объединенный
институт
ядерных
исследований
дубна

E2-86-119

P.P.Fiziev, Ts.Ya.Fizieva

THE CLASSICAL THREE-PARTICLE PROBLEM.
A Modification of the Delves Coordinates

Submitted to "Few-Body Systems"

1986

1. Introduction

The recent decades were marked by an intensive use of different versions of the hyperspherical coordinates in the quantum three-body problem (see refs. in paper[1]). One should emphasize an important step first obviously made by Delves[2] and implying a nontrivial renormalization of the Jacobi-particle radius-vectors; this procedure has been used in a number of subsequent papers[3]. In recent years the use of the Delves coordinates led to a considerable advance in the study of the quantum problem of three particles on a straight line interacting through phenomenological two-body potentials.[4]

As far as we know, except for ref.[5], there were no attempts to use these coordinates in studying the classical three-body problem with arbitrary masses m_i and pair potentials V_{ij} proportional to r_{ij}^{-1} where r_{ij} is the distance between particles with numbers $i,j = 1,2,3$.

Probing of the coordinates is to some extent easier within a classical version of the problem . It results in a further modification of coordinates and may provide some new results in the classical region, which are interesting themselves. Most of these results may also be transformed to a quantum version of the problem. It is also useful to study the classical problem for constructing a quasiclassical approximation and a path integral for the quantum case. We intend to study these problems in a sequel of papers beginning from this one.

2. Laboratory and moving frames of reference

In the laboratory reference frame Σ^{ℓ} after introducing the Jacobi coordinates (fig.1) the kinetic energy of the three-particle system is

$$T = \frac{1}{2}\mu_{1,2}\dot{\vec{R}}^2 + \frac{1}{2}\mu_{12,3}\dot{\vec{r}}^2 + \frac{1}{2}M\dot{\vec{R}}_{cm}^2 , \tag{1}$$

where

$$\mu_{1,2}^{-1} = m_1^{-1} + m_2^{-1} , \quad \mu_{12,3}^{-1} = (m_1+m_2)^{-1} + m_3^{-1} \tag{2}$$

are the Jacobi quasiparticle masses, and M is the system total mass.

The moving reference frame Σ is introduced so that in it

$$\vec{R}_{cm} = 0 , \tag{3}$$

$$\vec{R}R^{-1} = \vec{k} , \tag{4}$$

where $\vec{k} = (0,0,1)$ is the unit vector towards a positive direction of the axis OZ . At the angular velocity $\vec{\omega}$ of rotation Σ with respect to Σ^{ℓ} we have

$$T = \frac{1}{2}\mu_{1,2}\left[\dot{R}^2 + R^2(\vec{\omega}\times\vec{k})^2\right] + \frac{1}{2}\mu_{12,3}(\dot{\vec{r}} + \vec{\omega}\times\vec{r})^2 . \tag{5}$$

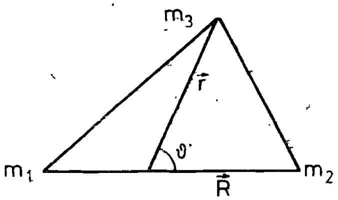


Fig.1. The triangle $\Delta(m_1, m_2, m_3)$.

We shall describe motion of the three-body system in the standard way (see, e.g., refs.[1,5,6]) using the solid state motion (translation and rotation of the triangle $\Delta(m_1, m_2, m_3)$ added by deformation (of the same triangle).

3. Coordinates of the triangle deformation

The triangle deformation $\Delta(m_1, m_2, m_3)$ will be described by the coordinates $(\rho, \varphi, \vartheta)$ introduced according to the formulae

$$R = \left(\frac{\mu}{\mu_{1,2}}\right)^{\frac{1}{2}} \rho \cos\varphi \qquad \in (-\infty, \infty) , \tag{6}$$

$$r = \left(\frac{\mu}{\mu_{12,3}}\right)^{\frac{1}{2}} \rho \sin\varphi \qquad \in (-\infty, \infty) , \tag{7}$$

$$\cos\vartheta = \vec{R}\cdot\vec{r}\,(Rr)^{-1} \qquad \in [-1, 1] , \tag{8}$$

where

$$\rho \in [0, \infty) , \quad \varphi \in [0, 2\pi) , \quad \vartheta \in [0, \pi] . \tag{9}$$

Our choice of a region of values of the variables differs from the standard one.[1-4] Its advantages are a continuous description of the whole space of variables $\{R, r\} = \mathbb{R}^{(2)}$, which is essential for a continuous description of all configurations of the system while moving along a straight line. Moreover, the angle φ differs from the Delves angle

$$\alpha = \frac{\pi}{2} - \varphi , \tag{10}$$

which turns out to be convenient in what follows.

For brevity we introduce the notation

$$s = \sin\varphi\ , \quad c = \cos\varphi\ , \quad \vec{\gamma} = \vec{r}\, r^{-1} . \tag{11}$$

Then, the kinetic energy in new variables acquires the form

$$T = \frac{\mu}{2}\{\dot{\rho}^2 + \rho^2[\dot{\varphi}^2 + s^2(\dot{\vec{\gamma}} + \vec{\omega}\times\vec{\gamma})^2 + c^2(\vec{\omega}\times\vec{k})^2]\} . \tag{12}$$

One can easily derive a formula for a moment of inertia $I_{\alpha\beta}$ of the system refs.[3,5]

$$I = tr\,\| I_{\alpha\beta} \| = \mu\rho^2 . \tag{13}$$

Simplicity of expressions (12) and (13) in comparison with the relevant ones in other variables (see, e.g., refs.[8,9]) enables one to understand better the meaning of the hyperradius ρ and still uncertain parameter μ : the reduced three-particle system mass.

4. Coordinates of the triangle rotation

To describe the triangle rotation $\Delta(m_1, m_2, m_3)$ as a solid state, we use the standard Euler angles Φ, Ψ, θ [7]. Then

$$\begin{aligned} \omega_x &= \dot{\Phi}\sin\theta\sin\Psi + \dot{\theta}\cos\Psi ,\\ \omega_y &= \dot{\Phi}\sin\theta\cos\Psi - \dot{\theta}\sin\Psi ,\\ \omega_z &= \dot{\Phi}\cos\theta + \dot{\Psi} . \end{aligned} \tag{14}$$

Supplementing the definition of the system Σ by the requirement that $\vec{\gamma}$ lies in the plane OXZ at an angle ϑ to OZ, i.e. $\vec{\gamma} = (\sin\vartheta, 0, \cos\vartheta)$.

Then

$$(\vec{\omega}\times\vec{k})^2 = \dot{\theta}^2 + \dot{\Phi}^2\sin^2\theta , \tag{15}$$

$$(\dot{\vec{\gamma}} + \vec{\omega}\times\vec{\gamma})^2 = (\dot{\vartheta} + \omega_y)^2 + (\omega_x\cos\vartheta - \omega_z\sin\vartheta)^2 . \tag{16}$$

5. Hamiltonian variables and the total angular momentum of the system

Performing a standard transition from generalised velocities $\{\dot{\rho}, \dot{\varphi}, \dot{\vartheta}, \dot{\Phi}, \dot{\Psi}, \dot{\theta}\}$ to generalised momenta $\{p_\rho, p_\varphi, p_\vartheta, p_\Phi, p_\Psi, p_\theta\}$,

we get for the kinetic energy of the system

$$T = (p_\rho^2 + \rho^{-2}\Lambda^2)/2\mu , \tag{17}$$

where

$$\Lambda^2 = p_\varphi^2 + s^{-2}L_1^2 + c^{-2}L_2^2 \tag{18}$$

is a large angular momentum of the system and

$$L_1^2 = c^{-2}p_\vartheta^2 + \sin^{-2}\vartheta\, p_\Psi^2 , \tag{19}$$

$$\begin{aligned} L_2^2 = {} & \sin^{-2}\theta\, p_\Phi^2 + (ctg^2\theta - 2\sin\Psi\, ctg\theta\, ctg\vartheta + ctg^2\vartheta)p_\Psi^2 + p_\theta^2 + \\ & + 2\sin^{-1}\theta(\sin\Psi\, ctg\vartheta - ctg\theta)p_\Phi p_\Psi - 2\sin^{-1}\theta\cos\Psi\, p_\Phi p_\vartheta + 2\cos\Psi\, ctg\theta\, p_\Psi p_\vartheta + \\ & + 2\cos\Psi\, ctg\vartheta\, p_\Psi p_\theta + 2\sin\Psi\, p_\theta p_\vartheta . \end{aligned} \tag{20}$$

For the total mechanical moment of the system $\vec{K}$:

$$\vec{K} = \vec{K}_{1,2} + \vec{K}_{12,3} = \mu_{1,2}\vec{R}\times(\dot{\vec{R}} + \vec{\omega}\times\vec{R}) + \mu_{12,3}\vec{r}\times(\dot{\vec{r}} + \vec{\omega}\times\vec{r}) \tag{21}$$

we get in the system Σ :

$$\begin{cases} K_x = \cos\Psi\, p_\theta + \sin^{-1}\theta\sin\Psi(p_\Phi - p_\Psi\cos\theta) ,\\ K_y = -\sin\Psi\, p_\theta + \sin^{-1}\theta\cos\Psi(p_\Phi - p_\Psi\cos\theta) ,\\ K_z = p_\Psi , \end{cases} \tag{22}$$

and in the system Σ^e :

$$\begin{cases} K_{xe} = \sin\Phi\, p_\theta + \sin^{-1}\theta\cos\Phi(\cos\theta\, p_\Phi - p_\Psi) ,\\ K_{ye} = -\cos\Phi\, p_\theta + \sin^{-1}\theta\sin\Phi(\cos\theta\, p_\Phi - p_\Psi) ,\\ K_{ze} = p_\Phi \end{cases} \tag{23}$$

It is seen that the components of $\vec{K}$ are independent of the deformation coordinates $(\rho, \varphi, \vartheta)$ and its square is

$$K^2 = p_\theta^2 + p_\Psi^2 + \sin^{-2}\theta(p_\Phi - \cos\theta\, p_\Psi)^2 = p_\theta^2 + p_\Phi^2 + \sin^{-2}\theta(p_\Phi\cos\theta - p_\Psi)^2 . \tag{24}$$

6. Particle interaction potentials

Consider systems of three particles with the potential

$$V = V_{12} + V_{23} + V_{31} , \tag{25}$$

where the two-particle potentials V_{ij} are

$$V_{ij} = r_{ij}^{-1}\alpha_{ij} ; \qquad \alpha_{ij} = \begin{cases} e_i e_j & : \text{for electricity} \\ -m_i m_j & : \text{for gravitation} \end{cases} \tag{26}$$

The distances r_{ij} between particles in our variables are

$$\begin{cases} r_{12} = \left(\frac{\mu}{\mu_{1,2}}\right)^{\frac{1}{2}} \rho |c| , \\ r_{23} = \left(\frac{\mu}{m_1+m_2}\right)^{\frac{1}{2}} \rho \sqrt{s^2 M/m_3 - 2\sqrt{m_1 M/m_2 m_3}\, s c \cos\vartheta + c^2 m_1/m_2} \\ r_{31} = \left(\frac{\mu}{m_1+m_2}\right)^{\frac{1}{2}} \rho \sqrt{s^2 M/m_3 + 2\sqrt{m_2 M/m_1 m_3}\, s c \cos\vartheta + c^2 m_2/m_1} . \end{cases} \tag{27}$$

Hence we get

$$V = \rho^{-1}\alpha(\varphi, \vartheta) , \tag{28}$$

where

$$\alpha(\varphi,\vartheta) = \sum_{i<j} (\rho r_{ij}^{-1}\alpha_{ij}) \tag{29}$$

depends only on the deformation angles φ and ϑ .

7. Reduction of the problem to the system with four degrees of freedom

For the system with the Hamiltonian

$$H = T + V , \tag{30}$$

where T is defined by expressions (17)-(20) and V by expressions (28), (29), the total moment $\vec{K}$ is conserved, i.e.,

$$[\vec{K}, H] = 0 , \tag{31}$$

where is the Poisson bracket. The reduction of the problem based on this fact goes back to the papers by Lagrange, Laplace and Jacobi (see refs.[7,9]). Taking into account that Φ is a cyclic coordinate we choose the system Σ^ℓ so that in it

$$\vec{K} = (0,0,K) . \tag{32}$$

Under such a choice the plane $0X^\ell Y^\ell$ coincides with the fixed Laplace plane, Ψ is the angle between this plane and that of the triangle $\Delta(m_1,m_2,m_3)$ and p_Φ =K=const. For various possibilities ($\sin\theta \neq 0$, or $\sin\theta = 0$), from eqs. (23) and (32) we get several types of motion:

I. Motions with $K \neq 0$ when, according to Weierstrass's-Sundman's theorem, the triple collision is impossible.

1. The general case of the three-dimensional motion of particles. Then, from (23) and (32) we find

$$L_2^2 = K^2 - p_\Psi^2 + 2\sqrt{K^2 - p_\Psi^2}\,(\sin\Psi \,\mathrm{ctg}\vartheta\, p_\Psi - \cos\Psi\, p_\vartheta) + \mathrm{ctg}^2\vartheta\, p_\Psi^2 , \tag{33}$$

$$T = \frac{1}{2\mu}\Big\{p_\rho^2 + \rho^{-2}p_\varphi^2 + \rho^{-2}s^{-2}(p_\vartheta^2 + \sin^{-2}\vartheta\, p_\Psi^2) + \rho^{-2}c^{-2}\big[(\cos\Psi\sqrt{K^2-p_\Psi^2} - p_\vartheta)^2 + (\sin\Psi\sqrt{K^2-p_\Psi^2} + \mathrm{ctg}\vartheta\, p_\Psi)^2\big]\Big\} . \tag{34}$$

Using the components of the vector $\vec{\ell}$ of the angular momentum of particle m_3 with respect to the c.m. of the system m_1,m_2:

$$\ell_1 = -\sin\Psi\, p_\vartheta - \mathrm{ctg}\vartheta\cos\Psi\, p_\Psi , \quad \ell_2 = \cos\Psi\, p_\vartheta - \mathrm{ctg}\vartheta\sin\Psi\, p_\Psi , \quad \ell_3 = p_\Psi ,$$

$$\ell^2 = p_\vartheta^2 + \sin^{-2}\vartheta\, p_\Psi^2 ,$$

we may express L_1^2 , L_2^2 and Λ^2 through $\vec{\ell}$:

$$L_1^2 = c^{-2}\ell^2 - s^2c^{-2}\sin^{-2}\vartheta\,\ell_3^2 , \qquad L_2^2 = K^2 - \ell_3^2 + \mathrm{ctg}^2\vartheta\,\ell_3^2 - 2\ell_2\sqrt{K^2-\ell_3^2} ,$$

$$\Lambda^2 = p_\varphi^2 + s^{-2}\ell^2 + c^{-2}[\ell_1^2 + (\sqrt{K^2-\ell_3^2} - \ell_2)^2] . \tag{35}$$

It should be noted that in this case $\cos\theta = p_\Psi/p_\Phi$ so that the degrees of freedom connected with the angles Φ and θ are completely eliminated from further consideration.

2. The case of plane motion. Then

$$p_\Psi = 0 , \quad \Psi = 0 , \tag{36}$$

so that the triangle $\Delta(m_1,m_2,m_3)$ remains always in the fixed Laplace plane. In this case $L_1^2 = p_\vartheta^2/c^2$, $L_2^2 = K^2 - 2Kp_\vartheta$ and

$$T = \frac{1}{2\mu}\Big\{p_\rho^2 + \rho^{-2}[p_\varphi^2 + s^{-2}p_\vartheta^2 + c^{-2}(K - p_\vartheta)^2]\Big\} . \tag{37}$$

3. The collinear motion at $K \neq 0$. It turns out that in this case the system has only one degree of freedom since $p_\theta = 0, \theta = \frac{\pi}{2}$, $p_\vartheta = K \sin^2 \varphi^{(K)}$, $\vartheta = 0, \pi$; $p_\varphi = 0$, and $\varphi^{(K)} = const$ is the solution of the equation

$$\frac{\partial}{\partial \varphi} \alpha(\varphi, \vartheta = 0, \pi) = 0 . \qquad (38)$$

Expressions for the kinetic energy include a potential of the centrifugal type

$$T = \frac{1}{2\mu} p_\rho^2 + \frac{1}{2\mu} \rho^{-2} K^2 , \qquad (39)$$

and the potential energy of the system is $V = \rho^{-1} \alpha^{(K)}$, where $\alpha^{(K)} = const$.

II. Motions with K≡0 when, according to Wierstrass's-Sundman's theorem, the triple collision is possible.

1. Plane motion. Then $L_z^2 = 0$ and

$$T = \frac{1}{2\mu} (p_\rho^2 + \rho^{-2} \lambda^2) , \qquad (40)$$

where

$$\lambda^2 = p_\varphi^2 + s^{-2} c^{-2} p_\vartheta^2 . \qquad (41)$$

2. Motion on a fixed straight line. In contrast with the collinear case with K≠0, now the system has two degrees of freedom and the relations

$$p_\vartheta = 0 \quad , \; \vartheta = 0 , \pi \qquad (42)$$

result in the simplest expression for the kinetic energy

$$T = \frac{1}{2\mu} (p_\rho^2 + \rho^{-2} p_\varphi^2) . \qquad (43)$$

It is to be noted that additional conditions of the type (36), (38) and (42) are not constrains in the phase space of the system but are the dynamic conditions limiting the choice of initial data. Their fulfillment results in special classes of solutions of the general three-particle problem.

8. The reduced mass of the three-particle system

a) at V≡0 the Hamiltonian of the three-particle system, like all relevant physical quantities are invariant with respect to the changes (at $\kappa > 0$)

$$\rho \longrightarrow \kappa \rho \quad , \quad \mu \longrightarrow \kappa^{-2} \mu , \qquad (44)$$

that leads to a physical uncertainty of the scales ρ and μ taken independently. In this connection Delves[2] (see also ref.[3]), basing on formal reasonings (simplicity of normalization of the wave function), proposed to fix μ by the formulae

$$\mu = \mu_D = \sqrt{m_1 m_2 m_3 / (m_1 + m_2 + m_3)} \qquad (45)$$

In ref.[10] another reduced mass was suggested

$$\mu = \mu_L = m_1 m_2 m_3 / (m_1 + m_2 + m_3)^2 . \qquad (46)$$

In literature, the choice of a value of μ is thought to be a nonphysical problem[1,2]: this is verified by the arguments concerning the moment of inertia of the system. The identity

$$r_{12}^2/\mu_3 + r_{23}^2/\mu_1 + r_{31}^2/\mu_2 = \rho^2 \mu M M^* / m_1 m_2 m_3 ,$$

where $\mu_i = m_i / M^*$ and M^* is some normalization mass allowing, without limitation on μ to require

$$\mu M M^* = m_1 m_2 m_3 .$$

Then, under normalization $M^* = \mu$ we get (45) and under normalization $M^* = M$ (46). The choice of μ is still formal problem.

b) At the electrical two-particle interaction $V = V^{(e)}$ the Hamiltonian (30) is invariant with respect to the changes

$$\rho \longrightarrow \kappa \rho , \quad \mu \longrightarrow \kappa^{-2} \mu , \quad e \longrightarrow \kappa^{\frac{1}{2}} e . \qquad (47)$$

There is such an invariance in the N-particle problem at any $N \geqslant 2$. However, we are not free of changing the physical meaning of charge determined from the two-particle problem. Consequently, the energy values $E = H$ of the three-particle problem allow one to fix a physically correct value of μ at the value of charge e determined from the two-particle problem.

c) Masses μ_D and μ_L behave differently as one of $m_{1,2,3}$ tends to ∞. For instance, as $m_3 \to \infty$, $\mu_D \to \sqrt{m_1 m_2}$, whereas $\mu_L \to 0$ that is inadmissible from the physical point of view. This property can be used while choosing a correct expression for μ.

Consider a system like He^4 with two light masses $m_1 = m_2 = m$ and one heavy mass $m_3 = M \gg m$. Then

$$\mu_D = m (1 + 2m/M)^{-\frac{1}{2}} = m[1 - (m/M) + 3(m/M)^3 - \dots] \qquad (48)$$

differs considerably from

$$\mu_L = m(m/M)(1+2m/M)^{-2} = m(m/M)(1+\cdots) ,$$

and one can easily compare their agreement with the experimental data.

Under these conditions μ_0 with an accuracy up to $O_2(m/M)$ coincides with the two-particle reduced mass

$$\bar{\mu} = mM/(m+M) = m[1-(m/M)+(m/M)^2-\cdots] . \qquad (49)$$

Comparison with the experimental data allows a unique choice of mass μ as a proper expression for μ_0 .*)

---

*) Since the data are available for the quantum three-particle problem, we shall only outline in this paper some reasonings. Consider a special class of motions for which $\varphi \equiv 0$ ($\langle\hat{\varphi}\rangle \equiv 0$ for the quantum case). Then, the heavy mass M remains all the time in the c.m.s. of m_1, m_2 (fig.2), that is well fulfilled in He^4 for the states of two electrons properly chosen (without taking into account spin).

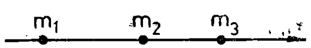


Fig.2. The case $\varphi \equiv 0$.

From (27) - (29) it is seen that $V = \gamma e^2/\rho$ where $\gamma = const \sim Z = 2$, i.e. the problem becomes hydrogen-like. Neglecting the contribution of the angular degrees of freedom to zero level of the system, we get an approximate estimate for the ionization potential of He^4

$$E \sim \gamma^2 e^4 \mu .$$

On the other hand, by the perturbation theory for He^4 12/, we get in the zeroth order

$$\Delta E^{(0)} = Z^2 e^4 \bar{\mu}/2 ,$$

that is in agreement with the experimental data with sufficient for our aims accuracy.

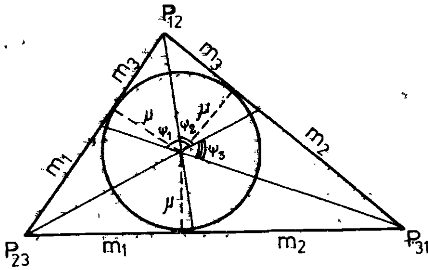
Since for He^4 μ_0 and μ_L differ by a factor of four and μ_0 coincides with $\bar{\mu}$ with an accuracy of 10^{-4} according to (48) and (49), then it becomes evident that the experimental data agree with the choice $\mu = \mu_0$.

9. The triangle of masses

Based on formula (45) we introduce the following geometric construction, the triange $\Delta(P_{12}, P_{23}, P_{31})$, fig.3. It is uniquely determined by the masses m_i that define its sides

$$P_{31}P_{23} = m_1 + m_2 , \quad P_{12}P_{31} = m_2 + m_3 , \quad P_{23}P_{12} = m_3 + m_1 .$$

Fig.3. The triangle $\Delta(P_{12}, P_{23}, P_{31})$.



As $m_i \geq 0$, these sides satisfy inequality for the triangle and $\Delta(P_{12}, P_{23}, P_{31})$ always exist.

Now we determine three angles $\psi_{1,2,3}$ by the formula

$$\psi_i = arctg(m_i/\mu) \in [0, \tfrac{\pi}{2}] \qquad (50)$$

and assume

$$t_i = tg\psi_i , \quad s_i = sin\psi_i , \quad c_i = cos\psi_i . \qquad (51)$$

Then from (45) and the summation rule of tangents, one can easily obtain

$$\psi_1 + \psi_2 + \psi_3 = \pi . \qquad (52)$$

It is clear from the elementary geometry that ψ_i are the angles between bisectors in $\Delta(P_{12}, P_{23}, P_{31})$. The reduced mass μ is the radius of the circle inscibed into $\Delta(P_{12}, P_{23}, P_{31})$ - fig.3.

This construction can easily be generalised to the problem with N particles when the triangle is changed by a figure in the relevant multidimensional space (tetrahedron, etc.) and the cicrle by the inscribed (hyper) sphere with radius $\mu = M(\frac{m_1}{M} \cdots \frac{m_N}{M})^{1/(N-1)}$ 2/, where $M = m_1 + \cdots + m_N$.

With the quantities introduced we can write in a simpler form the distances r_{ij} between particles

$$\begin{cases} r_{12} = \rho (s_3/s_1 s_2)^{\frac{1}{2}} \, |c| \\ r_{23} = \rho (s_1/s_2 s_3)^{\frac{1}{2}} \sqrt{s_2^2 s^2 - 2 s_2 c_2 s c \cos\vartheta + c_2^2 c^2} \\ r_{31} = \rho (s_2/s_1 s_3)^{\frac{1}{2}} \sqrt{s_1^2 s^2 + 2 s_1 c_1 s c \cos\vartheta + c_1^2 c^2} \end{cases} \qquad (53)$$

as well as the quantity $\alpha(\varphi,\vartheta)$ from (28), (29)

$$\alpha = \beta_{12}|c|^{-1} + \beta_{23}(s_2^2 s^2 - 2 s_2 c_2 s c \cos\vartheta + c_2^2 c^2)^{-\frac{1}{2}} + \beta_{31}(s_1^2 s^2 + 2 s_1 c_1 s c \cos\vartheta + c_1^2 c^2)^{-\frac{1}{2}}, \quad (54)$$

where

$$\beta_{ij} = (s_i s_j / s_k)^{\frac{1}{2}} \alpha_{ij} \quad ; \; i \neq j \neq k \neq i \, . \qquad (55)$$

Simpe exact relations of the geometric nature (50)-(55) are a new evidence in favour of the choice of μ in the form of (45).

10. Comparison with other choices of hyperspherical angles

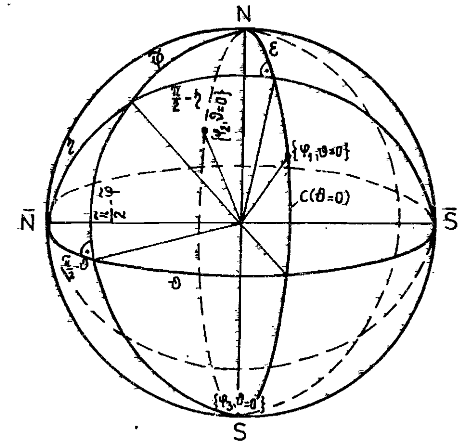
We have arrived at a version of hyperspherical angles, that somewhat differs from those described in literature (see refs.[1,3,5,6] and refs. therein). We consistently kept the description with a separated particle m_3 both in choosing the system of coordinates Σ and Euler angles $\{\Phi, \Psi, \Theta\}$ and in choosing the angles $\{\vartheta, \varphi\}$ for the description of the triangle deformation. Our expressions for the kinetic energy of the system are more convenient for solving the classical problem as they do not lead to discontinuous changes of angles described in ref.[5] This will be shown elsewhere.

The introduction of angular hyperspherical coordinates for the classical problem of three particles democratic with respect to three particles is presented in ref.[5] It is based on a modification of hyperspherical coordinates from ref.[3] and differs from ours both by the choice of a rotating system of coordinates and the choice of the triangle deformation angles.

For a detailed comparison of the two versions of the choice of the triangle deformation angles, it is convenient to represent them as angles on the two-dimensional sphere $\mathfrak{S}^{(2)}$ (fig.4). In order to put points $\{\varphi = 0, \frac{\pi}{2} \; ; \; \vartheta$ - arbitrary$\}$ (see eqs. (6)-(9) and fig. 1). in the poles of the sphere $\mathfrak{S}^{(2)}$, one should double the angle φ, i.e., pass to the angle

$$\tilde{\varphi} = 2\varphi \, .$$

Fig.4. The sphere $\mathfrak{S}^{(2)}$



Then, with varying in the interval $\tilde{\varphi} \in [0, 2\pi]$ the point on $\mathfrak{S}^{(2)}$ moves along the circumference which in its turn covers up once the sphere with varying ϑ in the interval $[0, \pi]$. The choice of (9) for $\varphi \in [0, 2\pi]$ leads to $\tilde{\varphi} \in [0, 4\pi]$, i.e., to the twofold covering of the sphere. This results in a continuous description of configurations of the system and as we shall see elsewhere corresponds to the construction of some two-sheet Riemann surface for the classical motion.

On the sphere $\mathfrak{S}^{(2)}$ there is a physically separated vertical circumference $C \; (\vartheta \equiv 0)$ corresponding to collinear configurations of particles. On C there are three points with angles $\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3$ that correpont to pair collisions of particles.

Under our choice of angles $\{\tilde{\varphi}, \vartheta\}$ the poles N and S spheres are on the circumference C as is shown in fig.4.

A democratic introduction of angles on the sphere $\mathfrak{S}^{(2)}$ corresponds to the rotation of axis NS to $\bar{N}\bar{S}$, i.e., the introduction of new angles $\{\eta, \varepsilon\}$ by formulae (see fig.4)

$$\cos 2\varphi = \sin\eta \cos\varepsilon \quad , \quad \cos\eta = \sin\vartheta \sin 2\varphi \, .$$

They are simply related with angles of the triangle deformation in refs.[1,3,5,13] and lead to

$$\alpha = \sum_{k=1}^{3} \beta_{ij} \sqrt{2} \left[1 - \sin\eta \cos(\varepsilon - \tilde{\varphi}_k)\right]^{-\frac{1}{2}}$$

for the function α from (28). A drawback of such a choice is the occurrence of nonphysical coordinate singularities at points

$\bar{N}, \bar{S}$ on $S^{(2)}$ with respect to which the circumpherence of collinear configurations $\mathcal{C}$ becomes the equator.

In a subsequent paper we shall apply the afore-mentioned results to a further study of the classical and quantum three-particle problem, in particular, the motion along the straight line when all dependences are extremely simplified.

Acknowledgement

The authors are deeply indebted to A.V.Matveenko for numerous discussions and elucidation of various questions in the three-particle problem the study of which was initiated by an interesting paper /14/. Thanks are also due to B.M. Barbashov, A.D.Donkov, V.B.Belyaev, Ya.A.Smorodinsky and V.A.Mescheryakov for fruitful discussions, help and support in preparing this paper.

References

1. Klar H.: J.Math.Phys. 26, 1621 (1985).
2. Delves L.M.: Nucl.Phys. 9, 391 (1959); 20, 275 (1960).
3. Smith F.T.: Chem.Phys. 31, 1352 (1959); Phys.Rev. 120, 1058 (1960); J.Math.Phys. 3, 735 (1962); Whitten R.C., Smith F.T.: J.Math.Phys. 9, 1103 (1968).
4. Hanke G., Manz J., Romelt J.: J.Chem.Phys. 73, 5040 (1980); Manz J., Shor H.: Chem.Phys.Lett. 107, 542 (1984).
5. Johnson B.R.: J.Chem.Phys. 79, 1906 (1983); 73, 5055 (1980).
6. Smorodinski Ya.A., Efros V.D.: Sov.J.Nucl.Phys. 76, 107 (1973); Kupperman A.: Chem. Phys.Lett. 32, 374 (1975); Schatz G.C., Kupperman A.: J.Chem.Phys. 65, 4642 (1976); Soloviev E.A., Vinitsky S.I.: J.Phys. B18, 557 (1985).
7. Landau L.D., Lifschiz E.M.: Mechanics. Moscow 1958 (in Russian)
8. Whittaker E.T.: A Treatise on the Analytical Dynamics. Cambridge 1965.
9. Zare K.: Celestial Mechanics, 24, 345 (1981).
10. Pars L.A.: A Treatise on Analytical Dynamics. London 1964.
11. Linderberg J.: in: New Horizons of Quantum Chemistry, Dordrecht-Boston-London: Redel Publ. Comp. 1983.
12. Bethe H., Salpeter E.: Quantum Mechanics of One- and Two-Electron Atoms, Berlin-Gottingen-Heidelberg, Springer 1957.
13. Hori S.: Suppl. Progr. Theor.Phys., Extra Number, 80 (1968).
14. Matveenko A.V.: Phys.Lett. B129, 11 (1983).

Received by Publishing Department
on February 28, 1986.

Физиев П.П., Физиева Ц.Я. E2-86-119
Классическая задача трех частиц.
Модификация координат Дельвеса

Рассматривается новый вариант гиперсферических координат в задаче трех частиц, который приводит к модификации координат Дельвеса. Получен вид кинетической и потенциальной энергии системы. Для разных случаев классического движения проводится редукция задачи в этих координатах. Рассматриваются свойства гамильтониана при масштабных преобразованиях. Из физических соображений отобрана формула для приведенной массы трехчастичной системы. Вводится треугольник масс и связанные с ним основные величины и соотношения.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1986

Fiziev P.P., Fizieva Ts.Ya. E2-86-119
The Classical Three-Particle Problem.
A Modification of the Delves Coordinates

A new version of the hyperspherical coordinates in the classical three-particle problem is considered, that leads to a modification of the Delves coordinates. A type of the kinetic and potential energy is obtained for the system. The problem is reduced in these coordinates for different cases of the classical motion. The Hamiltonian properties are considered under scale transformations. From physical reasonings a formula is chosen for the reduced mass of Three-body system. The triangle of masses and the relevant basic quantities and relations are introduced.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1986