сообщения
объединенного
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исследований
дубна

E2-86-176

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ON THE CONSTRUCTION OF OPERATOR EXPANSIONS AND EFFECTIVE THEORIES IN THE MS-SCHEME. General Formalism

1986

Operator product expansions (OPE) of various kinds[1] and effective theories (ET)[2,3] have become widely used for qualitative and semiquantitative predictions within QCD and other realistic field models (for a review see[4,5]). An important point in practical applications of these asymptotic expansions is the calculation of coefficient functions (CF) entering into them, which are the only piece amenable to evaluation in perturbation theory (PT). There exists a certain analogy between CF and ultraviolet renormalization constants because both of them originate from the region of large momenta[6]. As is well known, the most convenient scheme for evaluating ultraviolet (UV) counterterms is the minimal subtraction (MS) scheme[7], and its convenience is mostly due to the polynomiality of counterterms in masses[8]. This property leads to simple renormalization group (RG) equations and lays the ground for some powerful methods of computing RG parameters[9]. From the analogy mentioned above one may expect that a similar property with the same consequences should hold for CF's. Namely, one may hope that within the MS-scheme CF's can be expanded in powers of small dimensional parameters like momenta and masses without giving rise to infrared (IR) singularities. These expectations have been proved to be correct[10] (see also[6,11] for a study of some particular cases).

In short, the method of[10] consists in the following. The formal Taylor expansion of Green functions in powers of small momenta and masses produces IR divergences, which makes asymptotic series meaningless. However, as has been pointed out in[10], correct asymptotic estimates can be restored by adding, to the formal expansions, special IR counterterms. These counterterms compensate IR divergences and convert the whole series into a well-defined distribution. The expansions of Green functions thus obtained can be cast into a form of OPE and ET, the IR counterterms being interpreted as composite operators. The method of[10] is a general one and provides the complete treatment of Euclidean asymptotics of Green functions. However, a natural connection of CF's with high energies and the UV renormalization gets somewhat lost.

In a more familiar procedure of separating different dimensional scales via the ultraviolet R -operation with oversubtractions [12,13] this connection seems to be more transparent. Moreover, the latter approach provides a uniform treatment both for the ultraviolet renormalization and operator expansions. However, till now this formalism has been developed only for expansions in one large euclidean momentum[12,13] and large masses[3] and, as a rule, with the UV renormalization performed at a fixed momentum point, which is inconvenient for practical calculations. Becides, it is somewhat unclear whether in this approach one could construct CF's without infrared logarithms (even in the MS-scheme used for the UV renormalization).

In the present paper we apply the method of the ultraviolet R -operation[15] to construct expansions of Green functions in any number of large Euclidean momenta or/and masses. Then, short-distance expansions[12,13] and effective light theories[2,3] can be obtained as particular cases of the above limit. We start with the MS-renormalized Feynman integrals and obtain their expansions in terms of MS-renormalized quantities. In the next publication[17] we apply the technique developed here to obtain some widely used expansions and present arguments according to which in the MS-scheme the separation of large and small dimensional parameters can be made at zero momenta and masses with CF's free of IR logarithms.

The present work is organized as follows. In the first section we recall basic notions of the R-operation. The next section contains our derivation of an expansion of individual Feynman graphs and all definitions relevant to it. In the third section we obtain the corresponding expansion of the generation functional for various Green functions. In Appendices we collect prescriptions of the dimensional regularization and combinatorial formulae of functional technique.

1. R - Operation

On the whole our notation follows that of ref.[10].

A given Feynman graph G consists of vertices and lines connecting them. Each graph G corresponds to a momentum integral constructed via Feynman rules. Generally speaking, such integrals may contain divergences, and to make integral expressions meaningful, Feynman rules should be supplemented by a regularization. We shall use the dimensional regularization[14] that is the most suitable for our purposes. Some basic conventions of this regularization are given in

Appendix 1. In what follows the regularization is implicit.

A subgraph g of the graph G consists of vertices and lines (may be, not all) connecting them, chosen from the vertices and lines of G . A subgraph may coincide with G or be a single vertex. A subgraph g is properly contained in G if g is a subgraph and $g \neq G$ (g is a proper subgraph). At last, g is a nontrivial proper subgraph of G if $g \neq$ G and g is not a single vertex.

Call a set of subgraphs $\{g_\alpha\}$, $g_\alpha \subseteq G$, a partition of G, if $g_\alpha \cap g_\beta = \emptyset$ and each vertex of G belongs to one (and only one) $g \in \{g_\alpha\}$.

Then introduce the following Δ -operation defined on any subgraph g . By definition, put $\Delta(g) = 1$, if g is a single vertex. If g is a disconnected graph, then $\Delta(g) = 0$. In other cases $\Delta(g)$ is obtained by contracting g into a point. This point forms a new vertex with a Feynman rule uniquely determined by g and independent of $G(g \subseteq G)$. An exact meaning of $\Delta(g)$ will be concretized for the cases considered.

Now introduce (still purely combinatorially) the R-operation via the relation[15]

$$R\,G = \sum_{\{g_\alpha\}} \prod_\alpha \Delta(g_\alpha)\; G/\{g_\alpha\}, \tag{1}$$

where summation runs over all partitions of G. The graph $G/\{g_\alpha\}$ is a graph obtained from G by contracting all g_α into points with Feynman rules determined by $\Delta(g_\alpha)$.

Properties of the R-operation introduced up to this moment are sufficient for studying its action on the whole PT series (Green functions, S-matrix, etc.)[13,10]. The corresponding formulae are presented in Appendix 2.

A UV R-operation removing divergences that arise in the course of integration over large momenta can be imagined to be a particular case of the construction (1) with a given choice of $\Delta(g) = \Delta_{UV}(g)$. As is known, Feynman integrals are rendered finite if one subtracts UV divergences from all their one-particle-irreducible (1PI) components, i.e., those which cannot be made disconnected by removing one line. Therefore, define $\Delta_{UV}(g) = 0$ if g is not 1PI. This requirement confines the summation in (1) only over those $\{g_\alpha\}$, where g_α is either a single vertex or 1PI. Call such partitions UV-partitions $\{g_\alpha\}_{UV}$. The quantity $\Delta_{UV}(g)$ is a UV counterterm. The definition of $\Delta_{UV}(g)$ is not unique: a polynomial of external momenta of g can always be added[15], and how we shall fix this ambiguity determines the renormalization scheme. Here we consider two schemes: the MS scheme[7] and momentum subtraction scheme[15]. Rewrite (1) in the form

$$RG = {}'RG + \Delta(G). \tag{2}$$

Then the MS-scheme is given by the following recurrent relation

$$\Delta_{MS}(g) = - K\, {}'R_{MS}\, g, \tag{3}$$

where K is the operator that picks out poles arising when the dimensionality of space of the regularization $D = 4-2\epsilon$ tends to its physical value (4 for the Minkowski space). In full analogy, the momentum subtraction scheme is given by

$$\Delta_t(g) = - t\, {}'R_t\, g, \tag{4}$$

where t is the operator that picks out several first terms in the Taylor expansion at a fixed momentum point. The number of terms is determined by demanding convergence of the corresponding integrals and coincides with the momentum dimensionality dimg of g [15]. If dimg < 0, then $\Delta_t(g) = \Delta_{MS}(g) = 0$.

Let us expound some useful properties of counterterms:

a) they are always polynomial in the external momenta of g [15];

b) in the MS scheme they are polynomial in any dimensional parameters (like masses) and independent of the regularization parameter μ [8].

The latter property is very helpful for evaluating $\Delta_{MS}(g)$ and RG functions related to it because it allows one to nullify any dimensional parameter and, thus, to simplify considerably the integrals to be calculated[9].

Recurrent relations (3) and (4) can be solved explicitly[16]. Call two subgraphs nonoverlapping if they either are disjoint or one of them is a subgraph of the other. Consider any set of nonoverlapping subgraphs of G . In this set separate the system of maximal elements, i.e., elements such that there are no subgraphs from the set containing them as a proper subgraph. If the system of maximal elements comprises some partition of G, then we shall say that the initial set is a forest of G . A UV forest $F\{g\}_{UV}$ is a forest including only 1PI graphs and/or single vertices. Then (3) and (4) can be written down in the form

$$\Delta_{MS}(g) = - K_g \sum_{F\{\tau\}_{UV}} \prod_{\tau} (-K_\tau)\, g; \tag{5}$$

$$\Delta_t(g) = - t_g \sum_{\substack{F\{\tau\}_{UV} \\ \tau \neq g}} \prod_{\tau} (-t_\tau)\, g, \tag{6}$$

where K_τ and g are operators K and t acting only on the subgraph τ . Here summation runs over all possible forests of g except those containing g . Inserting these solutions into (1), we have

$$R_{UV}\, G = \sum_{F\{g\}_{UV}} \prod_{g} (-M_g)\, G, \tag{7}$$

where $M_g = K_g$ for the MS-scheme and $M_g = t_g$ for the momentum subtraction scheme. Summation runs over all forests of G .

2. Asymptotic Expansion of Individual Feynman Graphs

The structure of the R-operation turns out to be very convenient in analysing asymptotic properties of Feynman integrals. It has been used in[12,13] to prove the short-distance expansion of operator products and in[3] to obtain parameters of effective light theories originated from the large-mass expansion of the initial ones. In those papers, however, momentum subtraction was used for the UV renormalization, which complicates practical calculations. Moreover, in phenomenological applications one often deals with a more complicated asymptotic behaviour than one Euclidean large momentum or one large mass. Here we present an extension of the procedure[12] to the case of asymptotic expansions in large Euclidean momenta or/and masses, UV divergences being removed in the minimal way according to the MS-scheme. As a rule, our consideration will be aimed at the construction of a convenient and efficient calculational scheme for practical applications, so that sometimes our arguments will not be mathematically rigorous (but, of course, they can be made rigorous at the cost of some complications).

Let all external momenta and masses of lines of G be decomposed into two sets $\{Q, M\}$ and $\{q, m\}$. We are interested in the behaviour of $R_{MS}G$ at large values of Euclidean Q and M:

$$\frac{|Q|}{M} = const; \quad \frac{M}{m} \sim \frac{M}{|q|} \sim \lambda \to \infty; \quad R_{MS}\, G \xrightarrow[\lambda \to \infty]{} ? \tag{8}$$

Momenta Q are assumed to be Euclidean, i.e., each component of Q tends to infinity so that $qQ/Q^2 \sim \lambda^{-1}$. Many expansions considered in the literature are generated by limits that are particular cases of (8). For example, short-distance expansions correspond to $Q \to \infty$ and effective light theories arise at $M \to \infty$.

Notice an important role that 1PI graphs play in the course of the UV renormalization. Indeed, 1PI components contain all information about UV divergences of the graph, and removing UV poles from all 1PI components of the graph renders it UV finite. In the course of an asymptotic expansion procedure there also exists type of graphs playing an analogous role. Namely, these are components of the graph which absorbe all the dependence on its large parameters $\{Q,M\}$. It is clear that to obtain an asymptotic expansion, one has to expand only these components. Let us introduce some definitions.

Call a line of G hard if at least one of the following conditions are fulfilled: a) the line corresponds to a particle with a large mass M; b) a nontrivial combination of large momenta Q can flow through it. A subgraph $h \subseteq G$ is a hard subgraph (H - subgraph) if it contains at least one hard line and cannot be made disconnected by deleting one line that is not hard. A partition of G is hard (H -partition) if it contains only H-subgraphs and single vertices.

It is clear that H -subgraphs are analogs of 1PI subgraphs in the R_{UV}-operation. Indeed, as follows from the definition, any graph G can be represented in the form of a tree graph all lines of which are not hard and effective vertices are just H -subgraphs or 1PI subgraphs independent of $\{Q\}$ and $\{M\}$. Thus, it is sufficient to obtain an expansion of each H-subgraph because all components independent of Q and M come through the expansion procedure unchanged. It is clear also that each H -subgraph is represented by a tree skeleton graph with hard lines and 1PI effective vertices.

Introduce now an intermediate $R_H^{(N)}$-operation

$$R_H^{(N)} G = \sum_{\{g\}} \prod_g \Delta_H^{(N)}(g)\; G/\{g\} \equiv \sum_{\{g\}_H} \prod_g \Delta_H^{(N)}(g)\; G/\{g\}_H \,, \qquad (9)$$

where $\Delta_H^{(N)}$ is a particular case of the Δ -operation and $\Delta_H^{(N)}(g) \neq 0$ only if g is either a single vertex or an H -subgraph. Thus, the summation in (9) is effectively restricted by $\{g\}_H$ partitions, i.e., by H -partitions. Let us describe the algorithm of evaluating $\Delta_H^{(N)}(g)$. Consider a given H -subgraph $h \subseteq G$. The set of its external momenta consists of external momenta of G $\{Q, q\}$ and, perhaps, of some internal integration momenta of G $\{k\}$. Consider the Feynman integral $h(Q,q,k,M,m)$ corresponding to h and introduce the subtraction operator $t^{(N)}$ such that

$$(1-t^{(N)})\, h(Q,q,k,M,m) = o(\lambda^{1-N}). \qquad (10)$$

As $t^{(N)}$ one can choose the Taylor operator that picks out some first terms in the expansion of h in powers of q and k at some fixed point $\{q=q_h,\ k=k_h\}$, where $q_h \sim k_h \sim q$. The number of terms is equal to $\deg_{Q,M} h + N$, where

$$h(Q,q,k,M,m) = O(\lambda^{\deg_{Q,M} h}).$$

Rewriting now (9) in the form analogous to (2) and defining

$$\Delta_H^{(N)}(h) = -\,t^{(N)}\,{}'R_H^{(N)} h, \qquad (11)$$

we get (modulo $\ln\lambda$)

$$R_H^{(N)} G = \Delta_H^{(N)}(G) + {}'R_H^{(N)} G = (1-t^{(N)})\,{}'R_H^{(N)} G = O(\lambda^{-N}), \qquad (12)$$

where we choose G to be an H -graph. Of course, the estimate $O(\lambda^{-N})$ in the r.h.s. of (12) is formal because $R_H^{(N)} G$ can be divergent as $\epsilon \to 0$, nevertheless, at nonzero ϵ it is a well-defined quantity.

The meaning of (12) is fairly transparent: making a sufficient number of subtractions in H -subgraphs, one can obtain any desired behaviour of the l.h.s. of (12) as $\lambda \to \infty$. Note that the relation (12) gives the expansion of an unrenormalized graph G. Indeed, one H -partition consists of single vertices. The term corresponding to it in $R_H^{(N)} G$ is G itself so that we have

$$G = (1-R_H^{(N)})G + O(\lambda^{-N}) = -{}'\sum_{\{g\}_H} \prod_g \Delta_H^{(N)}(g)\; G/\{g\}_H + O(\lambda^{-N}), \qquad (13)$$

where the subscript means the absence of the term including only single vertices.

To simplify the notation, in what follows the index N will be omitted.

To solve the problem for the MS -renormalized graphs, note that the action of R_H is determined formally on each term of the renormalized expression $R_{MS}\, G$ (see (1) with $\Delta \equiv \Delta_{MS}$). Indeed, each term of the kind $\prod_g \Delta_{MS}(g) G/\{g\}_{uv}$ allows the interpretation in terms of Feynman integrals. The whole sum $R_{MS}\, G$ is UV -finite as $\epsilon \to 0$, so that we may expect that $R_H\,(R_{MS} G)$ will also be

UV-finite. Let us show that this is indeed the case. We have

$$R_H R_{MS} G = \sum_{\{g\}_{uv} \subseteq G} \; \sum_{\{h\}_H \subseteq G/\{g\}_{uv}} \prod_h \Delta_H(h) \prod_g \Delta_{MS}(g) \, (G/\{g\}_{uv})/\{h\}_H \tag{14}$$

Let us try to reexpress (14) in terms of explicitly UV -finite as $\epsilon \to 0$ quantities. Note that in (14) operations Δ_H and Δ_{MS} do not commute since subgraphs of H -partitions include in general vertices formed after shrinking all subgraphs of the UV-partition to points. One may attempt to regroup terms in (14) so that the summation would be again over partitions, that is, over disjoint subgraphs. For this purpose we will use the representation of R_H in a form analogous to (7).

Call an H -forest the forest including only H -subgraphs and single vertices. Denote it by $F\{\tau\}_H$. Then (11) can be cast into the form

$$\Delta_H(h) = -t_h \sum_{\substack{F\{g\}_H \\ g \neq h}} \prod_g (-t_g)\, h ;$$

$$R_H G = \sum_{F\{h\}_H} \prod_h (-t_h)\, G, \tag{15}$$

where $t_g \; (\equiv t_g^{(N)})$ acts only on g . Substituting (7) and (15) into (14), we get

$$R_H R_{MS} G = \sum_{F\{g\}_{uv} \subseteq G} \; \sum_{F\{h\}_H \subseteq G/\{g\}_{uv}} \prod_h (-t_h) \prod_g (-K_g)\, G. \tag{16}$$

The double sum in the r.h.s. of (16) can be written as a single sum over mixed forests. These are forests elements of which can be either H- or 1PI -subgraphs with the restriction that the latter do not contain H -subgraphs from the forest. If H -subgraphs $\{h\}$ and 1PI -subgraphs $\{u\}$ form the system of maximal elements of this forest, then their union $\{h\} \cup \{u\}$ (together with possible single vertices) forms a mixed partition of G . With the help of the mixed forest $F\{g\}_{uv,H}$ the sum in (16) can be rewritten in the form

$$R_H R_{MS} G = \sum_{F\{g\}_{uv,H}} \prod_g (-M_g)\, G \tag{17}$$

$$= \sum_{\{h\} \cup \{u\} \subseteq G} \Big(\prod_h (-t_h) \sum_{F\{g \subset h\}_{uv,H}} (-M_g) \Big) \Big(\prod_u (-K_u) \sum_{F\{v \subseteq u\}_{uv}} (-K_v) \Big)\, G$$

$$= \sum_{\{h\} \cup \{u\} \subseteq G,\; h \cap u = \phi} \prod_h \Delta_B(h) \prod_u \Delta_{MS}(u)\, G/(\{h\} \cup \{u\}),$$

where g may coincide with h only as 1PI -subgraphs; M_g is t_g for H -subgraphs and K_g for 1PI ones and

$$\Delta_B(h) = -t_h \sum_{F\{g \subset h\}_{uv,H}} (-M_g)\, h$$

$$= -t_h \sum_{\substack{\{\tau\}_H \cup \{u\}_{uv} \subset h \\ h \neq \tau,\; \tau \cap u = \phi}} \prod_{\tau, u} \Delta_B(\tau) \Delta_{MS}(u)\, h/(\{\tau\}_H \cup \{u\}_{uv}).$$

Relation (17) is the desired representation of (14) as a sum over partitions. The operation Δ_B differs from Δ_H by the inclusion of ultraviolet MS counter-terms of h . Indeed, from (17) we have

$$\Delta_B(h) = \Delta_H(R_{MS} h) = -t_h R_H R_{MS} h$$

$$= \sum_{\{u\}_{uv} \subseteq h} \Big(\prod_u \Delta_{MS}(u) \Big) \cdot \Delta_H(h/\{u\}_{uv}). \tag{18}$$

In (17) operations Δ_B and Δ_{MS} act independently because their actions are determined on disjont graphs ($h \cap u = \phi$). Note that in this relation $\Delta_{MS}(u)$ subtracts only divergences of the initial graph G . However, the action of Δ_B on H -subgraphs produces new vertices which can give rise to UV singularities having nothing to do with those of G. Nevertheless, the whole expression (17) turns out to be UV -finite. To see that this is indeed the case, add to the expression $\sum_{\{u\}_{UV}} \prod_u \Delta_{MS}(u) G/\{h\}_H$ resulting after the action of $\prod_h \Delta_B(h)$ terms needed to make it UV-finite. These terms supplement the sum to the complete R_{MS} operation $R_{MS}(G/\{h\}_H)$. The whole expression should be unchanged, so that we have to redefine Δ_B so as to absorb the difference $(\sum_{\{u\}_{uv}} \prod_u \Delta_{MS}(u) - R_{MS}) \cdot G/\{h\}_H$. It can be done in a full analogy with formulae (14)-(18). Denote the quantities thus redefined as Δ_R . Clearly, $\Delta_R - \Delta_B$ contains only terms with subtractions of divergences associated with shrinking to points subgraphs $\{h\}_H \subseteq G$. Making such a rerrangement we get:

$$R_H R_{MS} G = \sum_{\{h\}_H \subseteq G} \prod_h \Delta_R(h)\, R_{MS}(G/\{h\}_H); \tag{19}$$

$$\Delta_R(h) = -t_h \sum_{\{\tau \neq h\}_H} \prod_\tau \Delta_R(\tau)\, R_{MS}(h/\{\tau\}_H).$$

Quantities Δ_R are renormalized ones as compared to the "bare" Δ_B . The explicit relation between Δ_R and Δ_B is given by

$$\Delta_R(h) = \sum_{F\{\tau\le h\}_H} \prod_{\tau\neq\tau_{min}} (-\Delta_{MS}(\bar{\tau}(F))) \prod_{\tau_{min}} \Delta_B(\tau_{min}). \quad (20)$$

Here $F\{\tau\}_H$ is an H -forest of the graph h , and $\{\tau_{min}\}$ is the system of its minimal elements, i.e., elements which have no subgraphs from the forest. Then, $\bar{\tau}(F)$ is the so-called reduced graph obtained from τ by shrinking to points all elements of the forest F properly contained in τ . Note also that to obtain (19),(20), we have used the fact that Δ_{MS} commutes with t_h because the former is a polynomial in all dimensional parameters.

The R_{MS}-operation is linearly in Δ_R because of their UV-finiteness. This finiteness can be established by induction. If the only H -subgraph of h is h itself, then $\Delta_R(h) = -t_h R_{MS} h$ and this is an explicitly UV -finite quantity (we assume here that t_h are chosen so that their action does not lead to infrared singularities and postpone the discussion of other possibilities to the next publication[17]. Then the induction with respect to the number of H -subgraphs can be done via (19). Thus, Δ_R are UV -finite. As a result, we obtain that the r.h.s. of (19) is also UV -finite as a sum of explicitly UV -finite terms.

Now return to our problem and notice that due to (12) $R_H R_{MS} G$ is $O(\lambda^{-N})$ as $\lambda\to\infty$. This allows us to rewrite $R_{MS} G$ in a form analogous to (13):

$$R_{MS} G = -\sum_{\{h\}_H}' \prod_h \Delta_R(h)\; R_{MS}\left(G/\{h\}_H\right) + O(\lambda^{-N}), \quad (21)$$

and after that the problem of expanding the MS -renormalized Feynman integral is almost solved. The only fact preventing us from saying so is an incomplete factorization of large parameters $\{Q, M\}$ in $\Delta_R(h)$ The point is that graphs in $R_{MS}(G/\{h\}_H)$ can still contain H -subgraphs and thus be dependent on $\{Q,M\}$. Nevertheless, all terms on the r.h.s. of (21) have at least one H -subgraph contracted into a point as compared to G, and this allows one to use (21) for a complete reduction of all H -subgraphs. This reduction can be performed by applying (21) recursively to those graphs, that still contain the dependence on $\{Q,M\}$. The recursion stops when all terms on the r.h.s. of this expression do not contain any H -subgraphs and all large parameters turn out to be factorized in Δ_R . The final result of this procedure can be represented in the form

$$R_{MS} G = \sum_{\{h\}_H^c} \prod_h C_R(h)\; R_{MS}\left(G/\{h\}_H^c\right) + O(\lambda^{-N}) \quad (22)$$

$$\equiv R_{MS}\left(R_{as}^{(N)} G\right) + O(\lambda^{-N}). \quad (22)$$

Here the sum goes only over complete H -partitions $\{h\}_H^c$. They are complete in a sense that the reduced graphs $G/\{h\}_H^c$ do not contain H -subgraphs. The R_{MS}-operation is assumed to be linear in C_R , because the latter are UV -finite. Coefficients C_R can be expressed via some combinations of Δ_R :

$$C_R(h) = \sum_{h\in F\{\tau\}_H^c} \prod_\tau (-\Delta_R(\bar{\tau}(F))), \quad (23)$$

where $F\{\tau\}_H^c$ is a complete H -forest, that is a forest, minimal elements of which form a complete H -partition of h . This relation is rather useless in practice and we prefer to obtain a recurrent formula for C_R directly from (22). Rewrite the latter equation as

$$R_{MS} G = C_R(G) + R_{MS}\, {}'R_{as} G + O(\lambda^{-N}).$$

Applying the operator t_G to both its sides and recalling (10) we get

$$C_R(G) = t_G R_{MS}((1-{}'R_{as})G) = t_G R_{MS}\Big(G - \sum_{\{h\neq G\}_H^c} \prod_h C_R(h)\, G/\{h\}_H^c\Big) \quad (24)$$

This formula determines $C_R(G)$ via C_R defined on proper subgraphs of G and, hence, allows one to compute $C_R(G)$ recursively starting from h with only one complete H -partition coinciding with h itself, so that $C_R(h) = t_h R_{MS} h$. It is worth noting that the reduction of H -subgraphs can be performed not only in (21), but also in (17) and (13). As a result, functions C_B and C_H arise naturally. They can be related to Δ_B and Δ_H via relations similar to (23) and obey the following recurrent equations:

$$C_B(G) = t_G\Big(R_{MS} G - \sum_{\{h\neq G\}_H^c} \prod_h C_B(h) \sum_{\{u\cap h=\phi\}_{uv}} \prod_u \Delta_{MS}(u)\, G/\{h\}_H^c \cup \{u\}_{uv}\Big);$$

$$C_H(G) = t_G\Big(G - \sum_{\{h\neq G\}_H^c} \prod_h C_H(h)\cdot G/\{h\}_H^c\Big). \quad (25)$$

In turn, functions C_R , C_B , C_H and Δ_H are related to each other (cf. (18) and (20)):

$$C_R(G) = \sum_{G\in F\{\tau\}_H^c} \prod_{\tau\neq\tau_{min}} (-\Delta_{MS}(\bar{\tau}(F))) \prod_{\tau_{min}} C_B(\tau_{min});$$

$$C_B(G) = \sum_{\{u\}_{uv}} \prod_u \Delta_{MS}(u)\cdot C_H(G/\{u\}_{uv}); \quad (26)$$

$$C_H(G) = \sum_{G \in F\{\tau\}^c_H} \prod_{\tau} (-\Delta_H(\bar{\tau}(F))), \tag{26}$$

where τ_{min} are minimal elements of the forest $F\{\tau\}^c_H$. Thus, C_R can be completely expressed in terms of Δ_{MS} and Δ_H.

Formulae (22)-(24) give the solution of the problem formulated at the beginning of this section.

The expansion (22) has the form of an operator expansion. Indeed, its r.h.s. contains the sum of renormalized graphs $G/\{h\}^c_H$. A new vertex formed by shrinking h to a point may be identified with an operator insertion. This identification is possible because $C_R(h)$ is a power series in all external momenta of h except $\{Q\}$ (including some integration momenta of G, see (10)). Every term of this series uniquely corresponds to the Feynman rule generated by some operator vertex, which allows the interpretation in terms of an operator expansion. A more detail treatment of this possibility is given in[17] and in the next section.

3. Expansions of Full Green Functions

Hitherto we have dealt with individual Feynman graphs, whereas in applications it is often necessary to know the form of asymptotic series for complicated objects, such as S -matrices, Green functions, etc., which include the summation over an infinite number of Feynman integrals. To sum up expansions of individual integrals and to cast the results into a form of OPE or ET, one can resort to a very convenient combinatorial technique developed in[10,13]. For our purpose the formulae obtained in[10] are most useful because they are directly applicable to the MS -renormalized quantities (see Appendix 2).

Our consideration will be based on the following observation. Both R_{MS} and R_{as} have the structure of relation (1), i.e., both of them are particular cases of the algebraic construction of the R -operation at a certain choice of $\Delta(g)$. To achieve a complete analogy, it is convenient to extend the summation in eq.(22) to all partitions and to readdress the function of separating only admissible ones (UV -partitions for the R_{MS}-operation and complete H -partitions for the R_{as} -operation) to Δ_{MS} and C_R setting them zero in all wrong cases. In what follows just this possibility is understood.

Eq.(22) can be decomposed into successive actions of R_{as} and R_{MS} :

$$R_{MS} G = R_{MS}(R_{as} G) + O(\lambda^{-N}), \tag{27}$$

where R_{MS} does not affect C_R as has been explained above. Equality (27) permits us to divide the whole problem into two steps: first, one can resolve the combinatorics of the R_{as} -operation and then MS -renormalize the obtained expressions.

Consider a theory described by the interaction Lagrangian $\mathcal{L}_{int}(\varphi)$ and the corresponding generating functional (see eq.(A2.8). Adding to $\mathcal{L}_{int}$ terms $\sum_n J_n O_n$ with the sources $\{J_n\}$ of local composite operators $\{O_n\}$ composed of the fields φ and their derivatives, we obtain the generating functional $G(J, \{J_n\})$ of Green functions of operator products $\prod_\alpha O_\alpha$:

$$\langle T \prod_\alpha O_\alpha(x_\alpha)\, e^{L+J\varphi}\rangle_0 = \left(\prod_\alpha \frac{\delta}{\delta J_\alpha(x_\alpha)}\Big|_{J_n=0}\right) G(J,\{J_n\})$$

$$= \left(\prod_\alpha \int \frac{dK_\alpha}{(2\pi)^D} e^{iK_\alpha x_\alpha} \frac{\delta}{\delta \mathcal{J}_\alpha(K_\alpha)}\Big|_{J_n=0}\right) G(J,\{J_n\}),$$

where $J_\alpha(x_\alpha) = \int dK_\alpha\, e^{-iK_\alpha x_\alpha} \mathcal{J}_\alpha(x_\alpha)$. For applications one usually needs asymptotics of Fourier transorms of expressions like the above ones at some momenta of $\mathcal{J}_\alpha(K_\alpha)$ and masses of φ tending to infinity, so that we should study $G(J,\{J_n\})$ in that limit. The limit corresponds to the situation when Fourier transforms of some sources J_n are localized at large momenta $\{Q\} \to \infty$ and some fields (denote them by ϕ) have large masses $M \sim |Q|$ (see eq.(8)). Usually, it is sufficient to consider Green functions without heavy external particles and we will assume that J corresponds to fields with a finite mass.

Being expanded in J , J_n and coupling constants of $\mathcal{L}_{int}$, the renormalized functional $R_{MS}\, G(J,\{J_n\})$ is representable as a series of Feynman integrals, momenta and masses of which satisfy conditions (8). Therefore, to obtain the asymptotics of $R_{MS}\, G(J,\{J_n\})$ one can apply relation (27) diagram by diagram. Consider at first the effect of the R_{as} -operation. It is a particular case of (1), so that we can use formula (A2.10) with Δ replaced by C_R :

$$R_{as} G(J,\{J_n\}) = \langle T exp\left(C_R\left(Te^{L(\varphi,\phi)+\sum_n J_n O_n + J\varphi} - 1\right)\right)\rangle_0, \tag{28}$$

where the action of C_R is nontrivial only on H -subgraphs and single vertices. Moreover, R_{as} is defined as a sum only over the complete H -partitions. To pick out these partitions, one can demand C_R to be zero on graphs, the contraction of which into a point gives a vertex with at least one hard line. Such vertices correspond to operators containing heavy fields ϕ or carrying large momenta and cannot be produced by shrinking to points graphs of complete H -partitions. Further, we will consider only the case when all large momenta enter a diagram through operator insertions so that $\mathcal{J}$ carries only a finite momentum. As a consequence, all graphs with $\mathcal{J}$ at the ends of external lines are not hard and C_R nullifies them. The only exception is the single vertex $\mathcal{J}\varphi$. Taking into account all these remarks, we get:

$$R_{as}\, G(\mathcal{J},\{\mathcal{J}_n\}) = \langle T\exp\Big(C_R\big(Te^{L(\varphi,\Phi)+\sum_n \mathcal{J}_n O_n} - 1\big) + \mathcal{J}\varphi\Big)\rangle_0 . \qquad (29)$$

The action of C_R contracts graphs to vertices. As has been explained in the previous section, such vertices correspond to insertions of some local operators. Formally, this fact can be expressed by the equation

$$C_R\big(Te^{L(\varphi,\Phi)+\sum_n \mathcal{J}_n O_n} - 1\big) = \sum_r C_R^r(\{\mathcal{J}_n\},M,m,\mu)\, O_r(\varphi) \equiv L_{eff}(\{\mathcal{J}_n\},\varphi), \qquad (30)$$

where C_R^r are CF's; $\{O_r(\varphi)\}$ is a complete set of local operators composed of light fields and their derivatives and μ is the renormalization parameter (see Appendix I). The origin of the dependence of C_R^r on $\mathcal{J}_n$ is the same as that of their dependence on coupling constants. Inserting (30) into (29), we get

$$R_{as}\, G(\mathcal{J},\{\mathcal{J}_n\}) = \langle T\exp\big(L_{eff}(\{\mathcal{J}_n\},\varphi) + \mathcal{J}\varphi\big)\rangle_0 . \qquad (31)$$

The r.h.s. of this relation is the unrenormalized effective generating functional of the theory in the asymptotic region (8) with C_R^r being its effective renormalized couplings. As it is states by eq.(27), it can be renormalized by applying the R_{MS}-operation. Using formulae (A2.11) and (A2.15) with L_{eff} instead of L, we have

$$R_{MS}R_{as}\, G(\mathcal{J},\{\mathcal{J}_n\}) = \langle T\exp\Big(\Delta_{MS}\big(T\exp(L_{eff}(\{\mathcal{J}_n\},\varphi)+\mathcal{J}\varphi) - 1\big)\Big)\rangle_0 ; \qquad (32)$$

$$\Delta_{MS}\big(Te^{L_{eff}+\mathcal{J}\varphi} - 1\big) = \sum_r C_B^r(\{\mathcal{J}_n\},\epsilon,\ldots)\cdot O_r(\varphi) + \mathcal{J}\varphi \equiv L_{eff}^R + \varphi\mathcal{J}, \qquad (33)$$

where C_B^r are "bare" CF's containing, in general, poles in ϵ . Note that we could obtain the same representation (with "bare" CF's) by applying to G($\mathcal{J}$, $\{\mathcal{J}_n\}$) relation (22) expressed in terms of functions C_B (see eqs. (25)).

Thus, we have the following asymptotic expansion

$$R_{MS}\, G(\mathcal{J},\{\mathcal{J}_n\}) = \langle T\exp\big(L_{eff}^R(\{\mathcal{J}_n\},\varphi) + \mathcal{J}\varphi\big)\rangle_0 + o(\lambda^{1-N}) \qquad (34)$$

that solves the problem we have been dealing with.

In a subsequent publication this general formula will be used for deriving some expansions useful in applications.

Acknowledgements. The author is grateful to Profs. V.A. Matveev, V.A.Meshcheryakov, D.V. Shirkov and A.N. Tavkhelidze for continuing support and valuable comments. I thank also K.G. Chetyrkin, S.A. Larin and A.V. Radyushkin for discussions and F.V. Tkachov for critical reading the manuscript.

APPENDIX 1. Parameter μ in the Dimensional Regularization

The usual recipe of the dimensional regularization consists in replacing all four-dimensional integration measures in the momentum space by measures of integration in a formal complex-valued D-dimensional space. Having in view RG applications we should take care of preserving all canonical dimensions and introduce for this purpose some unit of mass μ . Then we define the measure in the momentum space to be

$$\int dp = \mu^{2\epsilon}\int d^D p, \qquad \epsilon = \tfrac{1}{2}(4-D). \qquad (A1.1)$$

Then δ-functions and variational derivatives determined according to (A1.1) should satisfy the following natural relations

$$\int dp\, \delta(p) = 1 ; \qquad \frac{\delta\varphi(p)}{\delta\varphi(q)} = \delta(p-q) \qquad (A1.2)$$

and are given by

$$\delta(p) \equiv \mu^{-2\epsilon}\delta^D(p); \qquad \frac{\delta}{\delta\varphi(q)} \equiv \mu^{-2\epsilon}\frac{\delta^D}{\delta\varphi(q)} , \qquad (A1.3)$$

where the superscript D marks the usual (without μ) definitions of the corresponding quantities.

The position space measure dx can be introduced via the relation

$$(2\pi)^{-D}\int dx\, dp\, e^{ipx} = 1$$

and turns out to be

$$\int dx = \mu^{-2\epsilon}\int d^D x . \qquad (A1.4)$$

Using the above relation and the analogs of (A1.2) for the position space, we have

$$\delta(x) = \mu^{2\epsilon}\delta^D(x); \qquad \frac{\delta}{\delta\varphi(x)} = \mu^{2\epsilon}\frac{\delta^D}{\delta\varphi(x)} . \qquad (A1.5)$$

Definitions (A1.1), (A1.3) and (A1.5) fix automatically the way that μ enters into any dimensionally regularized Feynman integral. In particular, in the p- space each connected component of Green functions is accompanied by the δ^D-function expressing momentum conservation and containing the factor $\mu^{-2\epsilon}$. Then each momentum loop carries $\mu^{2\epsilon}$ in full accordance with conventional rules. Note, however, that the usual rules dealing with δ^D without μ turn out to be rather inconvenient in the context of the functional technique (see Appendix 2). For example, each time we should keep in mind the presence of an extra power of $\mu^{-2\epsilon}$ (with respect to loop ones) per connected component[10] that complicates the rule according to which the operation Δ acts. Further complications may arise when deriving RG -equations. Definitions introduced above are more successful from this point of view, and we use them throughout the paper.

APPENDIX 2. Functional Technique

The material of this appendix is based on[13,10,18].

The quantum expression for the Wick theorem

$$T F_T(\hat{\varphi}) = N F_N(\hat{\varphi}) \tag{A2.1}$$

can be represented in the following functional form

$$F_N(\varphi) = e^{\ell} F_T(\varphi), \tag{A2.2}$$

where $\hat{\varphi}$ and φ are quantum on-shell and classical off-shell fields; F_T and F_N are linear functionals of the field φ of the form

$$F_{T,N}(\varphi) = \sum_n \int F^n_{T,N}(x_1, \cdots, x_n) \cdot \varphi(x_1) \cdots \varphi(x_n),$$

so that $T\,F_T(\hat{\varphi})$ and $N\,F_N(\hat{\varphi})$ represent time and normal ordered functionals, respectively. At last,

$$\ell = \frac{1}{2} \int dx\, dy \frac{\delta}{\delta\varphi(x)} D^c(x,y) \frac{\delta}{\delta\varphi(y)};$$

$$D^c(x,y) = \langle T \hat{\varphi}(x) \hat{\varphi}(y) \rangle_0 = \int \frac{dp}{(2\pi)^D i} \frac{e^{-ip(x-y)}}{m^2 - p^2 - i\epsilon}.$$

Note that all the prescriptions of the previous appendix are operative in the above formulae.

Consider the functional

$$G(J,\varphi) = e^{L(\varphi) + J\varphi}, \tag{A2.3}$$

where $L(\varphi) = i\int L_{int}(\varphi)dx$, $L_{int}(\varphi)$ being an arbitrary polynomial in φ, and $J\varphi = \int dx\, J(x)\varphi(x)$, J being a source of the field φ. The unrenormalized generating functional of the Green functions of the theory is determined by

$$G(J) \equiv G_N(J, \varphi = 0) = \langle T e^{L(\hat{\varphi}) + J\hat{\varphi}} \rangle_0, \tag{A2.4}$$

where G_N, in turn, is given by (A2.2) with $F_T = G(J,\varphi)$.

Using the conventional diagrammatic representation for coefficient functions of $G(J)$, we can introduce the action of the R-operation (1) on G:

$$R\,G(J) = R\left[e^{\ell} G(J,\varphi)\big|_{\varphi=0}\right] = R\langle T G(J,\hat{\varphi})\rangle_0. \tag{A2.5}$$

The combinatorics of the above expression can be explicitly resolved and it turns out that the action of R is equivalent to adding some counterterms to $L(\varphi)$:

$$R\,G(J) = \langle T\exp\left(\Delta\left(T e^{L(\hat{\varphi}) + J\hat{\varphi}} - 1\right)\right)\rangle_0$$

$$= \langle T\exp\left(L(\hat{\varphi}) + J\hat{\varphi} + \Delta\left(T\exp\left(L(\hat{\varphi}) + J\hat{\varphi}\right) - L(\hat{\varphi}) - J\hat{\varphi} - 1\right)\right)\rangle_0. \tag{A2.6}$$

The last line reflects the fact that $\Delta(g) = 1$ for simple vertices.

For the particular case of the UV -renormalization we have $\Delta_{UV}(g) = 0$ if g is not 1PI, so that Δ_{UV} nullifies all graphs containing vertices $J\varphi$ (except the simple vertex $J\varphi$). As a consequence, we get

$$\Delta_{UV}\left(T\exp\left(L(\hat{\varphi}) + J\hat{\varphi}\right) - L(\hat{\varphi}) - J\hat{\varphi} - 1\right) = \Delta_{UV}\left(T e^{L(\hat{\varphi})} - L(\hat{\varphi}) - 1\right)$$

and, for example

$$R_{MS}\,G(J) = \langle T\exp\left(L(\hat{\varphi}) + J\hat{\varphi} + \Delta_{MS}\left(T e^{L(\hat{\varphi})} - L(\varphi) - 1\right)\right)\rangle_0. \tag{A2.7}$$

Replacing $L(\varphi)$ by $L(\varphi) + \sum_i J_i \cdot A_i$ where A_i have a structure similar to $L(\varphi)$ and expanding (A2.7) with respect to J_i, we can obtain renormalized expressions for composite operators and their products, for example,

$$R_{MS}\langle TA(x)e^{L}\rangle_0 = \langle T\Delta_{MS}(TA(x)e^{L})e^{L_R}\rangle_0 ;$$

$$R_{MS}\langle TABe^{L}\rangle_0 = \langle T(\Delta_{MS}(TAe^{L})\Delta_{MS}(TBe^{L}) + \Delta_{MS}(ABe^{L}))e^{L_R}\rangle_0 \quad (A2.8)$$

and so on. Here L_R and Δ_{MS} are local quantities and can be represented as a sum over the set $\{O_n\}$ of monomials composed of φ and its derivatives as follows

$$\Delta_{MS}(Te^{L}-1) \equiv L_R = \sum_n Z_n O_n(\varphi);$$

$$\Delta_{MS}(TAe^{L}) = \sum_n Z_{An} O_n(\varphi); \quad (A2.9)$$

$$\Delta_{MS}(TABe^{L}) = \sum_n Z^n_{AB} \cdot O_n(\varphi).$$

Functions Z in these relations are pure poles in ϵ and polynomial in all external momenta and masses (in particular, Z are independent of μ). Substituting (A2.9) into (A2.8), we have finally

$$R_{MS}\langle TAe^{L}\rangle_0 = \sum_n Z_{An} \langle TO_n e^{L_R}\rangle_0 ; \quad (A2.10)$$

$$R_{MS}\langle TABe^{L}\rangle_0 = \sum_{n,m} Z_{An} Z_{Bm} \langle TO_n O_m e^{L_R}\rangle_0 + \sum_n Z^n_{AB} \langle TO_n e^{L_R}\rangle_0 \quad (A2.11)$$

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Received by Publishing Department
on March 28, 1986.

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Предложен метод построения асимптотических разложений фейнмановских интегралов и полных функций Грина по большим евклидовым импульсам и /или/ массам. Метод основан на свойствах ультрафиолетовой R - операции с перевычитаниями и может быть прямо применен к MS-перенормированным величинам при разложении их по MS-перенормированным операторам.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1986

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A method of constructing asymptotic expansions of individual Feynman integrals and full Green functions in large Euclidean momenta or/and masses is presented. The method is based on properties of the ultraviolet R-operation with oversubtractions and is directly applicable to MS-renormalized quantities, expanding them in terms of MS-renormalized operators.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1986