ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

ДУБНА

E2 - 8594

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1975

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Submitted to *Physics Letters*

Of late, there has been an obvious progress in understanding the relativistic quark model. It has been proved that even for noninteracting quarks the so-called current and constituent quarks are different objects [1]. For noninteracting quarks the exact transformation connecting constituent and current quarks (hereafter referred to as "constituent and current representations") has been constructed [2]. The current representation is used, for example, for phenomenological description of the deep-inelastic scattering of electrons on the polarized target [3]. But it must be pointed out that the approach very close to the current quarks was used in 1967 in the framework of the model of the quasi-independent quarks [4,5] and in Ref. [6], where the mixture of different SU(6) representations was studied.

In this paper we study the question of the dynamics of current and constituent representations. We shall construct the equations for the wave functions in constituent and current representations. Solving these equations (for example, for "oscillator potentials") one gets the transformation between current and constituent representations for interacting quarks.

The wave functions in the constituent representation are determined from the following equation:

$$(\ell^{(1)2} + \ell^{(2)2} + \ell^{(3)2} - M^2 - U)\, \Psi^{con}(\ell^{(1)}, \ell^{(2)}, \ell^{(3)}) = 0 \qquad (1)$$

and the Bargman-Wigner equations:

$$P\gamma^{(1)}\Psi^{con.} = P\gamma^{(2)}\Psi^{con.} = P\gamma^{(3)}\Psi^{con.} = M\Psi^{con.}, \qquad (2)$$

where M stands for the baryon mass, $\ell^{(i)}$ and P are the i -th quark and baryon momenta, $\gamma^{(i)}_{\mu}$ is the i -th quark γ -matrix, U is the potential binding quarks into the baryon. The Bargman-Wigner equations determine the spinor parts of the wave function and Eq. (1) determines the relative motion of quarks.

For many applications it is convenient to use /7/ the harmonic oscillator potential:

$$U = \Omega \sum_{i<j} (x^{(i)} - x^{(j)})^2 + m^2 , \qquad (3)$$

Ω and m are the parameters of the potential. They are connected with the parameters α_0 and α' of the Regge trajectory.

We shall remove the relative times from the bound states. Covariantly this can be done by using the solutions of Eq. (1) on the hyperplanes /8,9/:

$$P(\ell^{(i)} - P/3) = 0. \qquad (4)$$

The conditions (4) are called the Markov-Yukawa ones. The wave functions are now normalized by:

$$\int \bar{\Psi}(\ell^{(1)},\ell^{(2)},\ell^{(3)})\Psi(\ell^{(1)},\ell^{(2)},\ell^{(3)})\delta[P(\ell^{(2)}-P/3)] \times$$
$$\times \delta[P(\ell^{(3)}-P/3)]\delta(\ell^{(1)}+\ell^{(2)}+\ell^{(3)}-P)\,d\ell^{(1)}d\ell^{(2)}d\ell^{(3)} = \text{const.} \qquad (5)$$

The Markov-Yukawa conditions have been used for a number of physical problems /9,10/.

From now on we shall eliminate the relative times from the equations (not only from the solutions). From Eq. (1) in the c.m. rest system one gets the following equation

$$[m^2 + \frac{2}{3}M^2 + \vec{\ell}^{(1)2} + \vec{\ell}^{(2)2} + \vec{\ell}^{(3)2} - \Omega \sum_{i<j} (\vec{x}^{(i)} - \vec{x}^{(j)})^2]\Psi^{con}_{\vec{P}=0} = 0 \qquad (6)$$

if one omits the relative times $(\vec{x}^{(i)} - \vec{x}^{(j)})_0$ in (3), and for the time components of the quark momenta one uses M/3 following from the Markov-Yukawa conditions. The four-component vector $\mathring{a}$ stands for the vector a in the c.m. rest system. To get the wave function in the arbitrary frame one must apply the Lorentz transformation $U_{\vec{P}\neq 0}$ to $\Psi^{con}_{\vec{P}=0}$. This approach is very similar to the approach in Ref.[11].

Using this approach we inevitably come across the well-known difficulties with quark statistics. We think that this question can be solved using the three quark triplets[12].

Since the Bargman-Wigner equations (2) contain only the c.m. momentum P, the spinor parts of the wave function in this approach depend only on P. That is, if the wave function of the relative quark motion is the eigenfunction of the squared orbital momentum operator $\vec{L}^2$, the wave function is also the eigenfunction of $\vec{L}^2$ with the same eigenvalue $\ell(\ell+1)$. The baryon spin operator is $\vec{j}=\vec{L}+\frac{1}{2}\sum\vec{\sigma}^{(i)}$. Using the Clebsch-Gordan coefficients one can simply construct the wave function with definite quantum numbers ℓ, j, j_3. The reason why we have called the solutions of Eqs. (1) (or (6)) and (2) the constituent representation is that in this case in the c.m. rest frame the wave function has a definite value of orbital momentum.

But one can write the new equations which do not admit the solutions with definite angular momentum, i.e., the wave function is the mixture of the wave functions with different ℓ. These are the equations for the wave functions in the current representation. Later on, we shall use the equations of the quasi-independent quark model[4]:

$$(\ell^{(i)}\gamma^{(i)} - V^{(i)} - M/3)\Psi^{cur} = 0, \qquad (7)$$

as equations for the wave functions in the current representation. Equation (7) means that the bound state of three quarks can be described (maybe approximately) by three equations for separate quarks in some averaged potential $V^{(i)}$. We postulate that $V^{(i)}$ depends only on the relative coordinate of the i-th quark $X - x^{(i)}$, where X is the c.m. coordinate. Eq. (7) will be solved for following potentials:

a) for weak potential $V^{(i)}$; in this case one uses the perturbation theory;

b) for "spinor oscillator" potential; in this case equations can be solved exactly.

Let us consider the first case. As a weak parameter one can use $|\overset{\circ}{\ell}^{(i)}|/M$. One also postulates that the potential $V^{(i)} \sim \overset{\circ}{\ell}^{(i)2}/M$.

Now

$$\Psi^{cur}_{\vec{P}=0} = \prod_{i=1}^{3} \left(\frac{2}{3}M - \overset{\circ}{\vec{\ell}}^{(i)} \vec{\gamma}^{(i)}\right)\phi\left(\overset{\circ}{\vec{\ell}}^{(1)}, \overset{\circ}{\vec{\ell}}^{(2)}, \overset{\circ}{\vec{\ell}}^{(3)}\right) + O\left(\overset{\circ}{\vec{\ell}}^{(i)2}/M^2\right), \quad (8)$$

where ϕ contains only large components of spinors. It satisfies the equations:

$$\left(\overset{\circ}{\ell}^{(i)2} + \frac{1}{3} V^{(i)} M\right)\phi\left(\overset{\circ}{\vec{\ell}}^{(1)}, \overset{\circ}{\vec{\ell}}^{(2)}, \overset{\circ}{\vec{\ell}}^{(3)}\right) = 0. \quad (9)$$

According to (8), the spinor parts contain relative momenta, i.e., $\Psi^{cur}_{\vec{P}=0}$ has no definite orbital momentum even if ϕ has. But

$$\vec{j}\Psi^{cur}_{\vec{P}=0} = \prod_{i=1}^{3} \left(\frac{2}{3}M - \overset{\circ}{\vec{\ell}}^{(i)} \vec{\gamma}^{(i)}\right)\vec{j}\phi\left(\overset{\circ}{\vec{\ell}}^{(1)}, \overset{\circ}{\vec{\ell}}^{(2)}, \overset{\circ}{\vec{\ell}}^{(3)}\right) + O\left(\overset{\circ}{\vec{\ell}}^{(i)2}/M^2\right). \quad (10)$$

That is, if ϕ is the eigenfunction of $\vec{j}^2$ and j_3, $\Psi^{cur}_{\vec{P}=0}$ will also be the eigenfunction of $\vec{j}^2$ and j_3 with the same eigenvalues.

To get the wave function in the arbitrary frame one must act on $\Psi^{cur}_{\vec{P}=0}$ by the Lorentz transformation. This will provide the Lorentz covariance of the wave functions and will thus provide the angular conditions (if we construct the Lorentz-covariant current matrix elements). Relativistic current matrix elements will be constructed at the end of this paper.

It is very useful in many respects to study the wave functions in the $P_z \to \infty$ frame. That is why we shall study now the wave functions in the system, where the hadron momentum $\vec{P}$ is infinite and has only the third component /13/. The Lorentz transformation from the rest system to this one is:

$$U_{P_z\to\infty} = (P_z/M)^3 \prod_{i=1}^{3} (1-\sigma_3^{(i)}\gamma_5^{(i)}). \quad (11)$$

Using the γ -matrix properties one gets from (8) and (11)

$$\Psi^{cur}_{P_z\to\infty} = \prod_{i=1}^{3} (X^{(i)} - \vec{\gamma}_{\perp}^{(i)} \overset{\circ}{\vec{\ell}}{}_{\perp}^{(i)})\, U_{P_z\to\infty}\phi$$
$$\simeq \left(\frac{3}{2M}\right)^3 \prod_{i=1}^{3} X^{(i)} \exp\{ tg^{-1} [-\vec{\gamma}_{\perp}^{(i)} \overset{\circ}{\vec{\ell}}{}_{\perp}^{(i)} / x^{(i)}]\} \tilde{\Psi}_{P_z\to\infty}, \quad (12)$$

where $\tilde{\Psi}_{P_z\to\infty} = (\frac{2M}{3})^3 U_{P_z\to\infty}\phi$ and $\vec{a}_{\perp} = (a_1, a_2)$, $X^{(i)} = \frac{2}{3}M - \overset{\circ}{\ell}{}_3^{(i)}$. The operator standing in front of $\tilde{\Psi}_{P_z\to\infty}$ in (12) is the well-known Melosh transformation [2]. It is unitary in the first order of $\overset{\circ}{\ell}{}_3^{(i)}$.

But what is the equation for $\Psi_{P_z\to\infty}$? This wave function satisfies the Bargman-Wigner equations by definition. If in Eq. (5) one chooses instead of the oscillator potential the new one $U = \frac{1}{3}[V^{(1)} + V^{(2)} + V^{(3)}]M + m^2 + \frac{2}{3}M^2$, one will get Eqs. (9). That is, $\Psi_{P_z\to\infty}$ is the wave function in the constituent representation and Eq. (12) gives us the transformation between current and constituent representations in the $P_{z\to\infty}$ system. The wave functions (12) have been used by one of us for constructing current algebra [9].

The next example is the spinor oscillator potential. The equations of quasi-independent quarks (7) have in this case the form:

$$(\overset{\circ}{\ell}{}^{(i)}\gamma^{(i)} - \overset{\circ}{\vec{\ell}}{}^{(i)}\vec{\gamma}^{(i)} - \mu)\Psi^{cur}_{\vec{P}=0} = i\omega^{(i)}\vec{Y}^{(i)}\vec{\alpha}^{(i)}\Psi^{cur}_{\vec{P}=0}, \quad (13)$$

where $\omega^{(i)} = \Omega_-\theta(-K^{(i)}) - \Omega_+\theta(K^{(i)})$, $\Omega_\pm > 0$, $K^{(i)} = \mp(j + \frac{1}{2})^{(i)}$ is the sum of the orbital momentum and the spin of the i -th quark, $\overset{\circ}{\ell}{}_0^{(i)} = \frac{1}{3}M$, , $Y^{(i)} = x^{(i)} - X$ is the relative coordinate of the i -th quark. This equation has been dealt with in detail in Ref. [14].

One can diagonalize Eq. (13), if

$$\Psi^{cur}_{\vec{P}=0} = N \prod_{i=1}^{3} (\ell^{2(i)} - i\omega^{(i)} \vec{Y}^{(i)} \vec{\alpha}^{(i)} + \mu) \chi^{(i)}_{\vec{P}=0} , \tag{14}$$

where N is such a normalization constant that $\bar{\Psi}^{cur}\Psi^{cur} = \prod_{i=1}^{3} \bar{\chi}^{(i)}_{\vec{P}=0} \chi^{(i)}_{\vec{P}=0}$. This means that the transformation from $\chi_{\vec{P}=0}$ to Ψ^{cur} is unitary. $\chi^{(i)}_{\vec{P}=0}$ has only large components of the spinor. Nonzero components of $\chi^{(i)}_{\vec{P}=0}$ are:

$$\chi^{(i)}_{\vec{P}=0} = NR_{k,\ell}\, \Omega^{\ell}_{jm} \qquad \text{for} \quad P = (-1)^{j+\frac{1}{2}}, \tag{15}$$

$$\chi^{(i)}_{\vec{P}=0} = NR_{k+1,\ell}\, \Omega^{\ell}_{jm} \qquad \text{for} \quad P = (-1)^{j-\frac{1}{2}} .$$

In Eq. (15) Ω^{ℓ}_{jm} stands for spherical spinors /15/ and $R_{k,\ell}$ is the radial part of the wave function of the 3-dimensional harmonical oscillator. The mass spectrum is:

$$M^2/9 = \mu^2 + 4\Omega_{\pm} \ (j+k+1), \ P = (-1)^{j\pm\frac{1}{2}}. \tag{16}$$

(In Eqs. (15) and (16) we have omitted the quark indices i). Using the Clebsch-Gordan coefficients one can construct from $\chi^{(i)}_{\vec{P}=0}$ the wave function $\chi_{\vec{P}=0}(\ell, j, j_3, \alpha)$ with total orbital momentum ℓ, spin j, its third component j_3 and other quantum numbers α. The orbital part of this wave function $R = \prod R_{k^{(i)}\ell^{(i)}}$ satisfies Eq. (5), because the potential (3)(we shall put $\Omega_+ = \Omega_-$)is:

$$U = 3\,\Omega \sum_{i=1}^{3} Y^{(i)2} + C. \tag{17}$$

$\chi_{\vec{P}=0}(\ell, j, j_3, \alpha)$ satisfies the Bargman-Wigner equation, because it contains only large components of quark spinors. That is (for the major trajectory $k^{(i)} = 0$) $\chi_{\vec{P}=0}(\ell, j, j_3, \alpha)$ is the wave function in the constituent representation. Performing the transformation

$$\Psi^{cur}_{\vec{P}=0}(j,j_3)=N\prod_{i=1}^{3}(\overset{\circ}{\ell}^{(i)}-i\omega^{(i)}\vec{Y}^{(i)}\vec{a}^{(i)}-\mu)\chi_{\vec{P}=0}(\ell,j,j_3), \tag{18}$$

one gets the wave function with definite values of j and j_3 but an indefinite value of the orbital momentum operator. The wave functions $\Psi^{cur}_{P=0}(\ell,\ell_3)$ are normalized according to Eq. (4).

In the $P_z\to\infty$ system the wave functions (18) are:

$$\Psi^{cur}_{P_z\to\infty}=N\prod_{i=1}^{3}\sqrt{X_{\pm}^{(i)2}-(\vec{a}_{\perp}^{\pm(i)})^2}\exp\{tg^{-1}[-(\vec{\gamma}_{\perp}^{(i)}\vec{a}_{\perp}^{(i)\pm})/X_{\pm}^{(i)}]\}\chi_{P_z\to\infty} \tag{19}$$

where $\chi_{P_z\to\infty}$ was proved to be the wave function in the constituent representation, and Eq. (19) gives the transformation from the constituent representation to the current one. Because of our normalization of the wave functions this transformation is unitary. In Eq. (19) we have used the following notations:

$$\vec{a}^{\pm(i)}=\overset{\circ}{\vec{\ell}}^{(i)}\pm i\Omega_{\pm}\vec{Y}^{(i)},$$

$$X_{\pm}^{(i)}=M/3+\mu-a_3^{\pm(i)}.$$

One sees that the difference between (12) and (19) is only that

$$\overset{\circ}{\vec{\ell}}^{(i)}\to\vec{a}^{\pm(i)}.$$

We have considered specific (three-dimensional) relativistic equations for three-quark bound states. As mentioned above, we must construct also the relativistically covariant current matrix elements [16] to satisfy the angular conditions. Now we shall construct such matrix elements. The nonrelativistic matrix element has the form:

$$(\vec{P}'\,|\,J_\mu(0)\,|\,\vec{P})=3\int\prod_{i=1}^{3}d\vec{\ell}^{(i)}d\vec{\ell}'^{(i)}\,\delta(\sum_{j=1}^{3}\vec{\ell}^{(j)}-\vec{P})\,\delta(\sum_{j=1}^{3}\vec{\ell}'^{(j)}-\vec{P}')$$

$$\times \delta(\vec{\ell}^{(2)} - \vec{\ell}'^{(2)})\, \delta(\vec{\ell}^{(3)} - \vec{\ell}'^{(3)})\, \bar{\Psi}(\vec{\ell}'^{(1)}, \vec{\ell}'^{(2)}, \vec{\ell}'^{(3)})$$

$$\times \Gamma_\mu^{(1)} \frac{\lambda_a^{(1)}}{2} \Psi(\vec{\ell}^{(1)}, \vec{\ell}^{(2)}, \vec{\ell}^{(3)}), \qquad (20)$$

where [17] $\Gamma_\mu^{(1)} = \gamma_\mu^{(1)}$ or $\gamma_5^{(1)}\gamma_\mu^{(1)}$. To generalize this matrix element for the relativistic case one must replace:

i) $d\vec{\ell}^{(i)} \to d^4\ell^{(i)}$;

ii) $\delta(\sum_{j=1}^{3} \vec{\ell}^{(j)} - \vec{P}) \to \delta^4(\sum_{j=1}^{3} \ell^{(j)} - P)$;

iii) remove the time components of quark momenta, i.e., write the Markov-Yukawa conditions $\prod_{k=2}^{3} \delta[(\ell^{(k)} - P/3)P]$ under the integral;

iv) find the appropriate generalization for nonrelativistic three-dimensional δ-function $\delta(\vec{\ell}^{(j)} - \vec{\ell}'^{(j)})$. For this one can use such a trick [9]:

$$\int_{-\infty}^{\infty} d\alpha\, \delta^4(\lambda - \alpha L) = \frac{1}{L_0} \delta(\vec{\lambda} - \frac{\lambda_0}{L_0} \vec{L}). \qquad (21)$$

As L one can use $\frac{1}{2M}(P+P')_\mu$. Now in the nonrelativistic case $|\vec{L}| \ll 1, L_0 \approx 1$ and (21) is $\delta(\vec{\lambda})$.

From i). - iv). we simply find the required relativistic generalization:

$$\langle P' \,|\, J_\mu(0) \,|\, P \rangle = \frac{3}{L_0^2} \int \delta^4(\sum_{j=1}^{3} \ell^{(j)} - P)\, \delta^4(\sum_{j=1}^{3} \ell'^{(j)} - P') \prod_{i=2}^{3} \times$$

$$\times \delta[(\ell^{(i)} - P/3)P]\, \delta[(\ell'^{(i)} - P'/3)P']\, \delta\,[\vec{\ell}^{(i)} - \vec{\ell}'^{(i)} -$$

$$-(\ell_0^{(i)} - \ell_0'^{(i)})\vec{L}/L_0] \times$$

$$\times \bar{\Psi}^{cur}(\ell'^{(1)}, \ell'^{(2)}, \ell'^{(3)})\, \Gamma_\mu^{(1)} \frac{\lambda_a^{(1)}}{2} \Psi^{cur}(\ell^{(1)}, \ell^{(2)}, \ell^{(3)}) \times$$

$$\times \prod_{k=1}^{3} d^4\ell^{(k)} d^4\ell'^{(k)}. \qquad (22)$$

At the end we would like to emphasize our main results. Firstly, we have shown the three-dimensional relativistic equations for the bound states of three quarks in the constituent and current representations. This dynamical approach makes it possible to obtain the transformation between constituent and current representations for interacting quarks. Secondly, we have constructed the relativistically-covariant current matrix elements. This automatically guarantees angular conditions.

In conclusion we would like to thank Dr. B.V.Struminsky for very helpful discussions.

Note: When this work was finished we get the preprint (E.Celeghini, TH-74/Al, Firenze, 1974) where the kinematics of Melosh transformation for mesons was studied.

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17. This vertex follows from Eq. (8). Minimal coupling following from Eq. (1) gives (e.g., for vector current)

Received by Publishing Department
on February 7, 1975.