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A NON-RELATIVISTIC MODEL OF TWO-PARTICLE DECAY. Formulation of the Problem

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1. Introduction

This paper is devoted to the analysis of a simple non-relativistic model of two-particle decay. Why have we addressed ourselves with such a problem ? In order to answer this question, recall first that a wide family of decay processes, spontaneous or induced, is observed in different areas ranging from particle to molecular physics. As a rule, they represent one of our primary sources of physical information.

On the other hand, the state of arts in the quantum theoretical description of decay processes does not always correspond to their important role. It is true that most of them can be treated effectively by simple methods whose origin can be traced back to the founding fathers of quantum mechanics[1,2]. However, a lot of problems arise immediately when we try to go beyond the accuracy of these methods, which represent in a sense the first-order approximation, and to formulate the decay theory on rigorous grounds. A sketch of such a theory with an extensive bibliography can be found in Chaps.1-3 of Ref.3, but a number of open problems persist. In this situation, we regard a thorough treatment of various decay models as a way which can give us a deeper insight and stimulate development of the theory.

According to our opinion, there are two sorts of open problems which deserve a particular attention[4]. The first of them concerns a consistent relativistic description of decays. Starting from the first principles, one can construct a general quantum-kinematical framework[5,6], but the choice of the functions which determine the decay law and other measurable quantities has only an indirect justification based on similarities with non-relativistic systems and heuristic considerations. We met this problem in a recent study concerning the effects of localization on proton decay[7]. There is no model, up to our knowledge, of a relativistic quantum field theory, which would illustrate appearance of a Breit-Wigner-type ansatz on dynamical grounds. We intend to discuss this question later in more detail.

At the present time, however, we are going to work within the non-relativistic quantum mechanics. The situation is much better here; in many cases we are able to treat unstable systems rigorously, and at the same time more or less realistically. To be specific, let us mention the dilation-analytic technique/8-12/ or the tunneling decay models/13-16/. A common mathematical core of these and other studies is the perturbation theory of the eigenvalues which "dissolve" in the continuous spectrum once the perturbation is turned on. The second one of the important open questions mentioned above concerns the search for alternative techniques of solving this problem/4/.

After this introduction, let us turn to the contents of the present paper. We are going to treat in detail a two-particle decay with the simplest possible interaction Hamiltonian. It bears a close similarity to the lowest sector of the Lee model/17/. The decay of the V-particle have been discussed many times in this framework, e.g., in Refs.18-22, and the essential idea can be traced back to the Friedrichs' paper/23/. Hence the results we are going to derive are hardly surprising. Nevertheless, none of the existing treatments can be regarded as complete or entirely satisfactory, and we hope to fill the gaps. At the same time, a careful analysis of this problem represents a good starting point to discussion of more complicated decay models.

The model is described in Section 2. In the next section, we discuss its transformation properties with respect to the Galilei group $\vec{\mathcal{G}}$. Using the standard representation theory of $\mathcal{G}$, one can write easily the projective representation of $\mathcal{G}$ which suits to our problem. We deviate, however, from the standard formulation/24/ in the matter of time translations. Usually the representations are regarded as acting on functions of coordinates and <u>of time</u> ; we prefer to express them by operators acting on the state Hilbert space only. It makes the proof of Theorem 3.1 a bit lengthy, but according to our opinion, it is a proper way how the spacetime transformations of a quantum system should be described. Next we find the conditions under which the model is Galilei-invariant ; further on we shall consider this case only.

After separating the centre-of-mass motion in Section 4, we turn to discussion of the reduced resolvent which contains the essential dynamical information. We show that under mild assumptions about the interaction Hamiltonian, it has a meromorphic structure. The unperturbed Hamiltonian has a simple eigenvalue embedded in the continuous

spectrum ; the corresponding pole shifts under influence of the perturbation to the second sheet of the analytically continued reduced resolvent. Further properties of the solution, such as spectral concentration, relation to the scattering theory, etc., will be discussed in a sequel to this paper.

2. Description of the model

We are going to discuss the situation when a heavy particle of mass M decays into two particles of non-zero masses m_1, m_2 ; all of them are assumed to be non-relativistic and spinless. The state Hilbert space of such a system is

$$\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_d \quad , \tag{2.1}$$

where $\mathcal{H}_u = L^2(\mathbb{R}^3)$ refers to the heavy particle and $\mathcal{H}_d = L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$ to the decay products. Its elements shall be written as

$$\Psi : \ \Psi(\vec{X}, \vec{x}_1, \vec{x}_2) = \begin{pmatrix} \psi_u(\vec{X}) \\ \psi_{12}(\vec{x}_1, \vec{x}_2) \end{pmatrix} \quad , \tag{2.2a}$$

where $\vec{X}, \vec{x}_1, \vec{x}_2 \in \mathbb{R}^3$. Since we are going to build the model in such a way that the centre-of-mass motion would be free, it is natural to identify the heavy particle coordinates with

$$\vec{X} = \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m_1 + m_2} \quad . \tag{2.3a}$$

In view of the Bargmann's superselection rule/24,25/, the masses must satisfy

$$M = m_1 + m_2 \quad ; \tag{2.3b}$$

otherwise the decay would not occur in a Galilean-covariant theory (cf.Remark 3.3 below). The relative motion describes in terms of

$$\vec{x} = \vec{x}_2 - \vec{x}_1 \quad , \quad m = \frac{m_1 m_2}{m_1 + m_2} \quad . \tag{2.4}$$

Using these variables, we can set $\psi_d(\vec{X}, \vec{x}) := \psi_{12}(\vec{x}_1, \vec{x}_2)$ and write the state vectors as

$$\Psi : \ \Psi(\vec{X}, \vec{x}) = \begin{pmatrix} \psi_u(X) \\ \psi_d(X, x) \end{pmatrix} \quad . \tag{2.2b}$$

Next one has to choose Hamiltonian of the model. In order to make the decay energetically possible, the free energy of the heavy particle must be shifted on a positive constant E . It corresponds to the energy released in the decay of a relativistic particle of rest mass $M_0 = m_1 + m_2 + E/c^2$; if $E \ll m_j c^2$ for $j = 1,2$, then the heavy-particle energy is

$$(M_0^2 c^4 + p^2 c^2)^{1/2} = m_1 c^2 + m_2 c^2 + E + \frac{\vec{p}^2}{2(m_1+m_2)} + O(c^{-2}) \quad .$$

Motivated by this argument, we choose

$$H_g = H_0 + gV \quad , \tag{2.5}$$

where the free Hamiltonian is

$$H_0 = \begin{pmatrix} E - \frac{1}{2M}\Delta_X & 0 \\ 0 & -\frac{1}{2m_1}\Delta_{x_1} - \frac{1}{2m_2}\Delta_{x_2} \end{pmatrix} \tag{2.6a}$$

and the interaction Hamiltonian will be specified a little later. The Laplacians here are understood as self-adjoint operators, i.e., with the domains consisting of those $\psi \in L^2(\mathbb{R}^3)$ for which $\Delta\psi$ exists in the sense of distributions (cf.Ref.27, Sec.IX.7). Furthermore, the operator $-(2m_1)^{-1}\Delta_{x_1} - (2m_2)^{-1}\Delta_{x_2}$ is easily seen to be e.s.a. (Ref.27, Sec.VIII.10). We rewrite the free Hamiltonian using the coordinates (2.3), i.e., as the self-adjoint operator

$$H_0 = \begin{pmatrix} E - \frac{1}{2M}\Delta_X & 0 \\ 0 & -\frac{1}{2M}\Delta_X - \frac{1}{2m}\Delta_x \end{pmatrix} \quad . \tag{2.6b}$$

In fact, the two operators are unitarily equivalent by mens of U : $(U\psi)(\vec{X},\vec{x}) = \psi(\vec{x}_1,\vec{x}_2)$, but we prefer to speak about the same operator in different coordinates. The interaction Hamiltonian is chosen in the simplest possible way, namely

$$V : \quad (V\psi)(\vec{X},\vec{x}) = \begin{pmatrix} \int_{\mathbb{R}^3} v(\vec{y})\psi_d(\vec{X},\vec{y})\, dy \\ v(\vec{x})\psi_u(\vec{X}) \end{pmatrix} \quad , \tag{2.7}$$

where $v \in L^2(\mathbb{R}^3)$ is a given real-valued function. Using Fubini theorem and Hölder inequality, one finds easily $\|V\psi\| \le \|v\|\|\psi\|$ for all $\psi \in \mathcal{H}$, where $\|v\|$ is the L^2-norm. The equality is achieved, e.g., if $\psi_d = 0$. Hence V is a bounded operator, $\|V\| = \|v\|$, which is, furthermore, symmetric due to the real-valuedness of v . We conclude :

Proposition 2.1 : For a real coupling constant g , the operator H_g is self-adjoint on the domain $D(H_g) = D(H_0)$ consisting of all $\psi \in \mathcal{H}$ for which $H_0\psi$ exists in the sense of distributions.

Before proceeding further, let us show how the state vectors look like in the p-representation. If $\vec{p}_1, \vec{p}_2$ are the light-particle momenta, then

$$\vec{P} = \vec{p}_1 + \vec{p}_2 \quad , \tag{2.8a}$$

$$\vec{p} = \frac{m_1\vec{p}_2 - m_2\vec{p}_1}{M} \tag{2.8b}$$

are conventionally the centre-of-mass and the relative momentum, respectively. As above, $\vec{P}$ is identified with momentum of the heavy particle. For an arbitrary $\psi \in \mathcal{S}(\mathbb{R}^3) \oplus \mathcal{S}(\mathbb{R}^6)$, we define

$$\hat{\psi}(\vec{P},\vec{p}) \equiv \begin{pmatrix} \hat{\psi}_u(\vec{P}) \\ \hat{\psi}_d(\vec{P},\vec{p}) \end{pmatrix} := \begin{pmatrix} (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-i\vec{P}.\vec{X}}\psi_u(\vec{X})\, d\vec{X} \\ (2\pi)^{-3} \int_{\mathbb{R}^6} e^{-i(\vec{P}.\vec{X}+\vec{p}.\vec{x})}\psi_d(\vec{X},\vec{x})\, d\vec{X}\, d\vec{x} \end{pmatrix} ; \tag{2.9}$$

this transformation extends by continuity to the operator $F : \mathcal{H} \to \mathcal{H}$. In other words, we define

$$F := F_3 \oplus F_6 \quad , \tag{2.10}$$

where F_n denotes the Fourier-Plancherel operator on $L^2(\mathbb{R}^n)$; this relation shows that F is unitary. In what follows, we shall mostly write $F\psi = \hat{\psi}$, having in mind that $\hat{\psi} = \underset{n\to\infty}{\mathrm{l.i.m.}}\, \hat{\psi}_n$ for some sequence with suitably regularized integrals if ψ does not belong to $L^2 \cap L^1$. The free Hamiltonian acquires in the p-representation form of the matrix multiplication,

$$FH_0F^{-1} = \begin{pmatrix} E + \frac{\vec{P}^2}{2M} & 0 \\ 0 & \frac{\vec{P}^2}{2M} + \frac{\vec{p}^2}{2m} \end{pmatrix} \quad . \tag{2.11}$$

As for the interaction Hamiltonian, we have to express the vector $FV\psi$. It holds obviously $F_6 v\psi_u = \hat{v}\hat{\psi}_u$. For a function $\psi_d : \psi_d(\vec{X},\vec{y}) = \sum_{k=1}^{n} \psi_{1k}(\vec{X})\psi_{2k}(\vec{y})$ with $\psi_{1k} \in \mathcal{S}(\mathbb{R}^3)$ and $\psi_{2k} \in L^2(\mathbb{R}^3)$, the first row in (2.7) equals

$$\sum_{k=1}^{n} \psi_{1k}(\vec{X})(v,\psi_{2k}) = \sum_{k=1}^{n} \psi_{1k}(\vec{X})(\hat{v},\hat{\psi}_{2k}) \quad .$$

Applying to this the operator F_3 , we get

$$\sum_{k=1}^{n}\psi_{1k}(\vec{P})(\hat{v},\hat{\psi}_{2k}) = \int_{\mathbb{R}^3} \hat{\hat{v}}(\vec{k})\psi_d(\vec{P},\vec{k})\,d\vec{k} \quad .$$

The function ψ_d of this form are, however, dense in $L^2(\mathbb{R}^6)$, and the operator FV is bounded ; thus the relation

$$(FV\psi)(\vec{P},\vec{p}) = \begin{pmatrix} \int_{\mathbb{R}^3} \hat{\hat{v}}(\vec{k})\psi_d(\vec{P},\vec{k})\,d\vec{k} \\ \hat{v}(\vec{p})\psi_u(\vec{P}) \end{pmatrix} \quad . \tag{2.12}$$

holds for all $\psi \in \mathcal{H}$.

3. Galilei group transformations

Let us recall first some basic facts about the Galilei group $\mathcal{G}$; more detailed exposition can be found, e.g., in Refs.24-26 . It is a ten-parameter Lie group whose elements are $\gamma = (b,\vec{a},\vec{v},R)$, where R is a 3×3 orthogonal matrix. They satisfy the composition law

$$(b',\vec{a}',\vec{v}',R')(b,\vec{a},\vec{v},R) = (b+b',R'\vec{a}+\vec{a}'+\vec{v}'b,R'\vec{v}+\vec{v}',R'R) \quad . \tag{3.1}$$

There is a one-to-one correspondence between the elements of $\mathcal{G}$ and spacetime transformations. In this way, the light-particle coordinates transform under $\gamma \in \mathcal{G}$ to

$$\vec{x}_j' = R\vec{x} + \vec{v}t + \vec{a} \quad , \quad j=1,2 \quad . \tag{3.2}$$

The same is true for their centre-of-mass position,

$$\vec{X}' = R\vec{X} + \vec{v}t + \vec{a} \tag{3.3a}$$

so its identification with the position of the heavy particle is Galilei-covariant. On the other hand, the relative coordinate transforms as

$$\vec{x}' = R\vec{x} \quad . \tag{3.3b}$$

Let us turn now to transformations of the state vectors. For the model under consideration, the following assertion holds :

Theorem 3.1 : There is a unitary projective representation of $\mathcal{G}$ on $\mathcal{H}$ defined by

$$(U(b,\vec{a},\vec{v},R)\psi)(\vec{X},\vec{x}) = \exp\{\tfrac{i}{2}M[\vec{v}^2 b + \vec{v}.(2\vec{X}-\vec{a})]\} \times \times (e^{iH_g t}\psi)(R^{-1}(\vec{X}+\vec{v}b-\vec{a}),R^{-1}\vec{x}) \tag{3.4}$$

for all $\gamma \in \mathcal{G}$ and $\psi \in \mathcal{H}$. Its multiplier equals

$$\omega(\gamma',\gamma) = \exp\{\tfrac{i}{2}M(\vec{v}'.R'\vec{a} - \vec{a}'.R'\vec{v} - R'\vec{v}.\vec{v}'b)\} \quad . \tag{3.5a}$$

Proof of the theorem requires an auxiliary assertion :

Lemma 3.2 : The relation

$$(e^{iH_g b'} e^{(i/2)M[\vec{v}^2 b + \vec{v}.(2\vec{X}-\vec{a})]} e^{-iH_g b}\psi)(\vec{X},\vec{x}) = = e^{(i/2)M[\vec{v}^2(b+b') + \vec{v}.(2\vec{X}-\vec{a})]}\psi(\vec{X}+\vec{v}b',\vec{x}) \tag{3.6a}$$

Proof : It is clearly sufficient to check the relation (3.6a) for $b=0$, $\vec{a}=0$. It is useful to pass to the p-representation. We denote $\hat{H}_g = FH_gF^{-1}$; then it is equivalent to

$$(e^{i\hat{H}_g b'} S_{M\vec{v}}\, e^{-i\hat{H}_g b'}\hat{\psi})(\vec{P},\vec{p}) = e^{(i/2)M\vec{v}^2 b' + ib'\vec{v}.\vec{P}}(S_{M\vec{v}}\hat{\psi})(\vec{P},\vec{p}) \quad , \tag{3.6b}$$

where $S_{\vec{k}}$ is the shifting operator

$$(S_{\vec{k}}\hat{\Phi})(\vec{P},\vec{p}) := \hat{\Phi}(\vec{P}-\vec{k},\vec{p}) \quad .$$

The equivalence follows from the relations

$$(FS_{-\vec{v}b'}\Phi)(\vec{P},\vec{p}) = e^{-ib'\vec{v}.\vec{P}}\hat{\Phi}(\vec{P},\vec{p}) \quad ,$$

$$(F\, e^{iM\vec{v}.\vec{X}}\Phi)(\vec{P},\vec{p}) = \hat{\Phi}(\vec{P}-M\vec{v},\vec{p}) \quad ,$$

which can be verified directly for $\Phi \in L^2\cap L^1$; for a general $\Phi \in \mathcal{H}$, one can always find a sequence $\{\Phi_n\} \subset L^2\cap L^1$ such that $\hat{\Phi}_n$ converges pointwise to $\hat{\Phi}$ a.e. in $\mathbb{R}^3\times\mathbb{R}^6$. Before proceeding further, we must introduce some more notation. Let $B_n(\alpha) := \{x\in\mathbb{R}^n : |x|\le\alpha\}$, then we denote

$$N(\alpha) = \{\psi\in\mathcal{H} : \text{supp}\,\hat{\psi}_u \in B_3(\alpha) \ , \ \text{supp}\,\hat{\psi}_d \in B_6(\alpha)\} \quad .$$

Further we define the operator $\hat{H}_{g\alpha} = \hat{H}_0 + g\hat{V}_\alpha$, where $\hat{V}_\alpha$ is given by (2.12) with v replaced by

$$\hat{v}_\alpha := \hat{v}\chi_{B_3(\alpha)} \quad .$$

It is easy to see that the sets $\hat{N}(\alpha) := FN(\alpha)$ fulfil the following conditions

$$\hat{H}_0\hat{N}(\alpha) \subset \hat{N}(\alpha) \ ,$$
$$\hat{H}_{g\beta}\hat{N}(\alpha) \subset \hat{N}(\max(\alpha,\beta)) \ , \tag{3.7}$$
$$S_{M\vec{v}}\hat{N}(\alpha) \subset \hat{N}(\alpha + M|\vec{v}|) \ .$$

For an arbitrary $\hat{\psi} \in N(\alpha)$, the following estimate is valid

$$\|\hat{H}_{g\beta}\hat{\psi}\|^2 \leq 2\|(E + \frac{\vec{P}^2}{2M})\hat{\psi}_u\|^2 + 2g^2\left\|\int_{\mathbb{R}^3} \hat{\hat{v}}_\beta(\vec{k})\hat{\psi}_d(.,\vec{k})\, d\vec{k}\right\|^2 + 2\left\|(\frac{\vec{P}^2}{2M} + \frac{\vec{p}^2}{2m})\hat{\psi}_d\right\|^2 +$$
$$+ 2g^2\|\hat{v}_\beta\hat{\psi}_u\|^2 \leq$$
$$\leq 2\left[(E + \frac{\alpha^2}{2M})^2 + g^2\|\hat{v}_\beta\|^2\right]\|\hat{\psi}_u\|^2 + 2\left[(\frac{\alpha^2}{2M} + \frac{\alpha^2}{2m})^2 + g^2\|\hat{v}_\beta\|^2\right]\|\hat{\psi}_d\|^2 \ .$$

Since $\|\hat{v}_\beta\| \leq \|\hat{v}\| = \|v\|$, we get the inequality

$$\|\hat{H}_{g\beta}\hat{\psi}\|^2 \leq C_\alpha\|\hat{\psi}\|^2 \tag{3.8}$$

with

$$C_\alpha^2 = 2(E + \frac{\alpha^2}{2M} + \frac{\alpha^2}{2m})^2 + 2g^2\|v\|^2 \ .$$

It is easy to deduce from here that $\hat{N}(\alpha)$ consists of analytical vectors of $\hat{H}_{g\beta}$ for any fixed α,β . In a similar way, we are going to check the formula (3.6b) expanding the exponential functions into power series. Assume $\alpha \geq \beta$, then the relations (3.7) give

$$\hat{H}^k_{g\beta}S_{M\vec{v}}\hat{H}^l_{g\beta}\hat{N}(\alpha) \subset N(\alpha + M|\vec{v}|)$$

and (3.8) yields for any $\psi \in N(\alpha)$ the estimate

$$\|\hat{H}^k_{g\beta}S_{M\vec{v}}\hat{H}^l_{g\beta}\hat{\psi}\| \leq C^k_{\alpha+M|\vec{v}|}\, C^l_\alpha \|\hat{\psi}\| \leq C^{k+l}_{\alpha+M|\vec{v}|}\|\psi\| \ .$$

Hence we are allowed to apply the Hausdorff-Baker-Campbell formula which gives

$$(e^{ib'\hat{H}_{g\beta}}\, S_{M\vec{v}}\, e^{-ib'H_{g\beta}}\hat{\psi})(\vec{P},\vec{p}) =$$
$$= \left(\sum_{n=0}^{\infty} \frac{(ib')^n}{n!}\, [\hat{H}_{g\beta},[\dots[\hat{H}_{g\beta},S_{M\vec{v}}]\dots]\ \hat{\psi}\right)(\vec{P},\vec{p}) =$$
$$= \sum_{n=0}^{\infty} \frac{(ib')^n}{n!}\,(\vec{P}.\vec{v} - \tfrac{1}{2}M\vec{v}^2)^n\,\hat{\psi}(\vec{P}-M\vec{v},\vec{p}) =$$
$$= e^{ib'(\vec{P}.\vec{v} - (1/2)M\vec{v}^2)}\,\hat{\psi}(\vec{P}-M\vec{v},\vec{p}) \ .$$

This is true for each $\alpha \geq \beta$. For an arbitrary $\psi \in \mathcal{H}$, we can therefore choose a sequence $\{\psi_n\}_3$ such that $\psi_n \in N(n)$ and $\hat{\psi}_n$ converges to $\hat{\psi}$ pointwise a.e. in $\mathbb{R}^3 \times \mathbb{R}^6$, and conclude that

$$e^{ib'\hat{H}_{g\beta}}\, S_{M\vec{v}}\, e^{-ib'\hat{H}_{g\beta}}\,\hat{\psi} = e^{(1/2)M\vec{v}^2b'+ib'\vec{v}.\vec{P}}\, S_{M\vec{v}}\,\hat{\psi} \ . \tag{3.9}$$

It remains to perform the limit $\beta\to\infty$. We have $D(\hat{H}_{g\beta}) = D(\hat{H}_g) = D(\hat{H}_0)$ and

$$\|\hat{H}_{g\beta}\hat{\psi} - \hat{H}_g\hat{\psi}\| \leq |g|\,\|\hat{v}_\beta - \hat{v}\|\,\|\hat{\psi}\| \to 0$$

as $\beta\to\infty$ for every $\hat{\psi} \in D(\hat{H}_0)$, so $\hat{H}_{g\beta} \to \hat{H}_g$ in the strong resolvent sense (Ref.27, theorem VIII.25). It in turn implies

$$\operatorname*{s-lim}_{\beta\to\infty}\ e^{-i\hat{H}_{g\beta}t} = e^{-i\hat{H}_g t} \tag{3.10}$$

for all $t \in \mathbb{R}$ (Ref.27, theorem VIII.21). The operator $S_{M\vec{v}}$ is bounded and operator multiplication is strongly sequentially continuous, hence the relations (3.6) follow from (3.9) and (3.10). ∎

<u>Proof of theorem 3.1</u> : It is easy to check that the operators (3.4) are unitary. We have to show that they form a projective representation of $\mathcal{G}$, i.e.,

$$U(\gamma')U(\gamma) = \omega(\gamma',\gamma)U(\gamma'\gamma) \ , \tag{3.5b}$$

where $\gamma'\gamma$ is defined by (3.1) and ω is a suitable multiplier. By definition,

$$(U(b',\vec{a}',\vec{v}',R')U(b,\vec{a},\vec{v},R)\psi)(\vec{X},\vec{x}) =$$
$$= \exp\left\{\tfrac{i}{2}M\vec{v}'.(\vec{v}'b'+2\vec{X}-\vec{a}')\right\}(e^{iH_gb'}U(b,\vec{a},\vec{v},R)\psi)((R')^{-1}(\vec{X}+\vec{v}'b'-\vec{a}',(R')^{-1}\vec{x}) \ ;$$

at the same time, lemma 3.2 gives

$$(e^{iH_gb'}U(b,\vec{a},\vec{v},R)\ )(\vec{Y},\vec{y}) =$$
$$= e^{(i/2)M[\vec{v}^2(b+b')+\vec{v}.(2\vec{X}-\vec{a})]}\,(e^{iH_g(b+b')}\psi)(R^{-1}(\vec{Y}+\vec{v}(b+b')-\vec{a},R^{-1}\vec{y}) \ .$$

Combining these two relations, we obtain after a straightforward calculation the relation (3.5b) with ω given by (3.5a). One can check easily that ω is really a multiplier, $\omega(\gamma'',\gamma')\omega(\gamma''\gamma',\gamma) = \omega(\gamma',\gamma)\omega(\gamma'',\gamma'\gamma)$. ∎

Remark 3.3 : Let us return for a moment to the relation (2.3b). Suppose that the heavy-particle mass is M' . In this case, the upper and lower components of ψ would transform in a different way : the upper one would acquire the phase factor $\exp \frac{1}{2}M'[\vec{v}^2 b + \vec{v}.(2\vec{X}-\vec{a})]$. Now one has to use the identity $(0,0,-\vec{v},I)(0,-\vec{a},0,I)(0,0,\vec{v},I)(0,\vec{a},0,I) = (0,0,0,I)$; it shows that the unit element of $\mathcal{G}$ would transform the vector ψ to

$$\psi' = \begin{pmatrix} e^{iM'\vec{a}.\vec{v}}\,\psi_u \\ e^{iM\vec{a}.\vec{v}}\,\psi_d \end{pmatrix} ,$$

which represents a different state unless $M' = M$.

What is the physical meaning of the representation U ? Consider an observer who describes a state of the system at an instant t by the vector $\psi_t = e^{-iH_g t}\psi$ referring to an initial condition $\psi_0 = \psi$. Another observer who uses the primed reference frame will describe the same state by $\psi'_{t'}$, which is related to ψ_t by

$$\psi'_{t'} = U(b,\vec{a},\vec{v},R)\psi_t \quad . \tag{3.11a}$$

For simplicity, assume that the clocks of the two observers are synchronized, i.e., $t' = t$. Then $\psi'_{t'} = U(-t,\vec{a}-\vec{v}t,\vec{v},R)\psi$ so the relations (3.3) and (3.4) give

$$\psi'_t(\vec{X}',\vec{x}') = e^{(1/2)M[2R^{-1}\vec{v}.\vec{X} + \vec{v}.(\vec{v}t+a)]}\,\psi_t(\vec{X},\vec{x}) \quad ; \tag{3.11b}$$

in particular, the squared moduli of the primed and non-primed components are the same.

Existence of the representation U alone does not, however, imply Galilean invariance. The latter requires in addition that the equations of motion are form-invariant under changes of the reference frame. Suppose that $\psi_t(.,.)$ is a rapidly decreasing function, i.e., $\psi_u \in \mathcal{S}(\mathbb{R}^3)$ and $\psi_d \in \mathcal{S}(\mathbb{R}^6)$, then (3.11b) together with the equation $i\frac{d}{dt}\psi_t = (H_0+gV)\psi_t$ give

$$i\frac{d}{dt}\psi'_t(\vec{X}',\vec{x}') = e^{(1/2)M[-\vec{v}^2 t + \vec{v}.(2\vec{X}'-\vec{a})]}\Big\{\big(\tfrac{1}{2}M\vec{v}^2\psi_t - iR^{-1}\vec{v}.\vec{\nabla}_X\psi_t + H_0\psi_t\big)(R^{-1}(\vec{X}'-\vec{v}t-\vec{a}),R^{-1}\vec{x}') + g\begin{pmatrix} \int_{\mathbb{R}^3} \bar{v}(\vec{y})\psi_{t,d}(R^{-1}(\vec{X}'-\vec{v}t-\vec{a}),y)\,d\vec{y} \\ v(R^{-1}\vec{x}')\psi_{t,u}(R^{-1}(\vec{X}'-\vec{v}t-\vec{a})) \end{pmatrix}\Big\}.$$

The first three terms are nothing else than $(H_0\psi'_t)(\vec{X}',\vec{x}')$. In the last one, we use rotational invariance of the Lebesgue measure obtaining in this way

$$i\frac{d}{dt}\psi'_t(\vec{X}',\vec{x}') = (H_0\psi'_t)(\vec{X}',\vec{x}') + g\begin{pmatrix} \int_{\mathbb{R}^3} \bar{v}(R^{-1}\vec{y}')\psi_{t,d}(\vec{X}',\vec{y}')\,d\vec{y}' \\ v(R^{-1}\vec{y}')\psi_{t,u}(\vec{X}') \end{pmatrix} .$$

It is clear now that the vector function $t \mapsto \psi'_t$ fulfils Schrödinger equation of the same form as $t \mapsto \psi_t$ does if $v(\vec{x}) = v(R\vec{x})$. Galilean invariance requires therefore the function v to be spherically symmetric, $v(\vec{x}) = v(R\vec{x})$ for all $R \in O(3)$.

In what follows, we treat the Galilean-invariant case only ; we assume $v(\vec{x}) = v_1(r)$, where $r = |\vec{x}|$ and v_1 is a real-valued function from $L^2(\mathbb{R}^+, r^2 dr)$.

4. Separation of the centre-of-mass motion

The state Hilbert space (2.1) decomposes naturally into the tensor product of spaces referring to relative and centre-of-mass motion, $\mathcal{H} = \mathcal{H}^{cm} \otimes \mathcal{H}^{rel}$. With the usual license, we write

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes (\mathbb{C} \oplus L^2(\mathbb{R}^3)) \quad , \tag{4.1}$$

where the bilinear mapping $\otimes : L^2(\mathbb{R}^3) \times (\mathbb{C} \oplus L^2(\mathbb{R}^3)) \to \mathcal{H}$ is defined by

$$(\psi \otimes \binom{\alpha}{\varphi})(\vec{X},\vec{x}) := \begin{pmatrix} \alpha\psi(\vec{X}) \\ \psi(\vec{X})\varphi(\vec{x}) \end{pmatrix} \quad ;$$

one can check easily that it has the required properties /28/. The Hamiltonian (2.5) can be then expressed as

$$H_g = H_0^{cm} \otimes I + I \otimes H_g^{rel} \quad , \tag{4.2}$$

where $H_0^{cm} = -\frac{1}{2M}\Delta_X$ and $H_g^{rel} = H_0^{rel} + gV$ with

$$H_0^{rel} = \begin{pmatrix} E & 0 \\ 0 & -\frac{1}{2m}\Delta_x \end{pmatrix} , \tag{4.3a}$$

$$V = \begin{pmatrix} 0 & (v,.) \\ v & 0 \end{pmatrix} \quad ; \tag{4.3b}$$

we omit here the superscript "rel" for convenience. The operators H_0^{cm} and H_g^{rel} are self-adjoint and the relation (4.2) implies /28/

$$e^{-iH_g t} = e^{-iH_0^{cm} t} \otimes e^{-iH_g^{rel} t} \tag{4.4}$$

for all $t \in \mathbb{R}$. Hence the total propagator decomposes naturally and its centre-of-mass part represents a free motion. For our purposes, only the relative part is important. We are interested in the situation, when the initial state of the system represents the undecayed heavy particle,

$$\Psi = \begin{pmatrix}\psi_u \\ 0\end{pmatrix} = \psi_u \otimes \begin{pmatrix}1 \\ 0\end{pmatrix} \tag{4.5}$$

at $t = 0$. Then the state vector factorizes at each $t > 0$,

$$\Psi_t \equiv e^{-iH_g t}\Psi = \psi^{cm}_{t,u} \otimes \begin{pmatrix}\psi^{rel}_{t,u} \\ \psi^{rel}_{t,d}\end{pmatrix} \quad , \tag{4.6a}$$

where

$$\begin{pmatrix}\psi^{rel}_{t,u} \\ \psi^{rel}_{t,d}\end{pmatrix} := e^{-iH_g^{rel} t}\begin{pmatrix}1 \\ 0\end{pmatrix} \tag{4.6b}$$

and

$$\psi^{cm}_{t,u}(\vec X) = (2\pi i t)^{-3/2} \underset{\alpha\to\infty}{\mathrm{l.i.m.}} \int_{|\vec Y| \le \alpha} e^{i|\vec X - \vec Y|^2/2t}\psi_u(\vec Y)\, d\vec Y \tag{4.6c}$$

(cf.Ref.27, Sec.IX.7). The same can be put in other words. The decay is described fully by the reduced propagator

$$U_t := pr_{\mathcal{H}_u} e^{iH_g t} \equiv E_u e^{-iH_g t} \restriction \mathcal{H}_u \ , \tag{4.7a}$$

where E_u denotes the projection onto $\mathcal{H}_u$ (Ref.3, Chap.1). However, the latter equals $I \otimes E_u^{rel}$, where E_u^{rel} projects onto the one-dimensional subspace $\mathcal{H}_u^{rel} \sim \mathbb{C}$ in $\mathcal{H}^{rel}$, and therefore the relation (4.4) implies

$$U_t = e^{-iH_0^{cm} t} \otimes pr_{\mathcal{H}_u^{rel}} e^{-iH_g^{rel} t} \ . \tag{4.7b}$$

Before proceeding further, let us mention how this decomposition looks like in the p-representation. The operator (2.10) expresses as

$$\mathbf{F} = F_3 \otimes \begin{pmatrix}1 & 0 \\ 0 & F_3\end{pmatrix} \tag{4.8}$$

and transforms H into $\hat H = \overline{\hat H_0^{cm} \otimes I + I \otimes \hat H_g^{rel}}$, where $\hat H_0^{cm}$ is a multiplication operator, $(\hat H_0^{cm}\hat\Psi)(\vec P) = (\vec P^2/2M)\hat\Psi(\vec P)$ and

$$\hat H_g^{rel} = \begin{pmatrix}0 & g(\hat v, .) \\ g\hat v & \frac{\vec p^2}{2m}\end{pmatrix} \ . \tag{4.9}$$

5. The reduced resolvent

In what follows, we shall be concerned mostly with the relative motion, and therefore we omit the superscripts "rel". Let us first recall that the reduced propagator (4.7) is determined by the reduced resolvent

$$R_u(z, H_g) := pr_{\mathcal{H}_u} (H_g - z)^{-1} \tag{5.1}$$

as

$$U_t\Psi = \int_{\mathbb{R}} e^{-i\lambda t}\, dF_\lambda \Psi \quad , \tag{5.2a}$$

where the vector-valued measure is given by

$$\tfrac{1}{2}(F([\lambda,\mu]) + F((\lambda,\mu)))\Psi = \frac{1}{2\pi i} \lim_{\eta \to 0+} \int_\lambda^\mu \left[R_u(\xi + i\eta, H_g) - R_u(\xi - i\eta, H_g)\right]\Psi\, d\xi \tag{5.2b}$$

(Ref.3, Sec.3.1) ; if $R_u(.,H_g)$ has a pole near the real axis, then the reduced propagator is dominated by the corresponding exponential term. In our case, the (relative) subspace $\mathcal{H}_u$ is one-dimensional, so $R_u(z,H_g)$ and U_t act simply as multiplication by numbers $r_u(z,H_g)$ and $u(t)$, respectively.

Since $E > 0$ by assumption, the unperturbed Hamiltonian has a simple eigenvalue E embedded in the non-simple continuous spectrum $\sigma_c(H_0) = \mathbb{R}^+$. The perturbation problem for this eigenvalue can be solved, because the interaction Hamiltonian fulfils the Friedrichs condition

$$E_d V E_d = 0 \quad . \tag{5.3}$$

We shall work in the p-representation, where H_0 and $(H_0 - z)^{-1}$ act as multiplication operators. A simple algebraic argument (Ref.3, Proposition 3.2.1) gives

Proposition 5.1 : The reduced resolvent (5.1) acts as multiplication by

$$r_u(z, H_g) = \left[-z + E + g^2 G(z)\right]^{-1} \quad , \tag{5.4}$$

where

$$G(z) := \int_{\mathbb{R}^3} \frac{|\hat v(\vec p)|^2}{z - \frac{p^2}{2m}}\, dp \tag{5.5a}$$

for $z \in \rho(H_g)$, in particular, for each non-real $z \in \mathbb{C}$.

Spectrum of the operator H_g can be found easily. In particular, $\sigma_{ess}(H_g)=\sigma_{ess}(H_0)=\mathbb{R}^+$, because the operator V is of rank two, and therefore relatively compact with respect to H_0 (Ref.27, Sec.XIII.4). The crucial observation is that $r_u(\cdot,H_g)$ may be continued analytically across $\mathbb{R}^+$, even if the full resolvent is not having a cut there. We shall prove that the perturbation shifts the pole corresponding to the unperturbed eigenvalue from the real axis to the second sheet of the analytically continued $r_u(.,H_g)$. Let us first collect the hypotheses concerning the function v :

<u>Assumptions 5.2</u> : (a) v is spherically symmetric. In that case, the same is true for $\hat{v}$, and we shall write $\hat{v}(\vec{p})=\hat{v}_1(p)$, having in mind that $\hat{v}_1$ is <u>not</u> Fourier image of v_1 ,

$$\hat{v}_1(p)=\left(\frac{2}{\pi}\right)^{1/2}\ \underset{\alpha\to\infty}{\text{l.i.m.}}\int_0^{\alpha}\frac{r\sin pr}{p}\,\hat{v}_1(r)\,dr\ . \tag{5.6}$$

The relation (5.5a) can be now rewritten as

$$G(z)=4\pi\int_0^{\infty}\frac{|\hat{v}_1(p)|^2p^2}{z-\frac{p^2}{2m}}\,dp\ . \tag{5.5b}$$

(b) the function $\lambda\mapsto|\hat{v}_1(\sqrt{2m\lambda})|^2\sqrt{2m\lambda}$ can be continued analytically to an open set $\Omega\subset\mathbb{C}$ containing the point E and such that $\Omega\cap\mathbb{R}\subset\mathbb{R}^+$, i.e., there is a holomorphic function $f:\Omega\to\mathbb{C}$ such that $f(\lambda)=|\hat{v}_1(\sqrt{2m\lambda})|^2\sqrt{2m\lambda}$ for $\lambda\in\Omega\cap\mathbb{R}$. For notational convenience, we write $f(z)=|\hat{v}_1(\sqrt{2mz})|^2\sqrt{2mz}$ for non-real z too.

(c) $\hat{v}_1(\sqrt{2mE})\neq 0$,

(d) the last assumption can be replaced by a stronger requirement, $\hat{v}_1(p)\neq 0$ for all $p>0$.

Now we shall prove two auxiliary assertions :

<u>Lemma 5.3</u> : Let the function $\lambda\mapsto|\hat{v}_1(\sqrt{2m\lambda})|^2\sqrt{2m\lambda}$ have a bounded derivative in an open set $J\subset\mathbb{R}^+$. Define

$$I(\lambda,v):=\mathcal{P}\int_0^{\infty}\frac{|\hat{v}_1(p)|^2p^2}{\lambda-\frac{p^2}{2m}}\,dp\ , \tag{5.7}$$

where $\mathcal{P}$ denotes principal value, then the function $I(.,v)$ is continuous in J .

<u>Proof</u> : Choose an arbitrary $\lambda_0\in J$. Due to the assumption, there are positive C,δ such that

$$\left||\hat{v}_1(p)|^2p-|\hat{v}_1(\sqrt{2m\lambda})|^2\sqrt{2m\lambda}\right|\le C\,|p-\sqrt{2m\lambda}\,| \tag{5.8}$$

holds for all $p,\sqrt{2m\lambda}\in(\sqrt{2m\lambda_0}-\delta,\sqrt{2m\lambda_0}+\delta)$. The integral (5.7) can be then written as

$$I(\lambda,v)=\sum_{k=1}^{3}I_k(\lambda,v)=\left(\int_0^{\sqrt{2m(\lambda-\rho)}}+\mathcal{P}\int_{\sqrt{2m(\lambda-\rho)}}^{\sqrt{2m(\lambda+\rho)}}+\int_{\sqrt{2m(\lambda+\rho)}}^{\infty}\right)\frac{|\hat{v}_1(p)|^2p^2}{\lambda-\frac{p^2}{2m}}\,dp$$

for some $\rho\in(0,\lambda)$. We can choose $\rho\in(0,\lambda_0)$ and $\delta_1\in(0,\frac{1}{2}\delta]$ in such a way that $|\sqrt{2m(\lambda\pm\rho)}-\sqrt{2m\lambda}|<\frac{1}{2}\delta$ holds for

$$\sqrt{2m\lambda}\in(\sqrt{2m\lambda_0}-\delta_1,\sqrt{2m\lambda_0}+\delta_1) \tag{5.9a}$$

so that

$$\sqrt{2m(\lambda\pm\rho)}-\sqrt{2m\lambda_0}\ . \tag{5.9b}$$

In what follows, we shall consider only those λ which fulfil the condition (5.9a). The integrals $I_k(\lambda,v)$, $k=1,3$, are finite and $I_k(.,v)$ are continuous at λ_0 due to the dominated-convergence theorem. It is sufficient therefore to consider the second integral. A simple integration yields

$$\mathcal{P}\int_{\sqrt{2m(\lambda-\rho)}}^{\sqrt{2m(\lambda+\rho)}}\frac{p\,dp}{\lambda-\frac{p^2}{2m}}=\lim_{\eta\to 0+}\left(\int_{\sqrt{2m(\lambda-\rho)}}^{\sqrt{2m\lambda}-\eta}+\int_{\sqrt{2m\lambda}+\eta}^{\sqrt{2m(\lambda+\rho)}}\right)\frac{p\,dp}{\lambda-\frac{p^2}{2m}}=$$

$$=\lim_{\eta\to 0+}\frac{\eta\sqrt{\frac{2\lambda}{m}}+\frac{\eta^2}{2m}}{\eta\sqrt{\frac{2\lambda}{m}}-\frac{\eta^2}{2m}}=0$$

so we have

$$I_2(\lambda,v)=\mathcal{P}\int_{\sqrt{2m(\lambda-\rho)}}^{\sqrt{2m(\lambda+\rho)}}\frac{|\hat{v}_1(p)|^2p-|\hat{v}_1(\sqrt{2m\lambda})|^2\sqrt{2m\lambda}}{\lambda-\frac{p^2}{2m}}\,p\,dp\ . \tag{5.10}$$

According to the conditions (5.8),(5.9), the following inequality holds

$$\left|\frac{|\hat{v}_1(p)|^2p-|\hat{v}_1(\sqrt{2m\lambda})|^2\sqrt{2m\lambda}}{\lambda-\frac{p^2}{2m}}\,p\right|\le\frac{2mCp}{p+\sqrt{2m\lambda}}\le 2mC\ .$$

thus the rhs of (5.10) makes sense as a Lebesgue integral and $I_2(.,v)$ is continuous at λ_0 by the dominated convergence theorem. ∎

<u>Lemma 5.4</u> : Adopt the assumptions (a) and (b). The function G_Ω defined by

$$G_\Omega(z) = \begin{cases} G(z) & \ldots \quad \mathrm{Im}\, z > 0 \\ 4\pi I(z,v) - 4\pi^2 im|\hat{v}_1(\sqrt{2mz})|^2\sqrt{2mz} & \ldots \quad z \in \mathbb{R}\cap\Omega \\ G(z) - 8\pi^2 im|\hat{v}_1(\sqrt{2mz})|^2\sqrt{2mz} & \ldots \quad z\in\Omega\ ,\ \mathrm{Im}\, z<0 \end{cases} \qquad (5.11)$$

is holomorphic in $\{z : \mathrm{Im}\, z > 0\} \cup \Omega$.

<u>Proof</u> : One has to check that within Ω , the relation

$$\lim_{\eta\to 0+} G(\lambda\pm i\eta) = 4\pi I(\lambda,v) \mp 4\pi^2 im|\hat{v}_1(\sqrt{2m\lambda})|^2\sqrt{2m\lambda} \qquad (5.12)$$

holds. We notice first that $I(\lambda,v)$ makes sense due to (b), because the assumption of the preceding lemma is fulfilled in that case. Hence one can choose a sufficiently small ρ and express $G(z)$ as a sum of three integrals in analogy with the above proof ; the dominated-convergence theorem implies $G_k(\lambda\pm i\eta) \to 4\pi I_k(\lambda,v)$ for $k=1,3$. Further we have

$$\lim_{\eta\to 0+} G_2(\lambda\pm i\eta) = 4\pi \lim_{\eta\to 0+} \int_{\sqrt{2m(\lambda-\rho)}}^{\sqrt{2m(\lambda+\rho)}} \frac{|\hat{v}_1(p)|^2 p - |\hat{v}_1(\sqrt{2mz_\pm})|^2\sqrt{2mz_\pm}}{\lambda\pm i\eta - \frac{p^2}{2m}}\, p\, dp\, +$$

$$+\ 4\pi \lim_{\eta\to 0+} |\hat{v}_1(\sqrt{2mz_\pm})|^2\sqrt{2mz_\pm} \int_{\sqrt{2m(\lambda-\rho)}}^{\sqrt{2m(\lambda+\rho)}} \frac{p\, dp}{\lambda\pm i\eta - \frac{p^2}{2m}}\ , \qquad (5.13)$$

where $z_\pm = \lambda\pm i\eta$. The first limit equals $4\pi I_2(\lambda,v)$ according to (5.10) and the dominated-convergence theorem. One obtains easily

$$\lim_{\eta\to 0+} \int_{\sqrt{2m(\lambda-\rho)}}^{\sqrt{2m(\lambda+\rho)}} \frac{p\, dp}{\lambda\pm i\eta - \frac{p^2}{2m}} = m \lim_{\eta\to 0+} \left[\ln(\rho\pm i\eta) - \ln(-\rho\pm i\eta)\right] = \mp i\pi m\ ,$$

while $|\hat{v}_1(\sqrt{2mz_\pm})|^2\sqrt{2mz_\pm} \to |\hat{v}_1(\sqrt{2m\lambda})|^2\sqrt{2m\lambda}$, so we arrive at the relation (5.12). Next we must show that the convergence in (5.12) is uniform with respect to λ in any finite interval $J \subset \Omega\cap\mathbb{R}$. We have

$$|G_1(z_\pm) - G_1(\lambda)| \le 4\pi \int_0^{\sqrt{2m(\lambda-\rho)}} |\hat{v}_1(p)|^2 \left| \frac{1}{z_\pm - \frac{p^2}{2m}} - \frac{1}{\lambda - \frac{p^2}{2m}} \right| p^2 dp \le$$

$$\le 4\pi|\eta| \int_0^{\sqrt{2m(\lambda-\rho)}} \frac{|\hat{v}_1(p)|^2 p^2}{\left|z_\pm - \frac{p^2}{2m}\right| \left|\lambda - \frac{p^2}{2m}\right|}\, dp$$

and the last integral can be estimated easily. The third integral can be treated in the same way ; we obtain

$$|G_k(z_\pm) - G_k(\lambda)| \le \rho^{-2}\, \|v\|^2 |\eta| \ . \qquad (5.14a)$$

for $k=1,3$. Consider now $G_2(z_\pm)$. In the second term of (5.13), $|\hat{v}_1(\sqrt{2mz_\pm})|^2\sqrt{2mz_\pm} \to |\hat{v}_1(\sqrt{2m\lambda})|^2\sqrt{2m\lambda}$ uniformly in view of (5.8), and we have the estimate

$$\left| \int_{\sqrt{2m(\lambda-\rho)}}^{\sqrt{2m(\lambda+\rho)}} \frac{p\, dp}{z_\pm - \frac{p^2}{2m}} \pm i\pi m \right| \le m \ln \frac{\rho\pm i\eta}{\rho\mp i\eta} \le \frac{m}{\rho}|\eta| \ . \qquad (5.14b)$$

It remains to cope with the first term in (5.13). It is easy to see that

$$F(p,\sqrt{2mz}) := \frac{|\hat{v}_1(p)|^2 p - |\hat{v}_1(\sqrt{2mz})|^2\sqrt{2mz}}{z - \frac{p^2}{2m}}$$

is holomorphic with respect to $p,\sqrt{2mz}$ if only they belong to Ω . Then $h^{(1)} \equiv \partial F(p,u)/\partial u$ is also holomorphic, and therefore bounded if p,u belong to a bounded subset of Ω . We have

$$F(p,\sqrt{2mz_\pm}) - F(p,\sqrt{2m\lambda}) = (\sqrt{2mz_\pm} - \sqrt{2m\lambda}) \int_0^1 h^{(1)}(p,\sqrt{2m\lambda} + t(\sqrt{2mz_\pm} - \sqrt{2m\lambda}))dt$$

so there is a positive K such that $|F(p,\sqrt{2mz_\pm}) - F(p,\sqrt{2m\lambda})| \le$ $\le K|\sqrt{2mz_\pm} - \sqrt{2m\lambda}|$ holds if $p\in(\sqrt{2m(\lambda-\rho)},\sqrt{2m(\lambda+\rho)})$, $\lambda\in J$ and ρ,η are small enough. Hence we have

$$\left| \int_{\sqrt{2m(\lambda-\rho)}}^{\sqrt{2m(\lambda+\rho)}} \left[F(p,\sqrt{2mz_\pm}) - F(p,\sqrt{2m\lambda})\right] p\, dp \right| \le 2m\rho K|\sqrt{2mz_\pm} - \sqrt{2m\lambda}| \le$$

$$\le \frac{(2m)^{3/2}\rho K}{\sqrt{\lambda_0}} \sqrt{\eta}\ ,$$

since $\lambda \ge \lambda_0 > 0$ for $\lambda\in J$. Combining this estimate with (5.14) we see that the convergence in (5.12) is uniform. Since $G(.)$ is easily seen to be holomorphic in the upper and lower complex halfplanes, the assertion follows from the edge-of-wedge theorem /28/. ∎

<u>Remark 5.4</u> : One has to check the uniform convergence, because the remark following theorem 2-13 of Ref.28 is not correct : a counter-example is represented by $F(z) = z\, e^{1/z^2}$.

Now we are in position to prove the main result of this section :

<u>Theorem 5.5</u> : Assume (a)-(c). Then there is a connected complex neighbourhood $\Omega_1 \subset \Omega$ of the point E and a positive ε such that for each $g\in(-\varepsilon,\varepsilon)$,

$$r_u^\Omega(z,H_g) := \left[-z + E + g^2 G_\Omega(z)\right]^{-1} \qquad (5.15)$$

represents analytic continuation of (5.4) to $\{z : \mathrm{Im}\, z > 0\} \cup \Omega$. The function $r_u(.,H_g)$ has just one singularity in Ω_1 , a simple pole at $z = z_p(g)$, where the function $z_p = \lambda_p - i\delta_p$ belongs to $C^{\infty}[-\varepsilon,\varepsilon]$ and its real and imaginary parts fulfil

$$\lambda_p(g) = E + 4\pi g^2 \mathcal{P}\int_0^{\infty} \frac{|\hat{v}_1(p)|^2 p^2}{E - \frac{p^2}{2m}}\, dp + O(g^4) \quad , \tag{5.16a}$$

$$\delta_p(g) = 4\pi^2 m g^2 |\, \hat{v}_1(\sqrt{2mE})|^2 \sqrt{2mE} + O(g^4) . \tag{5.16b}$$

Proof : The assertion concerning analytic continuation follows from Lemma 5.3 . Only possible singularities of (5.15) are zeros of the function $f(g,z) := z - E - g^2 G(z)$ defined for $g \in \mathbb{R}$ and z from the analycity domain of G . For small enough g , one can use the implicit-function theorem (cf.Ref.30, thms.III.28, III.31). The function f is infinitely differentiable with respect to both g and z , further we have $f(0,E) = 0$ and $(\partial f/\partial z)(0,E) = 1 \neq 0$. Then there is a neighbourhood $(-\varepsilon', \varepsilon')$ of the point $g = 0$ and a unique function $z_p \in C^{\infty}[-\varepsilon', \varepsilon']$ such that $f(g, z_p(g)) = 0$ for $|g| < \varepsilon'$, i.e., $z_p(g) = E + g^2 G_{\Omega}(z_p(g))$. Continuity of the partial derivatives of f implies particularly that $(\partial f/\partial z)(\cdot, z_p(\cdot))$ is continuous in $(-\varepsilon', \varepsilon')$, and therefore there is a positive $\varepsilon \leq \varepsilon'$ such that $(\partial f/\partial z)(g, z_p(g)) \neq 0$ for $g \in (-\varepsilon, \varepsilon)$. Consequently, $r_u(\cdot, H_g)$ has a simple pole at $z_p(g)$. The first few terms of the Taylor expansion of z_p can be easily calculated : we obtain

$$\left.\frac{dz_p}{dg}\right|_{g=0} = \left.\frac{d^3 z_p}{dg^3}\right|_{g=0} = 0$$

and .

$$\left.\frac{d^2 z_p}{dg^2}\right|_{g=0} = 2G_{\Omega}(E)$$

which imply (5.16). ∎

Remark 5.6 : In fact, we have proved the theorem using the assumptions (a) and (b) only. The assumption (c) is important, however, since it determines the leading order in the formula (5.16b) which yields the decay width. We shall return to this problem in a sequel to this paper.

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Диттрих Я., Экснер П. E2-86-209
Нерелятивистская модель двухчастичного распада. Формулировка задачи

Настоящая работа посвящена подробному рассмотрению нерелятивистской модели бесспиновой частицы, распадающейся на две более легкие частицы, которая похожа на описание распада V-частицы в модели Ли. Галилеевская ковариантность сформулирована надлежащим способом при помощи унитарного проективного представления, действующего на пространстве состояний модели. Отделив движение центра тяжести, мы выводим мероморфную структуру приведенной резольвенты. Дальнейшие свойства решения будут обсуждены во второй части работы.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1986

Dittrich J., Exner P. E2-86-209
A Non-Relativistic Model of Two-Particle Decay. Formulation of the Problem

In the present paper, we treat in detail a simple non-relativistic model of a spinless particle decaying into two lighter particles, which is similar to the Lee-model description of V-particle decay. Galilean covariance is formulated properly, by means of a unitary projective representation acting on the state space of the model. After separating the centre-of-mass motion, we deduce the meromorphic structure of the reduced resolvent. Further properties of the solution will be discussed in the second part of the paper.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1986