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CAUSALITY IN QUANTUM ELECTRODYNAMICS

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Introduction

In quantum electrodynamics the usual condition of causality is satisfied: the operators A_μ and ψ are locally commutative. But neither A_μ, nor ψ are observables. We know some of the local observables such as field intensities $E(x)$, $H(x)$ and current density $j_\mu(x)$. Their mutual commutations as well as commutations with A_μ, ψ are local. We show in sect.1. that there exist other densities which have physical sense (number operator density for electrons for example) and do not commute locally with $\vec{E}(x)$. Then the question arises: Is quantum electrodynamics a causal theory in the sense that signals velocities do not exceed that of light? The corresponding quantum mechanical meaning of causality is proposed in sect.2. It turns out that $\vec{E}(x)$, $\vec{H}(x)$ and $j_\mu(x)$ behave strictly causally in this sense. We show in sect.3, that the electron density (or the module of the coordinate wave function of the electron) propagates with superluminar velocity. So does the phase of the electron coordinate wave function. In sect.4 we consider the relation of the phase and the electron momentum. Discussion and conclusions are given in the final section.

1. Operator densities having nonlocal commutations.

The number operator is the fundamental physical quantity in the second-quantized field theory. It turns out that in the Lorentz gauge we cannot take the simple expression (see (8.80) in /3/)

$$\int d^3x\, \psi^{(-)+}(\vec{x})\, \psi^{(-)}(\vec{x}) = \int d^3x' \int d^3x''\, \psi^{+}(\vec{x}')\, \Pi^{(-)}(\vec{x}', \vec{x}'')\, \psi(\vec{x}'') \tag{1}$$

as the number operator for electrons because (1) is not gauge

invariant: when $\psi \to \psi \exp ie\chi$ it is transformed into
$\int d^3x' \int d^3x'' \psi^\dagger(\vec{x}') \Pi^{(-)}(\vec{x}',\vec{x}'') \psi(\vec{x}'') \exp ie [\chi(\vec{x}'') - \chi(\vec{x}')]$.
In eq. (1) $\psi^{(-)}(\vec{x})$ denotes that part of $\psi(\vec{x})$ which destroys the electron; $\Pi^{(-)}$ is the projection operator on this part: $\psi^{(-)} = \Pi^{(-)}\psi$.

Similar difficulties arise when constructing the electron momentum operator. The quantity $\int d^3x\, \psi^\dagger(-i\vec{\nabla})\psi$ (see § 18 in /1/) is not gauge invariant and therefore cannot be an observable. Gauge invariant operators

$$P_j = \int d^3x\, \psi^\dagger(\vec{x})[-i\nabla_j - eA_j(\vec{x})]\,\psi(\vec{x})$$

(see, i.e., App. VII in /2/) cannot be considered as canonical momenta since $[P_x, P_y] \neq 0$.

There are not such troubles in the Coulomb (radiation) gauge. In this gauge A_μ is replaced by the transverse part of the vector potential. It is gauge invariant quantity bacause it can be expressed in terms of its rotor $\vec{H}$: $\vec{A}_\perp(\vec{x}) \sim \mathrm{rot} \int d^3y\, \vec{H}(\vec{y})/|\vec{x}-\vec{y}|$. Therefore the electron field operator φ in the Coulomb gauge also is gauge-invariant. Not only gauge-invariant operators of the electron number and momentum can be constructed using φ , but also the corresponding densites.

Let us see what interpretation can be given to the density $N^{(-)}(\vec{x}) = \varphi^{(-)\dagger}(\vec{x})\varphi^{(-)}(\vec{x})$. For this purpose we calculate the expectation value of $N^{(-)}$ in the one-electron state $\alpha_n^+ \Omega$. The operator α_n is defined by the expansion of the Schrödinger operator $\varphi(\vec{x})$ in the complete set of proper functions u_n, v_p of the Hamiltonian $\mathcal{D}$ of the Dirac equation $i\dot{\varphi} = \mathcal{D}\varphi$ with arbitrary external potential :

$$\varphi(\vec{x}) = S_n\, u_n(\vec{x})\,\alpha_n + S_p\, v_p(\vec{x})\,\beta_p^\dagger \equiv \varphi^{(-)}(\vec{x}) + \varphi^{(+)}(\vec{x}) \qquad (2)$$

/see ch. 14 in /4//. S_n and S_p are sums and/or integrals over electron (n) and positron (p) indices. We have

$$\langle \alpha_n^+\Omega, \varphi^{(-)+}(\vec{x})\varphi^{(-)}(\vec{x})\alpha_n^+\Omega\rangle = \sum_{\alpha=1}^{4} u_n^*(\vec{x},\alpha)\, u_n(\vec{x},\alpha) . \tag{3}$$

In the right-hand side of (3) we have the density of the coordinate probability distribution in the considered state (spinor index is explicitly written out in (3)). The expectation value of $N^{(-)}(\vec{x})$ in the state $\alpha_n^+\alpha_m^+\beta_p^+\Omega$ turns out to be equal to the sum of densities $|u_n|^2 + |u_m|^2$. The expectation value of $N^{(-)}(\vec{x})$ is equal to zero if taken in states describing positrons and/or photons only. In the general case one gets not a sum of squares of the wave functions modules, but the diagonal element $(\vec{x}|\rho|\vec{x})$ of a density matrix.

In the same line one can show that the momentum density operator

$$\vec{P}(\vec{x}) = \frac{1}{2} :[\varphi^+(\vec{x})(-i\vec{\nabla})\varphi(\vec{x}) + (-i\vec{\nabla}\varphi)^+\varphi]: \tag{4}$$

provides an addional information about the phase of the coordinate wave function. For instance,

$$\langle \alpha_n^+\Omega, \vec{P}(\vec{x})\alpha_n^+\Omega\rangle = \sum_{\alpha} |u_n(x,\alpha)|^2\vec{\nabla}\beta_n(x,\alpha) , \quad u_n = |u_n| e^{i\beta_n} . \tag{5}$$

The locality properties of the commutation of the φ with a complete system of local operators determine these properties for $N^{(-)}$ and $\vec{P}(x)$. By definition any observable of the theory can be expressed in terms of the operators of this complete system. One can take locally commutative operators $\vec{E}(x)$, $\vec{H}(x)$, $\overline{\psi}(x)$, $\psi(x)$ as such a system. All their commutations with φ are local but

$$[\varphi(\vec{x},t), E_m(\vec{y},t)] = e\,\varphi(\vec{x},t)\,\frac{1}{4\pi}\frac{\partial}{\partial x_m}\frac{1}{|\vec{x}-\vec{y}|} . \tag{6}$$

In the Coulomb gauge (6) follows from

$$\vec{E}(\vec{x},t) = -\partial \vec{A}_{\perp}(\vec{x},t)/\partial t - \mathrm{grad} \int d^3x' \, \varphi^+(\vec{x}',t)\varphi(\vec{x}',t)/|\vec{x}-\vec{x}'|$$

/see §49 in /5//. In the Lorentz gauge (6) is obtained from

$$\varphi(\vec{x},t) = \psi(\vec{x},t) e^{-ie\mathcal{U}(\vec{x},t)} \quad ; \quad \mathcal{U}(\vec{x},t) = \frac{-1}{4\pi}\int d^3y \, \mathrm{div}\vec{A}(\vec{y},t)/|\vec{x}-\vec{y}| \quad (7)$$

(see §80 in /6/; we denote by e the electron charge). Note that one can get $N^{(-)}(x)$ and $\vec{P}(x)$ as functions of $\psi(x)$ using (7).

Because of (6), the commutations $[N^{(-)}, \vec{E}]$ and $[\vec{P}(x), \vec{E}]$ are nonlocal too. They can be named macrononlocal contrary to commutators $\{\bar{\psi}(\vec{x},t), \psi^{(-)}(\vec{y},t)\}_+$ or $[N^{(-)}(\vec{x},t), N^{(-)}(\vec{y},t)]$ which are exponentially small when $|\vec{x}-\vec{y}| >> \lambda_m$, λ_m being the Compton wavelength of the electron. According to (6), one cannot make a simultaneous precise measurement of $\vec{E}(y,t)$ and $N^{(-)}(\vec{x},t)$ when $\vec{x}$ and $\vec{y}$ are macroscopically separated. The reasoning like that given at the end of §48 in /5/ entails the question: does this fact mean that electrodynamics signals can travel faster than light?

2. Signal transmission in quantum mechanics.

We shall consider the following scheme of the signal transmission. There is an external current, localized in some volume V_S. It is switched on at $t=0$, being zero before. This is a source S of the signal. There is a detector in some volume V_D. It measures some local physical quantity. For instance, its response may be proportional to the integral of the electric field intensity $\vec{E}(x)$ over V_D (as in the case of probe charge). The dimensions of the V_D and V_S are supposed to be much less than the distance R between them.

Now we ought to stress an important difference between the classical and quantum descriptions of the signal transmission. One cannot suppose that the field $\vec{E}(x)$ inside V_D is equal to zero if source current was not switched on. The state with a precise (i.e.

zero) $\vec{E}(x)$ value is not a stationary one because $\vec{E}(x)$ does not commute with the Hamiltonian of the electromagnetic fields. By the same reason $\vec{E}(x)$ cannot have precise values in stationary states (i.e. proper vector of the total Hamiltonian $\mathcal{H}$) but must be described by a probability distribution.

We assume the following convention: the moment of the signal arrival is the moment of time t, when distribution over the detected local observable changes inside $V_{\mathcal{D}}$, when comparing with that distribution which it would have at t, if the source was not switched on. If the signal arrival moment is less than R/c, we shall say: the local observable has a noncausal behavior.

Consider the following difference: "the distribution of $E(x)$ at the moment t when the source was switched on minus the distribution of $E(x)$ at t when source was not switched on". To calculate it we find the momenta of this distribution difference, i.e. mean value, mean square and so on. For this purpose one must calculate the quantities of the kind

$$\langle \mathcal{U}(t,o)\phi, \hat{\mathcal{O}}(\vec{x})\,\mathcal{U}(t,o)\phi\rangle - \langle e^{-it\mathcal{H}(o)}\phi, \hat{\mathcal{O}}(\vec{x})\, e^{-it\mathcal{H}(o)}\phi\rangle \,, \tag{8}$$

where $\hat{\mathcal{O}}(\vec{x})$ can denote $E^{m}(\vec{x})$, $m = 1, 2, 3 \ldots$, $\vec{x} \in V_{\mathcal{D}}$ or $\mathcal{O}(\vec{x}) = \mathcal{N}^{(-)}(\vec{x})$ and so on. ϕ is an initial state vector of the system; $\mathcal{U}(t,o)$ is the operator of the system evolution; $i\partial_t \mathcal{U} = \mathcal{H}(t)\,\mathcal{U}$. The total Hamiltonian $\mathcal{H}$ does not depend upon time, if the source current is not switched on. In such a case $\mathcal{H}$ equals $\mathcal{H}(o)$ all the times and the evolution operator is $exp[-it\,\mathcal{H}(o)]$. We have

$$\langle \mathcal{U}(t,o)\phi, \hat{\mathcal{O}}\,\mathcal{U}(t,o)\phi\rangle = \langle \phi, \mathcal{U}^{+}\hat{\mathcal{O}}\,\mathcal{U}\phi\rangle \,,$$

where $\mathcal{U}^{+}\hat{\mathcal{O}}\,\mathcal{U}$ is the Heisenberg operator $\mathcal{O}_{\mathcal{I}}(t)$, which at $t=0$ conside with the Schrödinger operator $\hat{\mathcal{O}}$. So one can rewrite (8) as follows

$$\langle\phi, [\mathcal{O}_J(\vec{x},t) - \mathcal{O}(\vec{x},t)]\,\phi\rangle \;, \qquad \mathcal{O}(\vec{x},t) = e^{it\mathcal{H}(o)}\,\mathcal{O}(\vec{x})\,e^{-it\mathcal{H}(o)}\,. \tag{9}$$

Now the problem is to find the Heisenberg operators and their expectation values. All electrodynamical Heisenberg operators can be obtained if operators $A_\mu(\vec{x},t)$ and $\psi(\vec{x},t)$ are known. We show in Appendix that operator differences $A_{J,\mu}(\vec{x},t) - A_\mu(\vec{x},t)$ and $\psi_J(\vec{x},t) - \psi(\vec{x},t)$ equal exactly zero if $\vec{x} \in V_D$ and $t < R/c$. Because of $\vec{E}_J - \vec{E} = -\partial_t(\vec{A}_J - \vec{A}) - grad(A_{J0} - A_0)$ the operator difference $\vec{E}_J(\vec{x}_D,t) - \vec{E}(\vec{x}_D,t)$ and its expectation value in any state ϕ also vanishes at $t < R/c$. Because of $E_J^2 - E^2 = (E_J - E)E_J + E(E_J - E)$ the second moment of the $\vec{E}(\vec{x})$ distribution difference also vanishes and so do all higher momenta. Therefore the distribution difference itself is zero at $t < R/c$. This means that the electric field from the source current propagates with velocity not exceeding c. The magnetic field, the Pointing vector density and the current density

$$j_{J\mu}(\vec{x},t) - j_\mu(\vec{x},t) \sim \bar{\psi}_J\gamma_\mu(\psi_J - \psi) + (\bar{\psi}_J - \bar{\psi})\gamma_\mu\psi$$

also reveal causal behavior. The expectation value of $j_0(\vec{x})$ must be interpreted as the charge distribution density (similar to $N^{(-)}(\vec{x})$ and unlike to the meaning of the expectation value of $\vec{E}(\vec{x})$).

3. The acausal behavior of the electron density and phase.

Now let us calculate the change in the number operator density for electrons

$$N_J^{(-)}(\vec{x},t) - N^{(-)}(\vec{x},t) = \varphi_J^{(-)\dagger}[\varphi_J^{(-)} - \varphi^{(-)}] + [\varphi_J^{(-)} - \varphi^{(-)}]^\dagger\varphi^{(-)}\;. \tag{10}$$

One could calculate the Heisenberg operator $\varphi^{(-)}(\vec{x},t)$ starting from the equation for $\varphi(\vec{x},t)$ /it is nonlocal, see eqs. (69) and(70) in /7/. We shall adhere to a simpler way: we use the relation (7) between φ and ψ and then refer to the Appendix

$$\varphi(\vec{x},t)=\int d^3y\, \Pi^{(-)}(\vec{x},\vec{y})\,\varphi(\vec{y},t)=\int d^3y\, \Pi^{(-)}(\vec{x},\vec{y})\, e^{-ie\mathcal{U}(\vec{y},t)}\psi(\vec{y},t)\quad . \tag{11}$$

Here we use the projection operator $\Pi^{(-)}(\vec{x},\vec{y})=S_n\, u_n(\vec{x})u_n^{+}(\vec{y})$, for u_n see eq.(2). Using the expansion $\psi=\Sigma_n e^n\psi^{(n)}$ (see App.), we get

$$\begin{aligned}\varphi_I^{(-)}(\vec{x},t)-\varphi^{(-)}(\vec{x},t)\equiv\int d^3y\ \Pi^{(-)}(\vec{x},\vec{y})\{e[\psi_I^{(1)}(\vec{y},t)-\psi^{(1)}(\vec{y},t)]-\\ -ie[\mathcal{U}_I^{(0)}(\vec{y},t)-\mathcal{U}^{(0)}(\vec{y},t)]\,\psi_0(\vec{y},t)\}\end{aligned} \tag{12}$$

(the equality $\psi_I^{(0)}=\psi^{(0)}\equiv\psi_0$ is taken into account) The Green function $S=S^{(-)}+S^{(+)}$ and eqs. (A.11) and (2350) from /1/ allow one to write

$$\psi_I^{(1)}(y)-\psi^{(1)}(y)=$$

$$=\int_0^{y_0}dz_0\int d^3z\,(-i)\,S(y,z)\,[A_{I\mu}^{(0)}(z)-A_\mu^{(0)}(z)]\,\gamma_\mu\,\psi_0(z)\quad . \tag{13}$$

Here y denotes $\{\vec{y},y_0\}$. Taking into account the equation

$$-i\,S^{(-)}(y,z)\gamma_0=S_n\,u_n(\vec{y})\,u_n^{+}(\vec{z})\,e^{-iE_n(y_0-z_0)}=\{\psi_0^{(-)}(y),\ \psi_0^{+}(z)\}_+ \tag{14}$$

and orthonormalization relations $\int d^3x\; u_n^{+}(\vec{x})\,u_m(\vec{x})=\delta_{mn}$ we get

$$\int d^3y\ \Pi^{(-)}(\vec{x},\vec{y})\ S^{(-)}(y,z)=S^{(-)}(\vec{x},y_0,z)\quad . \tag{15}$$

Similar calculation gives $\Pi^{(-)}S^{(+)}=0$. Using (13), (15) and (A.16) one can reduce (12) to

$$\begin{aligned}\varphi_I^{(-)}(\vec{x},t)-\varphi^{(-)}(\vec{x},t)=\ e\int_0^t dz_0\int d^3z\,(-i)\,S^{(-)}(x,z)\,\gamma_\mu\,\psi_0(z)\int d^4z'\,\mathcal{D}_{ret}(z-z')\,J_\mu(z')-\\ -ie\int d^3y\ \Pi^{(-)}(\vec{x},\vec{y})\,\psi_0(\vec{y},t)\,W(\vec{y},t)\ ,\qquad x\equiv\{\vec{x},t\}\end{aligned} \tag{16}$$

$$W(\vec{y},t)=\frac{-1}{4\pi}\sum_{k=1}^{3}\partial/\partial x_k\int\frac{d^3z}{|\vec{y}-\vec{z}|}\int d^4z'\,\mathcal{D}_{ret}(z-z')\,J_k(z')\quad ,\qquad z_0=t \tag{17}$$

Inserting eq.(16) in eq.(10) we get in the first order in e:

$$\begin{aligned}N_I^{(-)}-N^{(-)}=e\{\varphi_0^{(-)+}(\vec{x},t)\int_0^t dz_0\int d^3z\,(-i)\,S^{(-)}(x,z)\,\gamma_\mu\,\varphi_0(z)\int\mathcal{D}_{ret}\,J_\mu\ +\ h.c.\}-\\ -ie\{\varphi_0^{(-)+}(\vec{x},t)\int d^3y\ \Pi^{(-)}(\vec{x},\vec{y})\,\varphi_0(\vec{y},t)\,W(\vec{y},t)-h.c.\}\quad .\end{aligned} \tag{18}$$

Note that from eq. (7) the equality $\psi^{(o)} = \varphi^{(r)} \equiv \varphi_0$ follows.

The first brace in eq. (18) contains the product of functions $S^{(-)}(x,z)$ and $\int d^4z' \, \mathcal{D}_{ret}(z-z') J_\mu(z')$, $0 \leq z_0 < t$. The second one is not zero only inside the cone $\mathcal{S}$ in fig.1. Let the point $(\vec{x},t)$, $\vec{x} \cong \vec{R}$, be outside $\mathcal{S}$. The function $S^{(-)}(x,z)$ is not zero inside $\mathcal{S}$ and therefore the first brace does not vanish when $t < R/c$. But $S^{(-)}$ is exponentially small inside $\mathcal{S}$. Indeed $S^{(-)}$ can be expressed in terms of $\Delta^{(-)}$ and $\Delta^{(-)} \sim exp(-\lambda/\lambda_m)$, $\lambda^2 = (\vec{x}-\vec{z})^2 - (x_0 - z_0)^2 > 0$, $\lambda_m = h/mc$, see §15 in /8/. So, the discussed product is not negligible at $t < R/c$ only when $t > (R-\lambda_m)/c$. But the electron wave function (the module of which is supposed to be measured by a detector in V_D) cannot be localized in a region with dimensions less than λ_m. So, it is meaningless to speak about the superluminar velocity having in mind the first brace.

The second brace in eq. (18) contains the function $\Pi^{(-)}(\vec{x},\vec{y}) = -iS^{(-)}(\vec{x},0;\vec{y},0)\gamma$ which is not small only when $|\vec{x}-\vec{y}| \leq \lambda_m$. The expansion

$$W(\vec{y},t) = W(\vec{x},t) + \Sigma_k (y_k - x_k) [\partial W(\vec{y},t)/\partial y_k]_{\vec{y}=\vec{x}} + \dots \tag{19}$$

will be suitable because $W(\vec{y},t)$ does not change appreciably when y changes by λ_m. The first term in (19) contributes nothing to the second brace because of the equality $\int d^3x \, \Pi^{(-)}(\vec{x},\vec{y}) \varphi_0(\vec{y},t) = \varphi_0^{(-)}(\vec{x},t)$. It turns out that in contrast to the first brace which decreases at $t < R/c$ exponentially when R increases the second brace behaves at $t < R/c$ as an inverse power of R. This assertion follows from a simple estimation of $\partial W/\partial x_k$ for $\vec{x} \cong \vec{R}$. Let us choose the axis $z \parallel \vec{R}$ and assume that the external current density $\vec{J}(\vec{z}, z_0)$ is directed along the axis X, is localized in the source volume V_S and is not zero only in a time interval $0 \leq z_0 \leq \mathcal{T}$. Then

$$\partial W(\vec{x},t)/\partial x_\kappa \cong \delta_{\kappa 1} \cdot I_1/c \cdot L_1/R \cdot ct/R \cdot c\tau/R \quad , \quad \vec{x} \cong \vec{R} \tag{20}$$

if $\tau \ll t < R/c$. In eq. (21) I_1 denotes the source current averaged over the internal $(0,\tau)$; L_1 is the dimension of V_S along the axis x .

We shall write an estimate for the change of the number of electrons in the detector volume $V_\mathcal{D}$ at $t < R/c$. We calculate expectation value of (18) in the state $\alpha_n^+ \Omega$ which describes the electron plane wave $u_n(\vec{x}) \sim \exp ip_1 x_1$, directed along the axis x ; $p_1 = m v_1/\hbar$; v_1/c is supposed to be $\ll 1$. To calculate the expectation value an estimation was made for the integral

$$\int d^3y \; \Pi^{(-)}(\vec{x},\vec{y}) \; u_n(\vec{y}) \; (y_\kappa - x_\kappa) \tag{21}$$

in the case when there is no external potential for electrons and

$$\Pi^{(-)}(\vec{x},\vec{y}) = -i \left[\left((\vec{\gamma}\vec{\partial}) - i\gamma_4 \partial_t - m \right) \gamma_4 \, \mathcal{D}_m^{(-)}(x-y) \right]_{x_0 = y_0} .$$

The final result is

$$\int_{V_\mathcal{D}} d^3x \, \langle \alpha_n^+ \Omega \,|\, N_\gamma^{(-)}(\vec{x},t) - N^{(-)}(\vec{x},t) \,|\, \alpha_n^+ \Omega \rangle / \langle \Omega, \Omega \rangle =$$

$$= -2 \frac{e}{\hbar c} \frac{v_1}{c} N \frac{\hbar}{mc} \, \partial W(\vec{R},t)/\partial x_1 = 2 \frac{e^2}{\hbar c} \frac{v_1}{c} N \cdot \frac{I_1}{|e|} \cdot \frac{\lambda_m}{c} \cdot \frac{L_1}{R} \cdot \frac{ct}{R} \cdot \frac{c\tau}{R} . \tag{22}$$

Here the constants $\hbar$ and c are written explicitly (usually they are taken to be equal to unity), N is the initial number of electrons in $V_\mathcal{D}$; $e^2/\hbar c = 1/137$; $\lambda_m = \hbar/mc$; $\lambda_m/c = 1{,}3 \cdot 10^{-21}$ sec, $|e| = 1{,}6 \cdot 10^{-19}$ coulomb. To estimate the obtained acausal effect we let $v_1/c = 0{,}1$, $I_1 = 10^6$ amper, $L_1/R = c\tau/R = 0{,}01$. Then the number of electrons in $V_\mathcal{D}$ (or their density) at the moment $t = 0{,}1\, R/c$ differs from the initial number by $0{,}01$ per cent. Only the ratios of R, L_1, τ are needed for the estimate, but not their specific values. A greater value of the current is advantageous even if greater value of V_S and L_1 will be required: note that the ratio L_1/R can be retained (by taking a larger R) and so can the ratios τ/R and t/R.

The estimate demonstrates that the effect is macroscopic and cannot be ascribed to uncertainty relations. But it is small. We can show that its ratio to the density charge at $t > R/c$ is determined mainly by the small parameter λ_m/R, which cannot exceed 10^{-13} for reasonable values of R.

Now let us see how the density $\vec{P}(\vec{x})$, see (4), changes. If one introduces ψ into (4), using (7), then one gets

$$\vec{P}(x) = \frac{1}{2}\left[\psi^+(x)(-i\vec{\nabla})\psi(x) + h.c.\right] - e\vec{A}_L\psi^+\psi, \tag{23}$$

$$(A_L)_i = L_{ij}A_j = \frac{-1}{4\pi}\frac{\partial}{\partial x_i}\sum_j\frac{\partial}{\partial x_j}\int d^3y\, A_j(\vec{y})/|\vec{x}-\vec{y}| . \tag{24}$$

Here A_L is the longitudinal part of A. The first term in (23) is "causal". We calculate the expectation value of (23) in the state $\alpha_n^+\Omega$ (notice that $\psi^+\psi = \varphi^+\varphi$) When $\vec{x} \cong \vec{R}$ and $t < R/c$ one gets in the first order in e:

$$\langle\alpha_n^+\Omega|\vec{P}_J(\vec{x},t) - \vec{P}(\vec{x},t)|\alpha_n^+\Omega\rangle = -\frac{e}{\hbar c}\vec{\nabla}W(\vec{x},t)\sum_{\omega}|u_n(\vec{x},\omega)|^2 \tag{25}$$

Comparing with eq. (5) we conclude that $-e/\hbar c\,\vec{\nabla}W$ is the change of the phase gradient of the electron wave function in V_D (as compared with that value of the gradient which it would have if the source was not switched on). This quantity varies slightly over the region V_D. Let us estimate (25) under the same conditions which were used for the estimation of eq.(22). Using eq.(20) we obtain that at the moment $t = 0.1R/c$ the change of the gradient (in the direction of the axis x) is equal to $\sim 10^{-3}$ radian per λ_m. The initial phase gradient of the plane wave $exp\, i p_l x_l$ is determined by the momentum and equals to $p_l = m v_l/\hbar$. This amounts to $0.1/\lambda_m$ at $v_l = 0.1c$ and $10^{-3}/\lambda_m$ at $v_1 = 10^{-3}c$. So the acausal change of the phase is appreciable for slow electrons. Meanwhile the relative change of such electron density is small because of the multiplier v_l/c entering eq. (22):

(relative change of density) $= \frac{v_l}{c} \cdot \lambda_m \cdot$ (change of the phase grad.) (26)

We can argue that (25) is also a small effect, similar to (22). The phase change can be measured by observing an interference picture. But this means measuring the change of the electrons density (its maxima and minima). This change is especially small just when the phase change is appreciable, see(26).

Does the result (25) imply, that the electron momentum has acausal behavior too? To discuss this question we express the result (25) in terms of the usual momentum of the single nonrelativistic electron.

4. Causal behaviour of the electron velocity contrary to its canonical momentum.

Let us consider the system: external current in V_s —quantized electromagnetic field — a single electron, which is bound by some potential $V(q)$ inside the region V_0 (located near the origin). There is a detector in V_D , which measures the distribution over the electron momentum p or coordinate q .

The system Hamiltonian is taken at first in the Coulomb gauge

$$\mathcal{H}(t) = \frac{1}{2m}[\vec{p} - e\vec{A}_\perp(\vec{q})]^2 + V(q) + \frac{1}{8\pi}\int d^3x\,[\vec{E}_\perp^2(x) + \vec{H}^2(x)] + \int_{V_s} d^3x\,(\vec{J}(\vec{x},t)\vec{A}_\perp(\vec{x})) + e\int_{V_s} d^3x\; J_0(\vec{x},t)/|\vec{x}-\vec{q}| \tag{27}$$

The electron is nonrelativistic and spinless /see § 13 in /2/ and § 17 in /9//. To obtain the equation for the Heisenberg operator

$q_J(t)$ we find at first

$$\partial_t \vec{q}_J(t) = -i\left[\dot{q}_J(t), \mathcal{H}^H\right] = \frac{1}{m}\left[\vec{p}_J(t) - e\vec{A}_{J\perp}(q_J(t),t)\right] . \tag{28}$$

Here $\mathcal{H}^H = \mathcal{U}^+(t,0)\mathcal{H}(t)\mathcal{U}(t,0)$ and is the same function of Heisenberg operators, as $\mathcal{H}(t)$ is of the Schrödinger ones, see (27). Further we calculate $[\partial_t q_J, \mathcal{H}^H]$ and get (compare §23 in /5/)

$$m^2 d^2\vec{q}_J(t)/dt^2 = grad\, V(q_J(t)) + e\,\vec{E}_J(\vec{q}_J(t),t) +$$
$$+\frac{e}{2}\left\{\left[\partial_t \vec{q}_J \times \vec{H}_J(q_J(t),t)\right] - \left[\vec{H}_J \times \partial_t \vec{q}_J\right]\right\} \tag{29}$$

$$\vec{E}_J(\vec{x},t) = -\partial \vec{A}_{\perp J}(\vec{x},t)/\partial t - grad\int d^3y\; J_0(\vec{y},t)/|\vec{x}-\vec{y}| . \tag{30}$$

The Lorentz force operator stands in the right-hand side of (29). Starting from (29) and the electromagnetic field equations one can show that [1)]

$$q_J(t) - q(t) = 0 \quad \text{when} \quad t < R/c . \tag{31}$$

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1) The proof of (31) is more complicated as compared with the Appendix (even if it is done only in the first order in e), The reason is that E and H in the r.h.s. of (29) depend upon the operator $q(t)$, rather than upon X. Eq. (31) is exact if the potential $U(q)$ is infinite outside $V_{\mathcal{D}}$, the proper functions of $p^2/2m +$ $+V(q)$ being zero outside $V_{\mathcal{D}}$. In this case one can prove (31) in all orders of perturbation theory. Eq. (31) is only approximately true for realistic potentials (then $q_J(t) - q(t)$ is exponentially small at $t < R/c$). Let us add, that despite of eq.(33), the difference $\vec{E}^{(0)}_J(\vec{x},t) - \vec{E}^{(0)}(x,t)$ of the operators (30) is a function of retarded integrals of $\vec{J}$ and J_0 and vanishes when $t < R/c$ (a direct proof is given in /10/).

This result was obtained in (11) for the oscillator potential $V(q)\sim q^2$ and in dipole approximation ($\vec{A}_\perp(q)$ in (27) is replaced by $A_\perp(0)$) . In return it was obtained without using perturbation theory. Eq.(31) means that the electron coordinate distribution does not change till the moment R/c /11/. There is no contradiction with (22) since r.h.s. of (22) vanishes when $c\to\infty$ /see also (26)/. Besides (22) is obtained for the case when no external potentials are present, while (31) is obtained for the bound electron (see footnote 1)

One obtains from (28) that

$$\vec{p}_J(t)-\vec{p}(t)-e\left[\vec{A}_{J\perp}(q_J(t),t)-\vec{A}_\perp(q(t),t)\right]=m\,\partial_t\left[q_J(t)-q(t)\right]. \tag{32}$$

The r.h.s. of (32) vanishes if $t<R/c$, but $A_{J\perp}-A_\perp$ does not. Indeed at $\vec{x}\in V_\mathcal{D}$, $t<R/c$ we have

$$A^{(0)}_{J\perp k}(\vec{x},t)-A^{(0)}_{\perp k}(\vec{x},t)=\sum_m(\delta_{km}-L_{km})\int d^4y\,\mathcal{D}_{ret}(x-y)J_m(y)\neq 0 \tag{33}$$

because of the nonlocal character of the longitudinal projection operator L_{km}, see (24). So, at $t<R/c$

$$p_{Jk}(t)-p_k(t)=-e\,L_{km}\int\mathcal{D}_{ret}\,J_m\equiv -e\,\partial_k\,W(\vec{R},t)\neq 0\ . \tag{34}$$

The canonical momentum p has acausal behavior, in contrast to the operator $m\partial_t q\equiv mv$. This means acausal behavior of the phase of the electron wave function in the coordinate representation /11/.

Eqs. (28), (29) and (34) are true also in classic theory [2]. However all classic observables (velocity, angular momentum and so on)

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[2] I am grateful to A. Shabad for drawing my attention to this point.

can be expressed in terms of the function $q(t)$. In quantum mechanics the canonical momentum p plays an independent role, being related to the phase of the wave function. This role cannot be played by the operator $m\vec{v}$, since the x, y, z - components of the velocity (28) do not commute. So $m\vec{v}$ cannot be represented by the operator $(-i\partial)$ and the plane wave cannot be written as $\exp i m\vec{v}\vec{x}$ /12/ [3]. However it seems that just the quantity $m\vec{v}$ is detected when measuring the track curvature.

The same results can be obtained using the Lorentz gauge formulation of the theory. One gets the Lorentz gauge Hamiltonian if one drops out in (27) the subindex $\perp$, replace the last two terms by $\int d^3x\, J_\mu A_\mu$ and substitute eq. (17. 7) from /9/ for the electromagnetic energy operator. One obtains just the same equation for $\ddot{q}(t)$ but now $\dot{E}$ in r.h.s. denotes $-\partial A/\partial t$ $-\text{grad}\, A_o$. The equation $m\partial_t q = p - eA$ now contains A instead of $A_\perp$ and, consequently, p has causal behaviour. But now p is not gauge-invariant and cannot be a physical operator. The velocity operator $\vec{v}_L = (\vec{p} - e\vec{A})/m$ is gauge-invariant, but $[v_{Lx}, v_{Ly}]$ is not zero again and therefore $m v_L$ has no relation to the phase. One can take, of course, the Coulomb gauge momentum $p_\perp$ and express it in terms of Lorentz gauge operators. The result will be $p_\perp = p_i - eA_i$, $A_{\perp i} = L_{ij} A_j$ (the derivation is not presented). If one has not other definitions of the gauge-invariant canonical momentum (besides $p_L - e A_L$)

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[3] Therefore the effect (34) must be considered as a quantum one (as also the effects (22) and (25)) though it does not vanish when $\hbar \to 0$ (see /13/ for other quantum effects which do not vanish when $\hbar \to 0$).

in the Lorentz gauge then one get the same result as in the Coulomb gauge.

5. Discussion and conclusion.

It was shown that the theory possesses observables such that a device measuring them can detect a superluminar signal velocity. Taking the effect (22) as an example let us discuss its theoretical preconditions.

The quantum causality criterium, as defined in sect.2, seems reasonable and unquestionable. One gets causal behavior in the sense of the criterium for such local observables as $\vec{E}(x)$, $\vec{H}(x)$ $j_\mu(x)$. We concentrate upon using of $N^{(-)}(x) = \varphi^{(-)\dagger}(x)\,\varphi^{(-)}(x)$ for the electron density operator. It is constructed with the help of the gauge-invariant electron-positron field operator φ , inherent to the Coulomb gauge [4]. Remind that the "causal" electron density $\psi^{(-)\dagger}(x)\psi^{(-)}(x)$, see (1), is forbidden as gauge-nonvariant [5].

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[4] It is of interest to mention that two-dimensional quantum electrodynamics is "causal": the function $\partial W/\partial x$ (and, consequently, the r.h.s. of (22), (25)) vanish at $t < R/c$ (compare /16/):

$$\partial W/\partial x = \partial/\partial x\,[\,\mathcal{U}_f^{(o)} - \mathcal{U}^{(o)}\,] = A_f^{(o)} - A^{(o)} = \int \mathcal{D}_{ret}\, J.$$

The function $\mathcal{U}$ from eq. (7) is the potential for A_L and in the two-dimensional case $\vec{A}$ coincides with A_L .

[5] One can state, that it is forbidden as an unphysical operator by the Lorentz condition (see Note 1 in App). This calls up a parallel. Theories which describe higher (>1) spin particles moving in external electromagnetic fields also have subsidiary conditions (which eliminate superfluous wave function components) and also suffer from the superluminar velocity of propagation /15/.

We do not know any gauge-invariant electron density operator having causal behavior. Mandelstam has constructed another gauge-invariant fermion field operator, not coinciding with φ, see /14/. However it also commutes nonlocally with E, see (3.11b) in /14/. Note further, that $N^{(-)}(x)$ is defined with the help of the coordinate, which naturally arises in field theory. We know other electron-position operators, i.e. see ch.4f in /3/. It seems however that the following assertion is true for any position operator q: a state which is localized in q is smeared in terms of x but over a region having dimensions not exceeding the electron Compton wavelenght. If so, this difference between q and x is not relevant, because in (22) one deals with a change of the particles number in macroscopic volume $V_{\mathfrak{D}}$. Despite the above the orthodox conclusion from the result (22) is that $N^{(-)}(x)$ must be rejected as the definition of the electron density operator, though another definition is unknown. In the case of such an observable as the electron momentum one can point out an operator having causal behavior: it is sufficient to claim that real devices measure not p but $mv = p - eA$, see sect. 4. The same resolution of our causal difficulty may be proposed for such observables like electron angular momentum or energy. For instance, one may relate with the nonrelativistic electron energy not the operator $p^2/2m + V(q)$ but the operator $(p-eA)^2/2m + V(q)$ though it is unusual.

This rejection of some theoretical observables may be considered as a possible formulation of our results:

"The local commutativity of A_μ and ψ does not prevent the superluminar signal velocity. A new principle must be introduced in the theory; signal velocity must not exceed c. Then just this principle (and not the other postulates of the quantum electrodynamics) forbids operators having acausal behavior".

In conclusion I stress that the smallness of the considered acausal effect means that they do not contradict any known experiment on light velocity [6]. There is the contradiction with the theoretical principle - relativistic causality, i.e. the synthesis of the usual causality and the special relativity.

I wish to express my thanks to D. Kirzhnits, A. Shabad and B. Valuev for discussions.

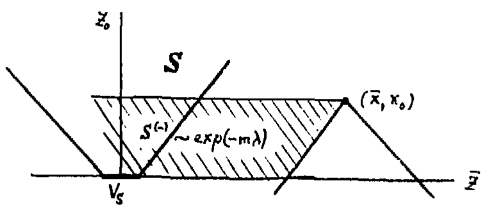

Fig.1.

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6) The considered effects may only give an idea of possible new experiments. We did not discuss real detectors but simply supposed the existense of desired devices.

Appendix

The notation of /1/ is assumed. When a constant external potential $V_\mu(\vec{x})$ and nonstationary external current $J_\mu(\vec{x},t)$ are present one has the following equations for electromagnetic potentials and spinor field

$$(\gamma_\mu \partial_\mu + m)\psi_J(x) = ie\left[A_{J\mu}(x) + V_\mu(\vec{x})\right]\gamma_\mu \psi_J(x) \tag{A.1}$$

$$\Box A_{J\mu}(x) = -\left[j_{J\mu}(x) + J_\mu(x)\right] \tag{A.2}$$

Here x denotes $\{\vec{x}, x_0\}$. The operators A_μ and ψ without subindex J (the external current is absent) satisfy the same equation, but J_μ being equal to zero. It is implied that j_μ in (A.2) denotes antisymmetrized operator $ie\bar{\psi}\gamma_\mu\psi$.

Solving (A.1) and (A.2) means that one knows how Heisenberg operators are expressed in terms of initial, i.e. Schrödinger, operators. The operators ψ_J, A_J and ψ, A must coincide at $t=0$ with the same Schrödinger operators (according to their definition, see sect. 2):

$$\psi_J(\vec{x},0) = \mathcal{U}^{-1}(0,0)\,\psi(\vec{x})\,\mathcal{U}(0,0) = \psi(\vec{x}) = \psi(\vec{x},0) \tag{A.3}$$

$$A_{J\mu}(\vec{x},0) = A_\mu(\vec{x}) = A_\mu(\vec{x},0) \; .$$

To solve (A.1), (A.2) let us expand ψ_J, A_J in a power series in the coupling constant e (G. Källen, see §23 in /1/ and ch. 8.7 in /4/).

$$\psi_J(x) = \sum_n e^n \psi_J^{(n)}(x)\;; \qquad A_{J\mu}(x) = \sum_n e^n A_{J\mu}^{(n)}(x) \tag{A.4}$$

We use analogous expansions for ψ and A. The initial conditions (A.3) will be satisfied if

$$\psi_J^{(0)}(\vec{x},0) = \psi^{(0)}(\vec{x},0) = \psi(\vec{x}) \;, \quad A_{J\mu}^{(0)}(\vec{x},0) = A_{\mu}^{(0)}(\vec{x},0) = A_{\mu}(\vec{x}) \tag{A.5}$$

and operators $\psi_J^{(n)}(\vec{x},0)$ and $A_{J\mu}^{(n)}(\vec{x},0)$ with $n \geq 1$ are equal to zero. We shall not consider the external current as small and write the following zero approximation for eqs (A.1) and (A.2):

$$(\gamma_\mu \partial_\mu + m)\,\psi_J^{(0)} = 0 \quad ; \quad \Box A_{J\mu}^{(0)} = -J_\mu \;. \tag{A.6}$$

The solutions of (A.6) are well known. The first equation is free. At all times $\psi_J^{(0)}(x) = \psi^{(0)}(x)$ and both are equal to free operator $\psi_0(x)$:

$$\psi_0(x) = -i\int d^3x'\, S(x-x')\,\gamma_0\,\psi(x') \;, \qquad x_0' = 0 \tag{A.7}$$

(see (8.67) in /3/). The second equation has the solution

$$A_{J\mu}^{(0)}(x) = A_{0\mu}(x) + \int \mathcal{D}_{ret}(x-x')\,J_\mu(x')\,d^4x' \tag{A.8}$$

$$A_{0\mu}(x) = \int d^3y\,\left[\partial\mathcal{D}(x-y)/\partial y_0\, A_\mu(y) - \mathcal{D}(x-y)\,\partial A_\mu(y)/\partial y_0\right] ; \quad y_0 = 0 \tag{A.9}$$

Indeed, (A.8) satisfies the equation $\Box A_{J\mu} = -J_\mu$ and the usual equaltime commutation relations (because they are satisfied by $A_{0\mu}$). When $x_0 = 0$ the operator $A_{J\mu}^{(0)}(x)$ coincides with the Schrödinger operator $A_\mu(\vec{x})$ (the second term in (A.9) at $x_0 = 0$ is equal to zero) We get

$$A_{J\mu}^{(0)}(x) - A_{\mu}^{(0)}(x) = \int \mathcal{D}_{ret}(x-x')J_\mu(x')d^4x' = \int d^3x'\, J_\mu(\vec{x}', x_0 - |\vec{x}-\vec{x}'|)/|\vec{x}-\vec{x}'| \;. \tag{A.10}$$

This difference is equal to zero if $\vec{x} \in V_D$ and $x_0 < R/c$ (remember that $J_\mu(x')$ is localized in V_S and is zero when $x_0' < 0$). Now we write the equations for operators $\psi_J^{(1)}, A_J^{(1)}, \psi^{(1)}, A^{(1)}$ and construct the following differences of these equations:

$$(\gamma_\mu \partial_\mu + m)(\psi_J^{(1)} - \psi^{(1)}) = i\left[(A_{J\mu}^{(0)} + V_\mu)\gamma_\mu(\psi_J^{(0)} - \psi^{(0)}) + (A_{J\mu}^{(0)} - A_\mu^{(0)})\gamma_\mu \psi^{(0)}\right], \quad (A.11)$$

$$\Box(A_{J\mu}^{(1)} - A_\mu^{(1)}) = -i\left[\bar{\psi}_J^{(0)}\gamma_\mu(\psi_J^{(0)} - \psi^{(0)}) + (\bar{\psi}_J^{0} - \bar{\psi}^{0})\gamma_\mu \psi^{(0)}\right]. \quad (A.12)$$

Here $\psi_J^{(0)} - \psi^{(0)} \equiv 0$ and $A_{J\mu}^{(0)} - A_\mu^{(0)}$ equal zero when $x_0 < R$. Therefore the right-hand sides of (A.11) and (A.12) vanish when $x_0 < R$. Because of zero initial values of $\psi_J^{(1)} - \psi^{(1)}$ and $A_J^{(1)} - A^{(1)}$ the solutions for these differences vanish when $x_0 < R$. For a formal proof one must express the solution of (A.11) and (A.12) in terms of their (zero) r.h.s. and (zero) initial conditions with the help of eqs at the end of §23 in /1/ and eqs. (A.7) and (A.9).

For the operators of the next (second) approximation one can obtain the equations of the kind (A.11) and (A.12). Their r.h.s. are expressed in terms of zero and first approximation operators. These r.h.s. vanish when $x_0 < R$ because they can be expressed in terms of the differences of these operators by analogy with (A.11) and (A.12) and these differences had been shown to be zeros at $x_0 < R$.

One gets by induction that the operator differences $\psi_J(\vec{x}_0, x_0) - \psi(\vec{x}_0, x_0)$ and $A_{J\mu}(\vec{x}_0, x_0) - A_\mu(\vec{x}_0, x_0)$ vanish in all order in e (and have not divergencies contrary to their constituents).

We conclude this Appendix with two Notes.

Note 1. The Lorentz condition must be added to eqs (A.1), (A.2) It has the form $\partial_\mu A_\mu(\vec{x}, x_0)\Phi = 0$ (Fermi form, see /6/) or $[\partial_\mu A_\mu(\vec{x}, x_0)]^{(-)}\Phi = 0$ (Gupta form). Because of $\Box\,\partial_\mu A_\mu = 0$ the condition is satisfied for any x_0 if it is satisfied for $x_0 = 0$ /2,6/ So, it is sufficient to take the initial vector ϕ in (8) from the space of allowable (by the Lorentz condition) vectors. The description of this space is given in /6/ for the Fermi form and in (17) for the Gupta form. We just use such a vector in (22).

The Lorentz condition requires in addition that physical observables must commute with the operators $\partial_\mu A_\mu$, $\partial_t \partial_\mu A_\mu$ (or with $[\partial_\mu A_\mu(\vec{x},0)]^{(-)}$) see §80 in /6/. But an operator do commute with these operators if it is gauge-invariant. Indeed, let us consider the unitary W, which realizes the gauge transformation

$$A'_\mu = W^{-1} A_\mu W = A_\mu + \partial_\mu \chi \quad ; \quad \psi' = W^{-1} \psi W = \psi \exp i e \chi \ . \tag{A.13}$$

The generator of W is expressed in terms of $\partial_\mu A_\mu$ and $\partial_t(\partial_\mu A_\mu) = \operatorname{div}\vec{E} - (j_0 + J_0)$, see (9.60) in /3/ (let us note that one can prove (A.13) in the case of interacting fields A_μ and ψ using the equal-time commutation relations).

<u>Note 2</u> We have shown, that at $t < R$ the <u>operator</u> differences $\psi_T - \psi$, $A_T - A$ vanish, the external potential V_μ being arbitrary. This means that the causal behavior of $E(x)$, $H(x)$, $j(x)$ is proved for any (allowable) initial state ϕ and any V_μ. It follows that this behavior takes place in much more complicated situations than those described in the beginning of sect 2. The state ϕ can describe electrons (free or bounded by V_μ), which are disposed between the source and the detector or are their constituents. One can assume that J_μ was a nonzero constant current till the moment $t = 0$ and at $t = 0$ it begins to alter somehow.

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