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# ON THE CONSTRUCTION OF OPERATOR EXPANSIONS AND EFFECTIVE THEORIES IN THE MS-SCHEME.

## Examples. Infrared Finiteness of Coefficient Functions

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This paper is the direct continuation of [1]. Recall that in [1] we have constructed asymptotic expansions of individual Feynman integrals and generating functional of Green functions in large Euclidean momenta or/and masses. For this purpose we have used the ultraviolet (UV) R-operation with oversubtractions in a special type of subgraphs, H-subgraphs. These are subgraphs which depend on large parameters and cannot be made disconnected by deleting a line with a small momentum and mass (see [1]). The resulting expansion has the form of operator expansion, UV-divergences being removed according to the MS-scheme [2].

In this publication we use the formulae of [1] to obtain asymptotic expansions of some full Green functions. Then we show that, with a special choice of subtraction operators, coefficient function (CF's) of such expansions can be made free from infrared (IR) logarithms, i.e.,they can be expanded in powers of small momenta and masses without arising IR-singularities, which makes IR cutoffs an unwarranted precaution. In Appendix we have explained the origin of "contact terms".

For all definitions concerning the R-operation, graph notation, etc.,we refer the reader to [1]. Formulae presented in [1] and used in this paper will be supplemented by index I, for example, (I.10), (I.20) and so on. For the explanation of quantities entering into them, see also [1].

1. The product of two currents at short distances (the Wilson expansion [3]).

Consider the generating functional of Green functions of the product $T(A(x)B(o))$ in the momentum space

$$R_{MS}\, T_{AB}(Q,\mathcal{J}) = \int dx\, e^{-iQx} R_{MS} \langle T\, A(x) B(o) e^{L+\mathcal{J}\varphi} \rangle_o \ , \qquad (1)$$

where $A$ and $B$ are local composite operators. Its expansion as $Q \to \infty$ ( $Q$ is assumed to be Euclidean) can be obtained from (I.34) by taking variational derivatives with respect to sources $\mathcal{J}_A$ and $\mathcal{J}_B$ of operators $A$ and $B$ and putting other $\mathcal{J}_n$ zero:

$$
\begin{aligned}
R_{MS} T_{AB}(Q,J) &= \sum_n C_B^n(Q) \langle T O_n(0) e^{L_R + J\varphi} \rangle_0 + o(Q^{1-N}) \\
&= \sum_n C_R^n(Q)\, R_{MS} \langle T O_n(0) e^{L+J\varphi} \rangle_0 + o(Q^{1-N}) \\
&\equiv \sum_n C_R^n(Q)\, R_{MS}\, T_{O_n}(J) + o(Q^{1-N}),
\end{aligned} \tag{2}
$$

where

$$C_R^n(Q) = \frac{\delta^2}{\delta J_A(Q)\, \delta J_B(0)} \cdot C_R^n(\{J_n\})\Big|_{J_n=0} \quad \text{(see eq. (1.30))};$$

$$C_B^n(Q) = \sum_K Z_{nK} \cdot C_R^K(Q),$$

and $Z_{nK}$ are renormalization constants of operators $\{O_n\}$. Note that derivatives $\delta/\delta J_A|_{J_n=0}$ and $\delta/\delta J_B|_{J_n=0}$ are equal to zero because they correspond to operators with a large incoming momentum forbidden by the completeness of H-partitions.

Note that there are no "contact" terms proportional to $\delta(Q)$ in the r.h.s. of (2) because we assume that all sources are localized at infinite $Q$. Such terms are necessary, however, to make the expansion integrable at $Q = 0$ with nonvanishing at this point probe functions/4/. Such a possibility is briefly discussed in Appendix. This remark concerns also all other expansions in large Euclidean momenta (see examples 2 and 3).

Let us make some comments on the derivation. As is clear from (1.22), to obtain an asymptotic series, one should make subtractions in all H-subdiagrams comprising complete H-partitions. As a result, we get the sum of terms with all possible reductions of complete H-partitions. In the case considered we have the only type of the complete H-partition, namely, that containing one H-subgraph and single vertices. This H-subgraph always coincides with some possible flow of $Q$ through the lines of a graph, and both operator vertices $A$ and $B$ belong to it (see Fig. 1b)). Contracting these vertices into a point, we obtain a local operator. This process is illustrated by Fig. 1a).

Functions $C_R(Q)$ contain all the information on the behaviour of (1) as $Q \to \infty$. They are polynomial in all small external momenta. Moreover, in the fifth section we shall show that they can be made polynomial in masses (see also /4,5/).



2. The product of three currents /4,6,7/.

Consider the expression

$$R_{MS} T_{ABC}(Q,q,J) = R_{MS} \int dx\, dy\, e^{-iQx - iqy} \langle T A(x) B(0) C(y) e^{L+J\varphi} \rangle_0, \tag{3}$$

where $A, B$ and $C$ are composite operators; $|Q| \to \infty$, $|q| \ll |Q|$, and $Q$ is Euclidean. The expansion of (3) at large $Q$ is used in various methods of dispersion sum rules/6,7/. As before, to obtain it, one should take derivatives of both sides of eq.(1.34) with respect to sources $J_A$, $J_B$ and $J_C$:

$$
\begin{aligned}
R_{MS} T_{ABC}(Q,q,J) = \sum_n C_R^n(Q)\, R_{MS} T_{O_n C}(q,J) + \sum_n C_R^n(Q,q)\, R_{MS} T_{O_n}(J) \\
+ o(Q^{1-N}),
\end{aligned} \tag{4}
$$

where

$$T_{O_n C}(q,J) = \int dy\, e^{-iqy} \langle T C(y) O_n(0) e^{L+J\varphi} \rangle_0,$$

$$C_R^n(Q,q) = \frac{\delta^3}{\delta J_A(Q)\, \delta J_C(q)\, \delta J_B(0)} C_R^n(\{J_n\}) \Big|_{J_n=0}$$

and $C_R^n(Q)$ are the same as in eq.(2). In this case the relation of "bare" CF's to renormalized ones is nonmultiplicative

$$C_B^n(Q,q) = \sum_m \left( C_R^m(Q,q) Z_{mn} - C_R^m(Q)\, Z_{Cm}^n(q) \right), \tag{5}$$

where $Z_{Cm}^n$ are additive renormalization constants of $T_{O_n C}$ (see eq. (A2.14) in /1/).

The presence of two sums in the r.h.s. of eq.(4) may be explained as follows. The complete H-partitions of diagrams contributing to $T_{ABC}$ may be of two different types. As in the previous case, they always contain only one H-subgraph but the latter may or may not include the vertex $C$. These two possibilities are shown in Figs. 1b) and 2b). They exactly correspond to the two sums on the r.h.s. of (4). The graphical representation of this expansion is given in Fig. 2a). Note that in full analogy with the previous case all derivatives like $\delta/\delta J_A|_{J_n=0}$, $\delta^2/\delta J_A \delta J_C|_{J_n=0}$, etc., do not contribute to the complete H-partitions.

All CF's are polynomial in small momenta and can be made polynomial in masses (see the fifth section and /4,7/).

In an analogous way from eq.(1.34) one can obtain the expansions of products of the form $T(\prod_i C_i(y_i) A(x) B(0))$ at $x \sim 0$ (the Fourier-conjugated Euclidean momentum $Q \to \infty$). The complete H-partition of diagrams contributing to this quantity contains only one H-subgraph. All such subgraphs correspond to all possible flows of the momentum $Q$ through the lines of a diagram and their reduction results in local (as in (2)), bilocal (as in (4)) and multilocal operators. In general, contact terms with $\delta(Q)$ are possible (see the remark at the end of the previous example).

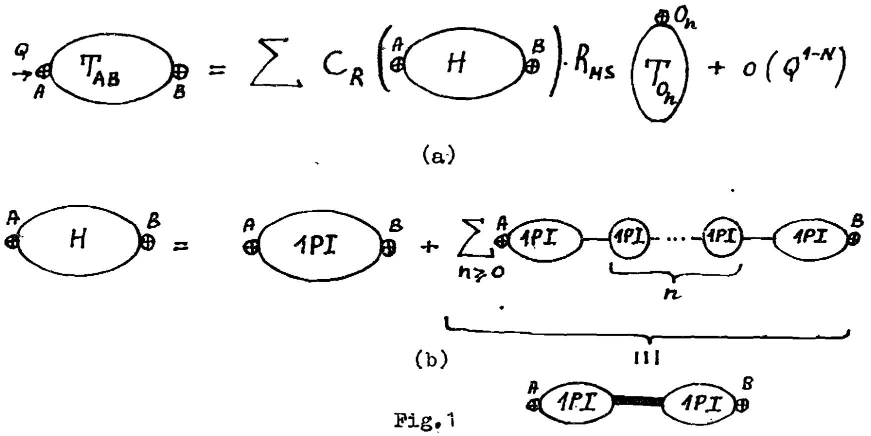

Fig.1

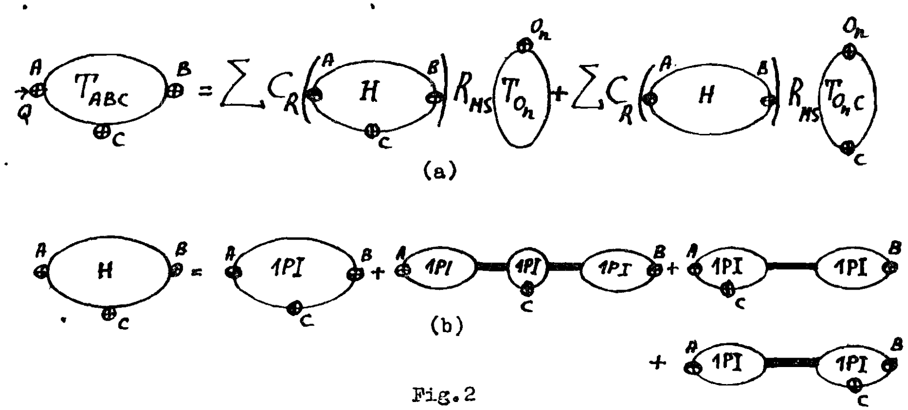

Fig.2

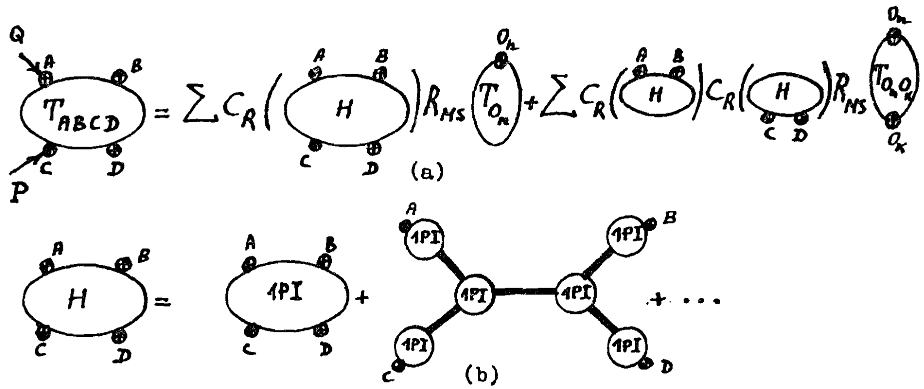

Fig.3



3. Consider the product of four composite operators /4,8/. In the momentum space its generating functional is

$$R_{MS} T_{ABCD}(Q,P,s,J) = R_{MS}\int dx\,dy\,dz\, e^{-iQx-iPy+i(P-s)z}\langle T A(x)B(0)C(y)D(z)e^{L+J\varphi}\rangle_0 \quad (6)$$

Let $P\sim Q\sim\lambda\to\infty$ be large Euclidean. This object is used for the description of two-virtual-photon deep inelastic scattering /8/. Let us obtain its expansion as $\lambda\to\infty$, $s \ll Q$.

Taking derivatives of both sides of (1.34) with respect to sources $J_A, J_B, J_C$ and $J_D$, we get:

$$R_{MS} T_{ABCD}(Q,P,s,J) = \sum_n C_R^n(Q,P,s)\, R_{MS} T_{O_n}(J) + \sum_{n,k} (C^n_{AB})_R(Q)\cdot (C^k_{CD})_R(P)\cdot R_{MS} T_{O_n O_k}(s,J) + o(\lambda^{1-N}), \quad (7)$$

where $(C^n_{AB})_R$ and $(C^h_{CD})_R$ coincide with CF's of expansions of T-products $T(AB)$ and $T(CD)$ at $Q,P\to\infty$, whereas

$$C_R^n(Q,P,s) = \frac{\delta^4}{\delta J_A(Q)\,\delta J_B(0)\,\delta J_C(P)\,\delta J_D(s-P)}\, C^h_R(\{J_n\})\big|_{J_n=0}$$

The connection between $C_R^n$ and "bare" ones is anologous to (5).

The origin of (7) can be understood as follows. There are two types of complete H-partitions of the diagrams contributing to $T_{ABCD}$, corresponding to two different flows of momentum $P$ and $Q$. The first one contains two disjoint H-subgraphs, the pairs of vertices $(A,B)$ and $(C,D)$ belonging to different graphs (Fig. 1b)). The second type includes only one H-subgraph absorbing all four operator insertions (Fig. 3b)). All other H-partitions are incomplete and correspond to derivatives $\delta/\delta J_A$, $\delta^2/\delta J_A\delta J_C$, $\delta^3/\delta J_A\delta J_B\delta J_C$, etc. The two sums in the r.h.s. of (7) are formed by terms with contractions of H-partitions of these two types. The structure of (7) is represented graphically in Fig. 3a).

The functions $C_R^n$ are power series in $s$ and can be made analytically dependent on masses (see /4/ and the fifth section).

4. Effective light theories

Consider the case when all the external momenta of Green functions are fixed, whereas the masses of some particles described by fields $\phi$ tend to infinity ($M\sim\lambda\to\infty$). In this particular case expansion (1.34) takes the form of effective light theories /9,4/. If the effect of large masses is fully factorized into CF's and there are only light particles in ET, then one can say that the decoupling of heavy particles occurs /9,10/.

Here we shall consider asymptotic expansions of the following objects

$$R_{MS}\, G(J) = R_{MS}\langle T e^{L(\varphi,\phi)+J\varphi}\rangle_0 ; \quad (8)$$

$$R_{MS}\, T_A(J) = R_{MS} \langle T A(0) e^{L(\varphi,\phi)+J\varphi}\rangle_0 ; \tag{9}$$

$$R_{MS}\, T_{AB}(q,J) = R_{MS} \int dx\, e^{-iqx} \langle T A(x) B(0) e^{L(\varphi,\phi)+J\varphi}\rangle_0 , \tag{10}$$

where $A$ and $B$ are local operators composed of fields $\varphi$ and $\phi$. The momentum $q$ (and the momenta of sources $J$ ) are small as compared to $M$ .

Expanding as usually (1.34) in sources of $A$ and $B$, we get:

$$R_{MS}\, G(J) = R_{MS} \langle T e^{L_{eff}(\varphi)+J\varphi}\rangle_0 + o(\lambda^{1-N}) = \langle T e^{L^R_{eff}(\varphi)+J\varphi}\rangle_0 + o(\lambda^{1-N}); \tag{11}$$

$$R_{MS}\, T_A(J) = \sum_n C^n_R(M)\, R_{MS} \langle T O_n(\varphi(0))\, e^{L_{eff}(\varphi)+J\varphi}\rangle_0 + o(\lambda^{1-N})$$
$$= \sum_{n,m} C^n_R(M)\, Z_{nm} \langle T O_m(\varphi(0))\, e^{L^R_{eff}(\varphi)+J\varphi}\rangle_0 + o(\lambda^{1-N}); \tag{12}$$

$$R_{MS}\, T_{AB}(q,J) = \sum_n C^n_R(q,M)\, R_{MS} \langle T O_n(\varphi(0))\, e^{L_{eff}(\varphi)+J\varphi}\rangle_0$$
$$+ \sum_{n,m} C^{A,n}_R(M)\cdot C^{B,m}_R(M)\, R_{MS} \int dx\, e^{-iqx} \langle T O_n(\varphi(x)) O_m(\varphi(0))\, e^{L_{eff}(\varphi)+J\varphi}\rangle_0 + o(\lambda^{1-N}), \tag{13}$$

where

$$L_{eff}(\varphi) = C_R\left(T e^{L(\varphi,\phi)} - 1\right) = L_{eff}(\{J_n\},\varphi)\big|_{J_n=0} \equiv \sum_r k^r_R(M)\cdot O_r(\varphi)$$

and operators $O_n(\varphi)$ include only light fields $\varphi$ . The renormalized (i.e. with the Ms-counterterms) effective action $L^R_{eff}(\varphi)$ is (see eq.(1.33))

$$L^R_{eff}(\varphi) = \sum_{r,s} k^r_R(M)\cdot \tilde{Z}_{rs}\cdot O_s(\varphi) \equiv \sum_s C^s_B\left(\{J_n=0\},\dots\right)\cdot O_s(\varphi), \tag{14}$$

where $\tilde{Z}_{rs}$ are renormalization constants.

Let us explain shortly these expressions. At first, consider eq.(11). Complete H-partitions of diagrams contributing to the l.h.s. of (11) contain all lines of heavy particles. The action of $C_R$ contracts all H-subgraphs of the complete H-partition to points, which produces insertions of light operators $O_n(\varphi)$. The r.h.s. of (11) contains the sum over all such partitions. Contributions of $O_n$ are analogous to finite counterterms, so that their contributions are equivalent to adding, to the Lagrangian, new vertices, which leads to the formation of the effective action. In comparison with (11), eqs. (12) and (13) contain new types of H-partitions with the insertion of $A$ or $B$ , which results in some new terms in expansions. All expansions include terms corresponding to the H-partitions of (11), so that in all these cases the effective action is present. Functions $k^r_R(M)$ are parameters of this action and contain all the dependence on heavy masses. Moreover, they can depend on some fixed parameters, e.g., masses of $\varphi$ , and the subtraction operator $t$ may be chosen so



that $k^r_R$ will be free from logarithms of the type $\ln(m/M)$ (see /4/ and the fifth section). If there are no particles with a heavy ($\sim M$) mass, that is, if there are no terms of the kind $CM^2\varphi^2$ in the effective Lagrangian, then the full decoupling occurs.

## 5. Infrared singularities of CF's

Proving the UV-finiteness of CF's in the limit of the removed regularization we have omitted all problems connected with IR divergences. Potentially they can appear because the recurrent formula (1.24) contains differential operators and their action might result in IR singularities (for example, these singularities emerge from differentiatinns with respect to external momenta at zero masses). If one uses the dimensional regularization for divergent expressions to make sense /11/, they manifest themselves as poles in $\epsilon$ as $\epsilon \to 0$.

We could avoid difficulties, introducing some IR cutoff. For example, it might be nonzero points of subtractions assotiated with $t_h$ or regulator masses introduced into IR dangerous lines. Then the limit $\epsilon \to 0$ will be well-defined, but CF's will acquire some undesirable properties. First of all, they will contain, in general, large logarithms of the type $\ln \mu_I/M$ where $\mu_I$ is the IR cutoff and $M$ is the scale parameter (see (1.8)). In applications such contributions might cause some trouble because in general one has to take care of their resummation. As a matter of fact, following this approach one includes some part of operator matrix elements responsible for the low-energy dynamics into CF's. However, in realistic quantum models like QCD the low-energy behaviour is unknown and cannot be described perturbatively. Hence, an incomplete separation of short and large distances (the presence of $\ln \mu_I/M$ is due to this reason) would be undesirable. Moreover, there exist some technical problems, because practical calculations become very complicated and, in fact, limited by the one-loop approximation.

Thus, we would like to combine two requirements to the scheme of separating short and large distances determined by operators $t_h$: the absence of IR logs in CF's and the simplicity of calculations. From the point of view of the latter condition, the scheme with zero points of subtractions with respect to small momenta and masses is the most useful. On the other hand, the action of $t_h$ so chosen might result in IR divergences. However, as will be shown in this section, this does not happen and CF's determined by $t_h \equiv (t_0)_h$ with zero points of subtractions are free of IR singularities, so that there is no need in IR cutoffs (note that an equivalent scheme based on different principles was proposed in /4/). Here we introduce also an intermediate scheme with subtractions at nonzero masses and some momenta, which is sometimes convenient.

The following simple example is a good illustration of the method and can be easily generalized. Consider the expansion of the diagram given in Fig. 4a) as $Q\to\infty$ . All parameters except $Q$ are fixed. Thin lines correspond to the massless field $\varphi$ , the thick one – to the field $\phi$ with a small nonzero mass. The diagram may be considered as one of the contributions to the Green function

$$T(Q,p,s,m) = i\int dx\, e^{iQx}\langle T\varphi(p)\varphi(s-p)\cdot\varphi\phi(x)\cdot\varphi\phi(0)\, e^{ig\int\varphi^3 dx}\rangle_0 ,$$

where $\varphi(p)$ is the Fourier transform of $\varphi(x)$ and $m$ is the mass of $\phi$ .

The limit $Q\to\infty$ is described by expansion (2). To construct the latter, one has to enumerate all complete H-partitions of the graph. For the case considered they coincide with all possible ways of flowing $Q$ through the diagram. There are three possibilities: through the massive line, through massless lines and through all lines. Thus, we see three terms in the Figure. In the first two terms the action of $C_R$ reduces to $t$ and the action of $R_{MS}$ is given just by $1-K$ . The structure of $C_R$ in the third term is determined by (1.24) and is represented in Fig. 4b).

Let operator $(t_0)_h$ pick out $N_h$ first terms in the expansion in powers of small external momenta and masses. Let us study the structure of IR singularities of $C_R$ determined by $t=t_0$ . It is clear that these singularities can emerge only in $C_R$ presented in Fig. 4b). Among others, the operator $t_0$ contains derivatives $\partial^n/\partial m^n|_{p,s,m=0}$ . Let us examine whether the result of their action on the square brackets is IR finite or not (see Fig. 4b)). Derivatives act on the massive line, and the result is $(Q-K)^{-2(n+1)}$. Being integrated over $K\sim+Q$, this factor leads to an IR singularity. The diagrams shown in Fig. 4c) contribute to this singularity. However, in the sum IR divergences cancel out. To see how this happens, represent the diagrams in the form given on the r.h.s. of Fig. 4c). We get that the sum is IR finite if the action of the operator $1-t_0$ gives an expression that is $\sim (\tilde K)^{2(n-2)}$ at $\tilde K\equiv K-Q\sim 0$ . It is not difficult to see, however, that

$$(1-(t_0)_h)\, h\,(p,s,\tilde K,Q) = O(\varkappa^{N_h+1}) ,$$

where $\varkappa\sim p,s,\tilde K$ and $N_h$ is some integer which depends on N linearly. At $p=s=0$ this expression is a function only of $\tilde K, Q$ and behaves as $(\tilde K)^{N_h+1}$ . Making N sufficiently large, we get $N_h+1\geq 2(n-2)$ for any $n$ . At such N all IR singularities cancel out, and derivatives of the type $\partial^n/\partial m^n|_{p,s,m=0}$ are finite. In full analogy one can show that the region $K\sim 0$ is also IR safe. Thus, the choice $t=t_0$ gives an IR finite expression.



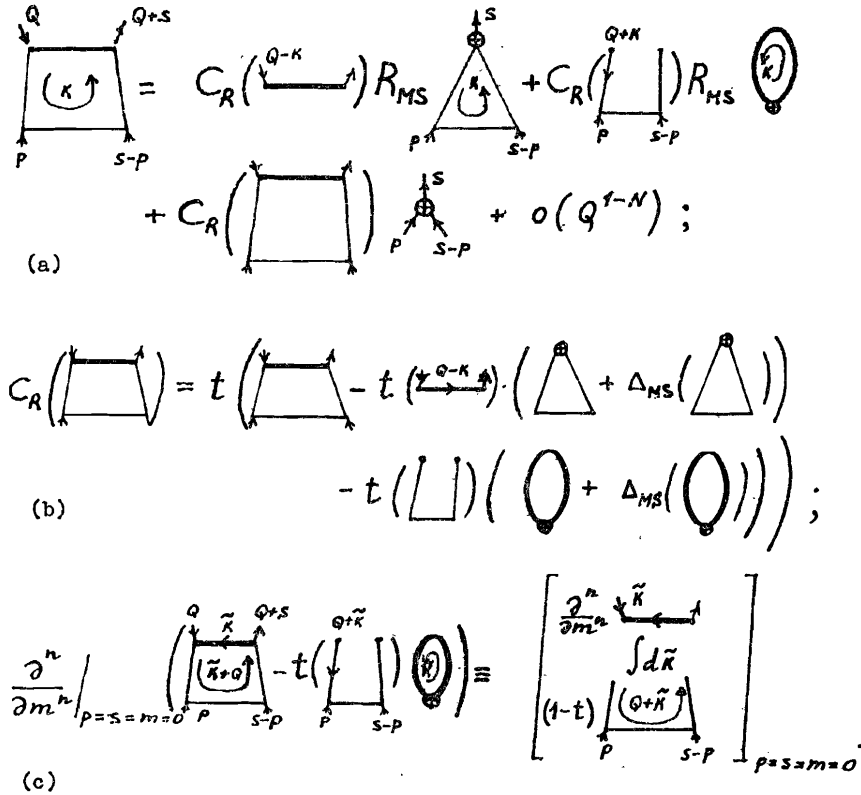

Fig.4

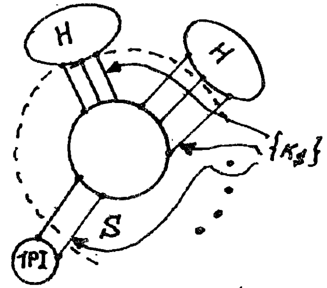


Fig.5

This simple reasoning can be generalized to any expansion of type (1.34). To obtain the generalization, it is convenient to introduce one more subtraction operator.

At first define more precisely the action of subtraction operators $t_h$ on a given graph.

Let us take an H-graph $G$. Fix some flow of external momenta $\{Q\}$ and $\{q\}$ through its lines and then choose some system of loop momenta $\{k\}$. The system $\{Q, q, k\}$ thus defined will be referred to as a base system. Then consider an H-subgraph $\sigma \subset G$. External momenta of $\sigma$ can be $Q, q$ and some loop momenta $k$. In general, the set of loop momenta of $\sigma$ does not necessarily coincide with any subset of loop momenta $k$. It makes us to introduce one more system of external and internal momenta. Let us consider some H-forest $F\{\sigma\}_H$ of the graph $G$. Choose the flow of $\{Q\}$ through $G$ so that some subset $\{Q\}_\sigma$ of large momenta enter some vertices of any graph $\sigma \in F\{\sigma\}_H$. If $\{Q\}_\sigma = \emptyset$, then $\sigma$ should contain lines with large masses. Note that different sets $\{Q\}_\sigma$ and different sets of the vertices, through which the momenta of the same $\{Q\}_\sigma$ flow, correspond to different H-subgraphs coinciding topologically with $\sigma$. Now introduce the system of loop momenta $\mathcal{k}$ according to the following induction procedure. The choice of $\mathcal{k}$ in the minimal elements of $F\{\sigma\}_H$ is arbitrary. Then consider some graph $\sigma \in F\{\sigma\}_H$ and choose its loop momenta so that some their subset coincide with loop momenta of the maximal subgraphs of $\sigma$. The resulting system $\{Q, q, \mathcal{k}\}_F$ of mapping $G$ by momenta can be introduced for any forest $F$ such that $G \in F$. Of course, momenta $\mathcal{k}$ can be expressed via $Q, q$ and loop momenta of the base system: $\mathcal{k} \equiv \mathcal{k}(Q, q, k)$.

Now, consider a given H-forest $F\{\sigma\}_H$ and the corresponding F-system $\{Q, q, \mathcal{k}\}_F$. Let $\sigma \in F\{\sigma\}_H$. External momenta of $\sigma$ are some subsets of $\{Q, q\}$ (denote them as $\{Q, q\}_\sigma$) and $\{\mathcal{k}\}$. Define the subtraction operator $(t_o)_\sigma$ acting on $\sigma$ as the Taylor operator expanding the integral corresponding to $\sigma$ in powers of $q$, $m$ and those $\mathcal{k}$ which are external for $\sigma$. Then, introduce the subtraction operator $\tilde{t}_\sigma$ that expands $\sigma$ only in powers of those $\{\mathcal{k}\}_\sigma$ which are external momenta of $\sigma$ and loop momenta of the graph in which $\sigma$ is a maximal subgraph. By definition, $\tilde{t}_G = 1$.

Given operators $(t_o)_\sigma$ and $\tilde{t}_\sigma$, we can introduce CF's corresponding to them. Denote these CF's as $C_{R_o}$ and $\tilde{C}_R$, respectively. Functions $C_{R_o}$ can be obtained via eq.(1.24), operators $t$ being replaced by $t_o$. For $\tilde{C}_R$ we have

$$\tilde{C}_R(G) = R_{MS}\Big(G - \sum_{\{\sigma\}^c_H} \cdot \prod_\sigma \tilde{t}_\sigma \tilde{C}_R(\sigma) \cdot G/\{\sigma\}^c_H\Big), \tag{15}$$



where $\tilde{t}_\sigma$ are defined with respect to the forest $F = G \cup \{\sigma\}^c_H$. In full analogy functions $\Delta_{H_0}$ and $\tilde{\Delta}_H$ are determined.

Notice the relation: $(t_o)_{\sigma_1} \cdot \tilde{t}_{\sigma_2} = (t_o)_{\sigma_1} \cdot (t_o)_{\sigma_2}$ if $\sigma_2 \subseteq \sigma_1$. Then we immediately have

$$(t_o)_G \, C_R(G) = C_{R_o}(G). \tag{16}$$

Our aim is to establish the IR finiteness of $C_{R_o}$. The above equation means that we can prove this fact if we show the existence of derivatives of the type $\partial^{n-r}_{q=0} \cdot \partial^{r}_{m=0} \tilde{C}_R(G)$ at $\epsilon = 0$ and at sufficiently large N. Here we shall present a heuristic consideration based on the dimensional analysis and power counting, which is quite sufficient for practical purposes.

As it follows from eq.(1.26), $\tilde{C}_R$ can be expressed entirely in terms of $\tilde{\Delta}_H$ and $\Delta_{MS}$ operations. As is well-known, $\Delta_{MS}$ is free from IR singularities and contains only UV poles. This fact allows us to restrict our attention to the consideration of IR properties of $\tilde{\Delta}_H$.

Let $G$ be an H-graph with the corresponding Feynman integral

$$G(Q, M, q, m) = \int dk \, I_G(Q, M, q, m, k), \tag{17}$$

where $I_G$ is an integrand and $\{k\}$ are loop momenta from the base system. The function $\tilde{\Delta}_H(G)$ can be represented as (see eq.(15))

$$\begin{aligned} &\tilde{\Delta}_H(G) = \int dk \, I_{\tilde{\Delta}}(Q, M, q, m, k); \\ &I_{\tilde{\Delta}}(Q, M, q, m, k) = \sum_{G \in F\{h\}_H} \prod_h (-\tilde{t}_h) \, I_G(Q, M, q, m, k). \end{aligned} \tag{18}$$

Here the summation runs over all H-forests containing the graph $G$.

Let us show that N (see (1.10)) can be chosen large enough to make IR finite all derivatives of the type $\partial^{n-r}_{q=0} \partial^{r}_{m=0} \tilde{\Delta}_H(G) \equiv \partial^n \tilde{\Delta}_H(G)$. The latter expression is determined by the integrand:

$$\partial^n I_{\tilde{\Delta}}(Q, M, k) = \sum_{G \in F\{\sigma\}_H} \prod_\sigma (-\tilde{t}_\sigma) \, \partial^n I_G(Q, M, k); \tag{19}$$

$$\partial^n \tilde{\Delta}_H(G) = \int dk \, \partial^n I_{\tilde{\Delta}}(Q, M, k)$$

(operators $\tilde{t}_\sigma$ commute with $\partial^n$). Potentially IR divergences can arise when (19) is integrated over any momentum region of $k$ where denominators of some lines of $\partial^n I_{\tilde{\Delta}}$ turn to zero. Denote the set of such lines by $S$. The nullification of momenta flowing through these lines corresponds to some fixed values $K^o_S$ of momenta $\{K_S\}$ from the set $\{k\}$. To prove the cancellation of IR divergences, we should establish the IR convergence of the integral

$$\int_{K_S \sim K^o_S} dK_S \, \partial^n I_{\tilde{\Delta}} \tag{20}$$

for any set $S$ /12/.

Suppose we have some set $S$ of lines with zero denominators. Without loss of generality we can assume that their nullification corresponds to $K_S=0$. Let the space of $K_S$ has the dimension $D_S$. Let us verify that $\partial^n I_{\tilde\Delta} \sim (K_S)^{1-D_S}$ as $K_S \to 0$ for some choice of $N$, so that the integral (20) is superficially convergent. To see that this is indeed true, rewrite (19) so that powers of $K_S$ could easily be counted. Our basic strategy will be some modification of the strategy of /13/.

Consider some term in (19) corresponding to the forest F and separate all lines of the corresponding Feynman graph into two classes. The first one comprises regular lines, that is lines whose denominators remain nonzero at $K_S=0$. These lines carry nonzero momenta or depend on large masses. The other class contains singular lines with zero at $K_S=0$ denominators. Of course, this separation should be performed after the application of operators $\tilde t_\delta$ in (19). Thus, first we should reparametrize all momenta of $G$ in terms of the F-system $\{Q,q,k\}_F$ attached to the forest under consideration, set $\{k\}_\delta=0$ for any $\delta\in F$, nullify $q$, $m$ and, at last, $K_S$ (which are functions of $k$), and only then examine whether denominators of the lines are zero or not.

Notice some properties of the reduced graph $\bar\delta(F)$ (see /1/, the second section). As a result of the above procedure, momenta of the lines belonging to $\bar\delta(F)$ can be some combinations of $\{Q\}$ and loop momenta $k$ of $\bar\delta(F)$, or zero. If, besides, a line is massless, then the latter case (and only it) corresponds to a singular line. Note also that the action of $\tilde t_\delta$ at arbitrary values of loop momenta of $\bar\delta(F)$ does not produce zero denominators, so that singular lines emerge only after putting $q=K_S(Q,q,k)=0$ and before this procedure their momenta are linear combinations only of $q$ and $K_S$.

Consider a given reduced graph $\bar\delta(F)$ corresponding to the graph $\delta\in F$. Denote the sets of regular and singular lines of $\bar\delta(F)$ as $\bar\delta_r(F)$ and $\bar\delta_s(F)$, respectively. The set $\bar\delta_r(F)$ can be represented as a union of its connected components $\rho_i$, $\bar\delta_r(F)=\bigcup_i \rho_i$. As it follows from the above consideration, $\rho_i$ are either H-subgraphs or 1PI-subgraphs all lines of which depend on nonzero loop momenta of $\bar\delta(F)$. Moreover, for every $\rho_i$ there exists $\tau_i\in\delta$ ($\tau_i\notin F$ if $\rho_i\neq\bar\delta$) such that $\bar\tau_i(F)=\rho_i$. Consider those $\tau_i$ which are H-subgraphs. Find all such $\tau_i$ for every $\delta\in F\cup G$ and form the set $\{\tau\}$ from them. The union $F_S=\{\tau\}\cup F$ is an H-forest. It may happen that different forests have the same $F_S$. Obviously, all such forests are subsets of $F_S$. Let $F_{min}$ be a minimal (with a minimal number of elements) forest corresponding to the given $F_S$. Let us describe the set $F_S\setminus F_{min}$. It consists of subgraphs $\tau$



such that $\bar\tau(F_S)=\bar\tau_r(F_S)$. Moreover, $\tau$ is a maximal element in some $\gamma\in F_S$ such that $\bar\gamma_r(F_S)$ is a union of $\rho_i$ and there exist $\delta_i$, which are not hard, such that $\bar\delta_i(F_S)=\rho_i$. From these two properties we immediately conclude that the set $\{\bar k\}_\tau$ of loop momenta of $\gamma$ from the $F_S$-system, which are external momenta of $\tau$, consists of linear combinations of momenta $\{K_S\}$.

With the help of $F_S$-forests the sum in (19) can be rewritten as follows:

$$\sum_{G\in F\{h\}_H}\prod_h(-\tilde t_h)\equiv\sum_{F_S}\sum_{F\subseteq F_S}\prod_{h\in F}(-\tilde t_h)=\sum_{F_S}\prod_{h\in F_S}\tilde f_h ,$$

where

$$\tilde f_h = \begin{cases}1-\tilde t_h, & \text{if } h\in F_S\setminus F_{min}\\ -\tilde t_h & \text{if } h\in F_{min}\end{cases}$$

so that we have

$$\partial^n I_{\tilde\Delta}=\sum_{F_S}\Big(\prod_{h\in F_S}\tilde f_h\Big)\partial^n I_G. \qquad (21)$$

Let us verify that each term in (21) is integrable over $K_S$ near $K_S = 0$.

Each term has the form $\left(\prod_{h\in F_S}\tilde f_h\right)\partial^n I_G$. There are two possibilities: (i) the product contains at least one factor $\tilde f_h = 1-\tilde t_h$; (ii) there are no such factors, that is, all $\tilde f_h=-\tilde t_h$. In case (i) $h\in F_S\setminus F_{min}$ and, hence, $\{K\}_h$ are linear functions only of $\{K_S\}$. Thus, $\tilde t_h$ expands $h$ in powers of $K_S$ near $K_S = 0$ and $1-\tilde t_h$ gives an additional factor $\sim(K_S)^{N+d_h+1}$. (Recall that to achieve the accuracy $o(\lambda^{1-N})$ of the expansion, operators $\tilde t_h$ should pick out first $(d_h+N)$ terms of the Taylor series in $\{\bar k\}_h$, where $d_h$ is the dimensionality of an $H$-subgraph $h$ at $\epsilon = 0$). Further, in case (i) the contribution to (20) can be written as (see Fig.5) $\int_{K_S\sim 0} dK_S\, h(K_S,Q,M)\cdot I_S(K_S)\cdot r(K_S)$, where $h$ is the contribution from the $H$-subgraphs ($\sim(K_S)^{d_h+N+1}$), $I_S$ is that from singular lines ($\sim(K_S)^{d_S}$), $d_S$ being the dimensionality of $I_S$), and $r$ is a possible contribution from regular 1PI graphs independent of large parameters (after the action of $\partial^n$ and integration over loop momenta the latter contribution depends only on $K_S$ and, hence, is $(K_S)^{d_r}$, where $d_r$ is its dimension). If $d_G$ is the dimension of $G$, then $d_G=D_S+d_S+d_h+d_r+n$ and $n\le d_G+N$. Thus, we have

$$h\cdot I_S\cdot r\sim(K_S)^{d_h+N+d_S+d_r+1},\quad d_h+N+d_S+d_r+1=d_G+N-D_S-n+1\geq 1-D_S .$$

This unequality is the desirable estimate for the IR convergence. In case (ii) all $h \in F_{min}$ so that $F_{min} \equiv F_S$ . In this case all lines of $G$ are regular. This can be proved by induction. If $\sigma_{min}$ is the minimal element of $F_{min}$ , then $\overline{\sigma}_{min} = \sigma_{min}$ and $(\overline{\sigma}_{min})_r = (\sigma_{min})_r$. If $(\sigma_{min})_r \neq \sigma_{min}$ , then $(\sigma_{min})_r$ should contain an $H$ -subgraph, because otherwise $\sigma_{min} = (\sigma_{min})_r \cup (\sigma_{min})_S$ is not H-sabgraph. But this $H$ -subgraph should belong to $F_S \setminus F_{min} = \emptyset$ , so that $(\sigma_{min})_r = \sigma_{min}$. Further, let $\gamma \in F_{min} \equiv F_S$ have regular lines in all its maximal components. If $\overline{\gamma}_r \neq \overline{\gamma}$ , then all connected components of $\overline{\gamma}_r$ should be 1PI -graphs independent of the large external parameters because $\gamma \in F_S$. Moreover, there exist subgraphs $\tau_i$ , which are not hard, such that $\bigcup_i \overline{\tau}_i = \overline{\gamma}_r$. But according to the induction hypothesis, maximal components of $\gamma$ are regular so that they should be elements of $F_S \setminus F_{min} = \emptyset$. Thus, $\overline{\gamma}_r = \overline{\gamma}$ and this proves the induction hypothesis.

Thus, we have shown that (21) is integrable at $K_S = 0$.

To summarise, CF's determined by operators $t_o$ are represented by power series in small momenta $q$ and masses $m$ and do not contain IR divergences.

Functions $\tilde{C}_R$ possess the same properties in a sense that they can be expanded in $q$ and $m$ into power series without giving rise to IR singularities.

Practical consequences of the cancellation of IR singularities in $CF$'s are twofold. First, it gives the possibility to use the mass-independent renormalization group technique[2]. Second, it leads to simple algorithms for evaluating $CF$'s in PT[14]. Here we present only formulae for evaluating $C_{R_o}$. These formulae can be considerably simplified compared to (1.24) and (1.25) by exploiting the properties of the dimensional regularization. As is well-known, in this regularization all massless integrals with zero external momenta are equal to zero. Thus, all completely reduced graphs should be set to zero after being expanded in powers of $q$ and $m$ . In doing so, we get:

$$C_{R_o}(G) = (t)_{qG} R_{MS} G - \sum_{\{G \neq h\}_H^c} \Big( \prod_h C_{R_o}(h) \Big) \cdot \Delta_{MS} \big( G / \{h\}_H^c \big) ; \qquad (22)$$

$$C_{B_o}(G) = (t)_{qG} R_{MS} G ; \qquad C_{H_o}(G) = (t)_{qG} \cdot G .$$

Note that "bare" CF's $C_{B_o}$ given by the second relation in (22) coincide with functions $\Delta^{as}(G)$ used in[4] and, thus, may serve as a starting point in comparing the two approaches (being rewritten in terms of $C_{B_o}$ , (1.22) coincides with the "EA-expansion" of[4]).

14

To obtain $CF$'s via (22), one should evaluate only MS -counterterms and integrals depending only on large variables $\{Q\}$ and $\{M\}$. For example, in the case of short-distance expansions (see examples 1 and 2 in the previous section) $CF$'s are expressed via massless propagator integrals with one external momentum $Q$ . Within dimensional regularization such integrals can be evaluated up to three loops[15]. In the case of effective light theories $CF$'s are determined by massive vacuum integrals and so on.

There exists, however, one more possible application of the technique developed here. Namely, it can be used for analysing Euclidean collinear singularities. These singularities arise in Euclidean Green functions at some zero external momenta. Indeed, for a while one can imagine these momenta to be nonzero but small as compared to other dimensional parameters. Such a reformulation of the problem gives a new scale and Green functions can be expanded in it with $CF$'s finite in the collinear limit. Thus, all collinear singularities turn out to be localized in operators.

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## Appendix. On the Origin of Contact Terms

We shall consider here a simple example when the graph G is a function of one large momentum $Q$ .

The expression $R_H G(Q)$ is not integrable at $Q=0$ with probe functions $\varphi(Q)$ which have at $Q=0$ zero of an insufficiently large order. Thus, applying the standard continuation procedure[16], we find that the following integral

$$\int dQ \Big( \varphi(Q) - \sum_{n=0}^{N-5} \frac{1}{n!} Q^n \varphi^{(n)}(0) \Big) \cdot R_H(Q) \qquad (A1)$$

is convergent for any $\varphi(Q)$ which is finite at $Q=0$ and grows more slowly than $Q^{N-5}$ as $Q \to \infty$ (recall that $R_H G(Q) = O(\frac{1}{Q^N})$ as $Q \to \infty$ and at $Q=0$ it has the behaviour $\frac{1}{Q^{N-1}}$ with possible logarithmic corrections). Using the identity

$$\varphi^{(n)}(0) = (-1)^n \int ds\, \delta^{(n)}(s)\, \varphi(s)$$

we can rewrite (A1) as

15

$$\int dQ\,\varphi(Q)\left[R_H G(Q) - \sum_{n=0}^{N-5} (-1)^n \frac{1}{n!}\,\delta^{(n)}(Q)\int ds\cdot s^n R_H G(s)\right] \equiv \qquad \text{(A2)}$$
$$\equiv \int dQ\,\varphi(Q)\left[R_H G(Q) + \sum_{n=0}^{N-5}\delta^{(n)}(Q)\cdot c_n\right].$$

It is not difficult to show by using asymptotic properties of $R_H G(Q)$ that the functional (A2) is $O(\lambda^{-N})$ (up to logs) under the rescaling $Q\to\lambda Q$ of the expression in the brackets. Terms with $\delta$ - functions are contact terms. Thus, the distribution in the brackets can be used as a starting point of the combinatorial **analysis.**

Note that in our presentation we have omitted all contact terms because field sources $J_n(Q)$ playing the role of $\varphi(Q)$ are assumed to be localized at infinite $Q$ .

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Горишний С.Г. E2-86-177

О построении операторных разложений и эффективных теорий в MS-схеме. Примеры. Инфракрасная конечность коэффициентных функций


Метод построения асимптотических разложений фейнмановских интегралов по большим евклидовым импульсам и (или) массам, предложенный автором ранее, использован для получения ряда широко используемых разложений. Показано, что при определенном выборе вычитающего оператора коэффициентные функции разложений не содержат инфракрасных сингулярностей; приведены эффективные формулы для их вычисления.



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Gorishny S.G. E2-86-177

On the Construction of Operator Expansions and Effective Theories in the MS-Scheme. Examples. Infrared Finiteness of Coefficient Functions


The method of constructing asymptotic expansions of individual Feynman graphs in large Euclidean momenta or/and masses proposed in our previous paper is used to obtain some widely used expansions. We show that, with a special choice of subtraction operators, coefficient functions of these expansions do not contain infrared singularities, and present efficient formulae for their evaluation.



The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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