x)\ \zeta^*(y)$ for $A_\alpha(x)A_\beta(y)$ in eq.(8).
One may get higher-order gauge invariant Green functions by the substitution of the product of several factors $B_{\alpha_i}(x_i)$ and factors $\zeta(y_j)\,\zeta^*(z_j)$ for $A_\alpha(x)A_\beta(y)$ in eq. (8).

2.4.2. Let us define the gauge noninvariant electron propagator $G(x,y)$ by substitution $\psi(x)\psi^*(y)$ for $A_\alpha(x)A_\beta(y)$ in eq. (8). The change of variables (6) allows one to represent $G(x,y)$ as the product of the propagator $G_0(x,y)$ and the quantity

$$f(x,y) \equiv \int \delta\varphi\, e^{-S_2} \exp[ie(\varphi(x)-\varphi(y))] \Big/ \int e^{-S_2}\delta\varphi \tag{21}$$

which reduces to the calculation of the Gauss-type integrals (/1/, chap VIII, § 45.5). Analogously, one can express higher order gauge noninvariant Green functions in terms of the gauge invariant Green functions (item 2.4.I) and the propagator $\mathcal{D}^{\ell}$.

2.4.3. The difference between our consideration and that of the textbook /1/ (chapter VIII, § 45.5) is the following
 i) we have introduced the counter-term (3) into the Lagrangian,
 ii) we have conducted our consideration in the Euclidean metric, so that our quantities $B_\alpha(x)$ are gauge invariant unlike quantities $A^{tr}_\alpha(x)$ of the book /1/ which are defined in the pseudo-Euclidean metric and do transform under the gauge transformation

$$A^{tr}_\alpha(x) \to A^{tr}_\alpha(x) + \partial\chi/\partial x_\alpha, \quad \Box\chi = 0.$$

3. Thus, the Euclidean QED, as defined by the Lagrangian (5), describes in a gauge invariant way the interacting spinor and electromagnetic fields; besides, it describes a noninteracting longitudinal particle of the mass $M/\sqrt{d_\ell}$ (see eq. (20)).
Let us note that having introduced the counterterm (3) into the Lagrangian, one has to change substantially all the consideration of, e.g., Schwinger model.

Moreover, the description of QED via the Hamiltonian

$$H = \int d^3x \Big\{ \frac{1}{2} \Pi^t(x)^2 + \frac{1}{2}(\mathrm{rot}\, A^t)^2 + \psi^*(x)\gamma_4\Big[\sum_{j=1}^{3} \gamma_j\Big(\frac{\partial}{\partial x_j} - ieA_j^t(x)\Big) + m\Big]\psi - \frac{e^2}{2}\rho\frac{1}{\Delta}\rho \Big\}$$

(where $A^t(x)$ is a spatially transveral part of a vector-potential ( $\mathrm{div}\, A^t(x) = 0$ ), $\Pi^t(x)$ the momentum canonically conjugated to $A^t(x)$, $\rho = \psi^*(x)\psi(x)$, $\Delta$ the Laplace operator) which was sometimes universally recognized /1,3/, turns out to be wrong. This description does not provide the vanishing of the quadratical divergence of the inverse photon propagator for all values of the coupling constant $e^2$ and leads to some condition determining allowed values of $e^2$ /4/. It is to be noted, however, that we do not know how to extend our Euclidean metric consideration to the case of pseudo-Euclidean metric. Therefore, the above statement concerning the necessity to reject the Hamiltonian $H$ is not at all a rigorous one: it is only a conjecture.

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Заставенко Л.Г. E2-86-425

Член $\delta\mathcal{L} = -\frac{1}{2} M^2 A_\mu^2$ в лагранжиане эвклидовой квантовой электродинамики и калибровочная инвариантность


В рамках формализма континуальных интегралов дано простое доказательство утверждения, что введение в лагранжиан эвклидовой КЭД контрчлена $\delta\mathcal{L} = -\frac{1}{2}A_\mu^2 M^2$ /которое обычно производят для устранения квадратичных расходимостей обратного фотонного пропагатора/ не приводит к нарушению калибровочной инвариантности КЭД.



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Zastavenko L.G. E2-86-425

The Counterterm $\delta\mathcal{L} = -\frac{1}{2}M^2A_\mu^2$ in the Lagrangian of Euclidean Quantum Electrodynamics and the Gauge Invariance


The quadratic divergence of the inverse photon propagator in QED comples one to introduce the counterterm $\delta\mathcal{L} = -\frac{1}{2}A_\mu^2 M^2$ into the Lagrangian. Our work contains a simple proof that such an introduction does not break the gauge invariance of Euclidean QED. The gauge non-invariant part of $\mathcal{L}$ describes some non-interacting particle. We use the Feynman path integral formulation of QED.



The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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