СООБЩЕНИЯ
ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

ДУБНА

E2 - 8542

S.Berceanu

ON TAUBERIAN THEOREMS AND INCLUSIVE PROCESSES

1975

E2 - 8542

S.Berceanu*

ON TAUBERIAN THEOREMS AND INCLUSIVE PROCESSES

* Present address: Institute of Atomic Physics, Bucharest.

The experimental data on multiparticle production at high energy indicate that the values of the total average multiplicity and average transverse momentum are systematically smaller than the phase space allows[1]. However, theoretically it is not clear if the general principles (like unitarity and analyticity) control the behaviour of the avegare multiplicity [2,3] $\langle n \rangle$ and average transverse momentum $\langle p_\perp \rangle$. Therefore, it is useful to find general conditions for the unsaturation of kinematical limits of $\langle n \rangle$ and $\langle p_\perp \rangle$.

In this note, firstly we show that the diffractive processes cannot lead to the saturation of the kinematical bound of $\langle n \rangle$ (Sect. 1). In Sect. 2 we argue that scaling[4] upper bounds for the structure function of one-particle inclusive reactions implies the unsaturation of the kinematical bound of $\langle n \rangle$ and $\langle p_\perp \rangle$. Further, if $\langle n \rangle$ and $\langle p_\perp \rangle$ increase asymptotically slower than $\sqrt{s}$, then a separate conservation of the reduced center of mass (c.m.) energy and longitudinal momentum occurs on both c.m. hemispheres, where $\sqrt{s}$ denotes the c.m. energy. The results are also extended to m-particle inclusive reactions (Sect. 3).

The proofs in this paper are mainly based on applications of some Tauberian theorems which seem to be a powerfull tool for dealing with scaling properties.

1. Average Multiplicity of Diffractive Processes

We shall prove that the diffractive processes do not lead to the saturation of kinematical bound of $\langle n \rangle$.

Let us denote by σ_n the (exclusive) n-particle production cross section and by σ_{tot} the total cross section of a hadronic ab collision. Introducing the probabilities for n -particle production

$$p_n(s) = \sigma_n(s)/\sigma_{tot}(s),$$

the total avegare multiplicity is defined as

$$<n(s)> = \sum_{n=2}^{\sqrt{s}} n p_n(s). \tag{1}$$

In fact, in the sum from the r.h.s. of eq. (1) n runs till the integer part of $(\sqrt{s} - m_a - m_b)/m_\pi$ but we express $\sqrt{s}$ in pionic masses and neglect the masses m_a, m_b of the colliding particles a, respectively b.

By diffractive processes we mean processes where

$$\lim_{s\to\infty} p_n(s) = p_n > 0. \tag{2}$$

The average multiplicity of a diffractive process is obviously

$$<n_D> = \sum_{n=2}^{\sqrt{s}} n p_n, \tag{3}$$

where p_n are defined by eq. (2).

Now we shall use a Tauberian theorem (due to Ikeara[5]). If the Dirichlet series

$$D(\xi) = \sum_{n=0}^{\infty} a_n/n^{\xi}, \quad a_n > 0 \tag{4}$$

is convergent for $\mathrm{Re}\,\xi > 1$ and regular at $\xi = 1$, then

$$\lim_{x\to\infty} x^{-1} \sum_{n<x} a_n = 0. \tag{5}$$

Taking in the Dirichlet series (4) $a_n = np_n$, the conditions of the Ikeara theorem are fulfiled because

$$D(1) = \sum_{n=2}^{\sqrt{s}} p_n \leq 1.$$

Hence, with $x = \sqrt{s}$ in eq. (5) we obtain

$$\lim_{s\to\infty} \sum_{n=2}^{\sqrt{s}} np_n/\sqrt{s} = 0,$$

i.e.,

$$\lim_{s\to\infty} \langle n_D \rangle / \sqrt{s} = 0 , \tag{6}$$

and so we have proved our assertion.

We remember that some models [3] claim the saturation of the kinematical bound of $\langle n \rangle$. Eq. (6) shows that if the saturation occurs, then it could come only from non-diffractive processes. Details will be discussed elsewhere [6]. Here we want to stress also the utility of another Tauberian theorem [7] for diffractive processes. This theorem enables to establish (rigorously) the expected asymptotic behaviour of the multiplicity moments,

$$\langle n^k \rangle \underset{s\to\infty}{\sim} s^{(k-1)/2} , \quad k = 2, 3, \ldots , \tag{7}$$

for diffractive models [8] for which

$$p_n \sim 1/n^2 . \tag{8}$$

We express the Tauberian theorem [7] in a sufficient form for the present purpose. Let us suppose that the series

$$Q(t) = \sum_{m=0}^{\infty} q_m t^m , \quad q_m \geq 0 \tag{9}$$

converges for $0 \leq t < 1$. If L varies slowly [7] at ∞, $0 \leq \rho < \infty$, , and the sequence $\{q_m\}$ is monotonic, then the relation

$$\sum_{m=0}^{N} q_m \sim \frac{1}{\Gamma(\rho+1)} N^{\rho} L(N) , \quad N \to \infty \tag{10}$$

is equivalent to

$$q_N \sim \frac{1}{\Gamma(\rho)} N^{\rho-1} L(N) , \quad N \to \infty . \tag{11}$$

The series (9) with

$$q_m = m^k p_m , \quad k = 2, 3, \ldots \tag{12}$$

is evidently convergent for $0 \leq t < 1$. We proceed to evaluate the multiplicity moments for diffractive processes

$$<n^k> = \sum_{m=2}^{\sqrt{s}} q_m = \sum_{m=2}^{\sqrt{s}} m^k p_m , \quad k = 2,3,\ldots \quad . \tag{13}$$

Supposing for p_m the behaviour (8), q_m(eq.(12)) takes the form (11) for $\rho = k-1$ $(N=\sqrt{s})$. Then eq. (10) furnishes the behaviour (7) for the multiplicity moments (13) of diffractive processes.

2. Unsaturation of the Kinematical Bound of $<n$ and $<p_\perp>$

We begin with some notations and definitions. Let us denote by $f_c(x, p_\perp^2, s)$,

$$f_c(x, p_\perp^2, s) = \frac{2\pi}{\sigma_{tot}(s)} x_c^0 \frac{d\sigma_c}{dx dp_\perp^2} , \tag{14}$$

the structure function of the inclusive reaction

$$a + b \to c + \text{anything} . \tag{15}$$

Here x is the reduced c.m. longitudinal momentum $(p_{||})$, $p_\perp$ is the transverse momentum and x_c^0 the reduced c.m. energy E of the particle c with mass M_c ,

$$x_c^0 = 2E_c/\sqrt{s} = 2\sqrt{p_{||}^2 + p_\perp^2 + M_c^2}/\sqrt{s} = \sqrt{x^2 + 4(p^2 + M_c^2)/s} \ . \tag{16}$$

With these notations the total average multiplicity and total average transverse momentum can be written, respectively,

$$<n>_1 = \sum_c <n_c> = \sum_c \int \frac{f_c(x, p_\perp^2, s)}{x_c^0} dx dp_\perp^2 , \tag{17}$$

$$<p_\perp>_1 = \sum_c <p_\perp>_c <n_c> = \sum_c \int \frac{f_c(x, p_\perp^2, s)}{x_c^0} p_\perp dx dp_\perp^2 . \tag{18}$$

The index 1 specifies that we deal with average values for the one-particle inclusive reaction (15). Analogously are introduced the total average of $p_\perp^k$, $k = 2, 3, \ldots$ The generalization to m-particle inclusive reactions is sketched in Sect. 3.

A useful quantity in the following is the average

$$\langle p_\perp^k x^0 \rangle_1 = \sum_c \int p_\perp^k f_c(x, p_\perp^2, s)\, dx dp_\perp^2, \quad k=0,1,2,\ldots \,. \tag{19}$$

We need also the energy and longitudinal momentum sum rules[9], which in the present notations take the form

$$\langle x^0 \rangle_1 = 2\,, \tag{20}$$

$$\langle x \rangle_1 = 0\,. \tag{21}$$

With the notations

$$x_\perp = 2p_\perp / \sqrt{s}\,, \tag{22}$$

$$\langle N^k \rangle_1 = \langle n^k \rangle_1 / \sqrt{s}\,, \quad k = 1, 2, \ldots\,, \tag{23}$$

our purpose is to find conditions in which the following relations hold:

$$\lim_{s \to \infty} \langle N \rangle_1 = 0\,. \tag{24}$$

$$\lim_{s \to \infty} \langle x_\perp \rangle_1 = 0\,. \tag{25}$$

Now let us suppose that the structure function (14) has the scaling bound

$$\int f_c(x, p_\perp^2, s)\, dx \leq g_c(p_\perp) / 2p\,, \tag{26}$$

and in the central region has the behaviour

$$\max_{x \in [-\epsilon, \epsilon]} f_c(x, p_\perp^2, s) \leq \frac{\gamma}{(p_\perp^2 + M_c^2)[\ln(p_\perp^2 + M_c^2)]^{\beta}}, \quad \beta = 2 + \delta,\ \delta > 0\,. \tag{27}$$

It can be noted that the upper bound (27) for the behaviour at large $p_\perp$ in the cenrtal region is a weak condition for the structure function, which is verified by usual models[10] for large $p_\perp$.

We assert that if the structure functions (14) have the scaling upper bounds (26) and verify the conditions (27) at large $p_\perp$, then the total average multiplicity and total average transverse momentum increase slower than $\sqrt{s}$ when $s \to \infty$ (i.e., the relations (24), (25) hold). Also, if the condition (26) is verified, then

$$\lim_{x \to \infty} \langle x_\perp^2 \rangle_1 = 0. \tag{28}$$

For the proof the following Tauberian theorem[11] will be used. If

$$\lim_{x \to \infty} \int_1^x \frac{b(y)}{y} \left(1 - \frac{y}{x}\right) dy = \alpha , \tag{29}$$

where b is bounded real function, then

$$\lim_{x \to \infty} \int_1^x \frac{b(y)}{y} dy = \alpha . \tag{30}$$

Introducing in eq. (20) the bound (26) and applying the Tauberian theorem (29)-(30) with $b(y) = y g_c(y)$ and $x = \sqrt{s}$, we get readily

$$\lim_{s \to \infty} \int_0^{\sqrt{s}/2} dp_\perp\, p_\perp\, g_c(p_\perp) / \sqrt{s}/2 = 0 . \tag{31}$$

With the definitions (16) and (19) for $k = 1$, and the bound (26), eq. (28) is obtained immediately. Further, with the bounds (27) for $f_c(x, p_\perp^2, s)$ in the central region and eq. (31) in the fragmentation region, it can be verified that the relations (24) and (25) hold.

3. Asymptotic Left-Right Conservation of x and x^0

In this Section it will be supposed that eqs. (24) and (25) hold. We argue that separate left-right conservation of the reduced c.m. energy and longitudinal momentum occurs. Also we shall be concerned with the evaluation of the asymptotic behaviour of more general average quantities for the m-particle inclusive reactions

$$a+b \to c_1 + \dots + c_m + \text{ anything .} \tag{32}$$

Let us consider the average of a function $F_{c_1 \dots c_m}(\vec{p}_1, \dots, \vec{p}_m)$ on a given volume of the phase space

$$\langle F(R) \rangle_m = \sigma_{tot}^{-1}(s) \sum_{c_1, \dots, c_m} \int_R F_{c_1 \dots c_m}(\vec{p}_1, \dots, \vec{p}_m) \frac{d\sigma_{c_1 \dots c_m}}{d\vec{p}_1 \dots d\vec{p}_m} d\vec{p}_1 \dots d\vec{p}_m, \tag{33}$$

where $\vec{p}_i$ denotes the 3-momentum vector of the particle $c_i (i=1, \dots, m)$ and $d\sigma_{c_1 \dots c_m}/d\vec{p}_1 \dots d\vec{p}_m$ the differential cross section of the process (32). In the following we take F as the monoms

$$F_{c_1 \dots c_m} = \prod_{i=1}^{m} (p_{||}^{c_i})^{\alpha_i} (p_{\perp}^{c_i})^{\beta_i} (E^{c_i})^{\gamma_i}, \alpha_i, \beta_i, \gamma_i \geq 0 . \tag{34}$$

If in eq. (33) we introduce F with particular values α, β, γ from eq. (34), we get the averages (17)-(21). When $F_{c_1 \dots c_m} = 1 (m > 1)$, we get the binomial multiplicity moments in the R region of the phase space

$$\langle \nu(R) \rangle_m = \sigma_{tot}^{-1}(s) \sum_{c_1, \dots, c_m} \int_R \frac{d\sigma_{c_1 \dots c_m}}{d\vec{p}_1 \dots d\vec{p}_m} d\vec{p}_1 \dots d\vec{p}_m . \tag{35}$$

It may be remarked that the inequality

$$|p_{||}| \leq E_c \leq |p_{||}| + p_{\perp} + M_c \tag{36}$$

implies the same asymptotic behaviour of x^0 and x in the average (33) of the monoms (34), when eq. (25) holds.

Further we list 4 consequences of eqs. (24), (25) for various averages (33)-(34).

a. If eqs. (24), (25) hold, then

$$\lim_{s\to\infty} \langle x(r/l)\rangle_1 = 1 , \tag{37}$$

i.e., if the total average multiplicity and total average transverse momentum increase slower than $\sqrt{s}$ when $s\to\infty$, then a separate right (left) conservation of the reduced c.m. longitudinal momentum (and energy) occurs.

Adding the sum rules (20) and (21) and taking into account ineq. (36), we get easily

$$\langle x(r)\rangle_1 + \langle x_\perp\rangle_1 + \sum_c M_c \langle n_c\rangle/\sqrt{s} \geq 1 \geq \langle x(r)\rangle_1 .$$

If eqs. (24), (25) are true, then eq. (37) follows.

b. If eq. (25) holds, then

$$\lim_{s\to\infty} \langle x^\alpha x_\perp^\beta\rangle_1 = \begin{cases} 0, \text{ when } \beta > 0 \\ \lambda, \text{ when } \beta = 0 , \end{cases} \tag{38}$$

where

$$\begin{aligned} \lambda &= \gamma , && \text{if } \gamma = 0, 1 \\ 0 < \lambda &< 1 , && \text{if } 0 < \gamma < 1 . \end{aligned} \tag{39}$$

Here by γ we denote

$$\gamma = \lim_{s\to\infty} \langle x^2\rangle_1 .$$

The behaviour from eqs. (38), (39) can be checked applying the Schwartz inequality.

c. If eq. (24) holds, then

$$\lim_{s\to\infty} \langle N^k\rangle_1 = 0 , \; k = 1, 2, \ldots ; \; \lim_{s\to\infty} \langle \nu(R)\rangle_m / s^{m/2} = 0 , \tag{40}$$

where N is defined by eq. (23) and ν by eq. (35).

d. If eqs. (24), (25) hold, then

$$\lim_{s\to\infty} \frac{\langle p^k_{\|,\perp,0}\rangle_m}{(\sqrt{s}/2)^{m+k}} = 0 , \; m > 1, k \geq 1 , \tag{41}$$

where the index 0 refers to the energy.

For proving eq. (41), the energy-momentum sum rules [9] for m-particle inclusive reactions are needed:

$$(P-p_{c_1}-\dots-p_{c_{m-1}})\frac{d\sigma_{c_1\dots c_{m-1}}}{d\vec{p}_1\dots d\vec{p}_{m-1}}=\sum_{c_m}\int p_{c_m}\frac{d\sigma_{c_1\dots c_m}}{d\vec{p}_1\dots d\vec{p}_m}\,d\vec{p}_1\dots d\vec{p}_m,$$

where p_{c_i} denotes the energy-momentum 4-vector of the particle c_i $(i=1,\dots,m)$ and P is the energy-momentum 4-vector of the initial state.

In conclusion, the properties (37)-(41) are consequences of the relations (24) and (25). Using Tauberian theorems we succeeded in Sect.2 to prove that the scaling properties (26) and (27) of the structure functions are sufficient conditions for eqs. (24), (25). A more general justification of eqs. (24) and (25) will be presented elsewhere [6].

The author acknowledges Dr. A.C.Gheorghe for initiating him in the use of Tauberian theorems.

References

1. D.R.O.Morrison. Proceedings of the Fourth International Conference on High-Energy Collisions, Oxford, (1972);
   D.R.O.Morrison. Review lectures given at the Fifth Hawaii Topical Conference in Particle Physics, CERN/D.Ph.II/PHYS 73-46, CERN preprint (1973).
2. N.N.Khuri. Phys.Rev., D9, 1802 (1974).
3. S.Berceanu, to be published.
4. R.P.Feynman. Phys.Rev.Lett., 23, 1414 (1969).
5. S.Leng. Algebraic numbers, Izdat. Mir, Moscow, § VIII (1966) (in Russian).
6. S.Berceanu, A.C.Gheorghe, to be published.
7. W.Feller, An introduction to probability theory and its applications, Izdat. Mir, Moscow, Vol, II, § XIII, 513 (1967) (in Russian).
8. C.Quigg, J.H.Wang, C.N.Yang. Phys.Rev.Lett., 28,1290 (1972).
9. L.S.Brown. Phys.Rev., D5, 748 (1972).
10. S.D.Ellis, R.Thun. Ref. TH 1874-CERN, CERN preprint (1974).

11. N.Dunford, J.T.Schwartz. Linear operators, Part II, Izdat. Mir, Moscow, 165 (1966) (In Russian).

Received by Publishing Department
on January 20, 1975.