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# MATRIX CANONICAL REALIZATIONS OF THE LIE ALGEBRA $o(m, n)$.

## I. Basic Formulae and Classification

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# MATRIX CANONICAL REALIZATIONS OF THE LIE ALGEBRA $\sigma(m, n)$.
## I. Basic Formulae and Classification

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## 1. Introduction

In the previous paper we dealt with canonical realizations of the complexified Lie algebra of the orthogonal group in n - dimensional Euclidean space $O_{\mathbb{C}}(n)$ /1/. By canonical realization we understood there an isomorphism mapping $\tau$ of a given Lie algebra $G$ into the Weyl algebra $W_{2N}$, i.e. essentially into the algebra of polynomials in $N$ pairs of quantum canonical variables $p^i$, $q_i$, $i = 1,2,\dots,N$. Among the other results we proved there that in any canonical realization of Lie algebra $\sigma(m,n)$ of a pseudoorthogonal group $O(m,n)$, $m \geq n > 0$, in $W_{2N}$, $N = m+n-2$ (for the exception of the case $m+n = 6$), all Casimir operators are realized by constant multiples of the identity element x/ (we speak about Schur-realizations) and if $m+n > 6$ they depend on the quadratic ones in one of the two possible ways only (it is what we call "degeneration" of realization). It means that to remove partly or even fully the mentioned "degeneration" we must enlarge the number N of canonical pairs. In contrast to canonical realizations of $\sigma(m,n)$ in $W_{2N}$, $N = m+n-2$, the realizations with $N > m+n-2$ need not be necessarily the Schur-realizations. So we come naturally to the question whether some realizations of $\sigma(m,n)$ exist in which the "degeneration" is at least reduced and which are at the same time Schur-realizations.

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x/ As to the particular case $W_{2(m+n-3)} \subset W_{2(m+n-2)}$ see also /2/.

In this paper we solve this question positively in the generalized framework of the so-called matrix canonical realizations in which the generators of considered Lie algebra are expressed by matrices with elements from $W_{2N}$ (if the dimension of such matrices is $M$, we denote such a generalization of the Weyl algebra by the symbol $W_{2N,M}$ ).

The matrix canonical realizations represent one possible proper algebraical embedding of the Weyl algebra into a larger structure. It is known that another possibility is to embed the Weyl algebra into its quotient division ring. In both these cases the class of allowed functions, in terms of which the generators of a given Lie algebra can be expressed, is essentially enriched compared with the original Weyl algebra. Therefore the possibility of obtaining the wider class of realizations in these structures arises without necessity of changing the pure algebraical approach.

From the point of view of application to the representetion theory we shall introduce further concept of skew-hermitean realization though the representation aspects are not discussed in this paper. By the skew-hermitean matrix canonical realization we shall essentially imply the one which after replacement of $p^i$ and $q_i$ by their Schrödinger representatives passes to a skew-symmetric representation on a suitable Hilbert space.

The main result of this paper lies in the formulae describing recurrently two sets of matrix canonical skew-hermitean Schur-realizations of the Lie algebras $\sigma(m,n)$, $n \geq 1$. In some special cases these formulae coincide with earlier results of some authors(see,e.g. Richard /3/). Every realization from the first set is uniquely determined by some finite-dimensional irreducible skew-hermitean representation of the compact Lie algebra $\sigma(m-n)$[x/] and the finite sequence of $n$ real numbers. If the dimension of representation of $\sigma(m-n)$ is $M$ then $\sigma(m,n)$is realized in $W_{2n(m-1),M}$. Realizations from the second set are usual canonical realizations in $W_{2d(m+n-d-1)}$, $d=1,2,\dots,n-1$ ; they are characterized by $d$-tuple of real constants.

We shall introduce further the concept of related and non-related realizations by means of which all realizations described above will be classified. We prove that any two realizations chosen from both the sets are non-related if they differ either in characterizing tuples of real numbers or in the case of realizations of the first type with the same characterizing tuples, if the irreducible representations of the algebra $\sigma(m-n)$ are non-equivalent.

The exact formulation of all these statements is contained in Theorem 3. Its proof is based mainly on Theorem 1 where the basic recurrent formulae are included.

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[x/] As to case $m-n=0,1$ see following remark.

Theorem 2 shows that any skew-hermitean matrix canonical Schur-realization of a compact Lie algebra is usual matrix skew-hermitean representation, which generalizes the Joseph assertion /4/.

Again, as in our previous paper, all considerations are purely algebraical. As to the problem of "degeneration", i.e. mutual dependence or independence of Casimir operators in the described realizations, we shall discuss these questions in the second part of this paper.

## 2. Preliminaries

A. The (complex) Weyl algebra $W_{2N}$ is the associative algebra with the identity element $\mathbb{1}$ over the field of complex numbers $\mathbb{C}$ ; its generating elements $q_i$, $p^i$, $i=1,\dots,N$, fulfill the usual canonical commutation relations

$$[p^i, p^j] = [q_i, q_j] = 0, \quad [p^i, q_j] = \delta^i_j \mathbb{1}.$$

As the consequence of the Poincaré-Birkhoff-Witt theorem the monomials

$$q^r (p)^s = q_1^{r_1} \dots q_N^{r_N} \cdot (p^1)^{s_1} \dots (p^N)^{s_N}$$

form the basis of $W_{2N}$ (see /5/, p.178), i.e. every element $w \in W_{2N}$ can be uniquely written in the form

$$w = \sum_{r,s} a_{r,s} \cdot q^r (p)^s ,$$

where $a_{r,s} = a_{r_1 \dots r_N s_1 \dots s_N} \in \mathbb{C}$.

B. The symbol $\sigma(m,n)$, $m \geqslant n \geqslant 0$, $m+n \geqslant 2$, denotes the Lie algebra of pseudoorthogonal group in $(m+n)$ - dimensional pseudoeuclidean space with the metric tensor $g_{\mu\nu}$, $\mu,\nu = 1,2,\dots,m+n$. If $L_{\mu\nu} = -L_{\nu\mu}$ denotes $1/2 \cdot (m+n)(m+n-1)$ elements of the basis of $\sigma(m,n)$ then the commutation relations hold

$$[L_{\mu\nu}, L_{\rho\tau}] = g_{\nu\rho} L_{\mu\tau} - g_{\mu\rho} L_{\nu\tau} + g_{\nu\tau} L_{\rho\mu} - g_{\mu\tau} L_{\rho\nu} \tag{1}$$

$\rho,\tau = 1,2,\dots,m+n$. If $n \geqslant 1$ we can assume without the loss of generality the metric tensor having the form $(g_{\mu\nu}) = diag(g_{11},\dots,g_{m+n-2,m+n-2},-1,1)$. In addition to the tensor basis $(L_{\mu\nu})$, the second one can be chosen

$$L_{ij}, \quad P_k = L_{k,m+n} + L_{k,m+n-1}, \quad Q_k = L_{k,m+n} - L_{k,m+n-1},$$
$$R = L_{m+n-1,m+n}; \quad i,j,k,l = 1,2,\dots,m+n-2,$$

in which the commutation relations (1) have the form:

$$\begin{aligned}
&[L_{ij}, L_{kl}] = g_{jk} L_{il} - g_{ik} L_{jl} + g_{jl} L_{ki} - g_{il} L_{kj}, \\
&[L_{ij}, P_k] = g_{jk} P_i - g_{ik} P_j, \quad [L_{ij}, Q_k] = g_{jk} Q_i - g_{ik} Q_j, \\
&[L_{ij}, R] = 0, \quad [R, P_k] = P_k, \quad [R, Q_k] = -Q_k, \\
&[P_i, P_j] = [Q_i, Q_j] = 0, \quad [P_i, Q_j] = -2(L_{ij} + g_{ij} R).
\end{aligned} \tag{2}$$

Note that the generators $P_1,\dots,P_{m+n-2}$ and $Q_1,\dots,Q_{m+n-2}$ form the bases of $(m+n-2)$-dimensional Abelian subalgebras of $\sigma(m,n)$.

C. Any irreducible finite-dimensional representation of the compact Lie algebra $\sigma(m,0) \equiv \sigma(m)$, $m \geqslant 2$, is, for the exception of $m=2$, equivalent to a skew-hermitean one. Any such representation of $\sigma(m)$ is uniquely determined by the so-called signature $\alpha=(\alpha_1,\dots,\alpha_{[\frac{m}{2}]})$, where the numbers $\alpha_1,\dots,\alpha_{[\frac{m}{2}]}$ for $m>2$ are either all integers or all half-integers such that $\alpha_1\geqslant\alpha_2\geqslant\dots\geqslant\alpha_{\nu-1}\geqslant|\alpha_\nu|$ if $m=2\nu$ and $\alpha_1\geqslant\alpha_2\geqslant\dots\geqslant\alpha_\nu\geqslant 0$ if $m=2\nu+1$.

In the case of commutative Lie algebra $\sigma(2)$ all irreducible skew-hermitean representations are one-dimensional and the generator $L_{12}$ is represented as $irE_1$, $r\in R$, where $E_1$ is identity operator. By the signature $\alpha$ of this representation we understand the one-point sequence $\alpha=(r\equiv\alpha_{[\frac{2}{2}]})$.

Two equivalent irreducible representations have the same signatures and to different signatures there correspond non-equivalent irreducible representations (/6/, p.518-519).

## 3. Basic concepts

Definition 1: Let $W_{2N}$ be the complex Weyl algebra in $N$ canonical pairs and let $Mat_M$ be the algebra of complex $M\times M$-matrices. The tensor product $W_{2N,M} = W_{2N}\otimes Mat_M$ we shall call matrix Weyl algebra.

It is clear that $W_{2N,1}\simeq W_{2N}'\simeq W_{2N}'\otimes E_M\subset W_{2N,M}$ ( $E_M$ is the unity $M\times M$-matrix) and $Mat_M\simeq 1\otimes Mat_M\subset W_{2N,M}$.

<u>Definition 2</u>: A matrix canonical realization of the Lie algebra $G$ is the homomorphism $\tau : G \to W_{2N,M}$ . The homomorphism $\tau$ extends naturally to the homomorphism mapping (denoted by the same symbol $\tau$ ) of the enveloping algebra $\mathcal{U}G$ of $G$ into $W_{2N,M}$. If $G$ is simple then any homomorphism is either trivial or it is an isomorphism of $G$.

Remember that involution on an associative algebra (over $\mathbb{R}$ or $\mathbb{C}$ ) is the mapping "+": $G \to G$ obeying the relations $(\alpha a + \beta b)^{+} = \bar{\alpha} a^{+} + \bar{\beta} b^{+}$ , $(ab)^{+} = b^{+} a^{+}$, $(a^{+})^{+} = a$ , $a, b \in G$ .

<u>Definition 3</u>: Let an involution "+" be defined on the algebra $W_{2N,M}$ . A matrix canonical realization of a real Lie algebra $G$ in $W_{2N,M}$ is called <u>skew-hermitean</u>, if all the elements of $G$ are realized by skew-hermitean expressions, i.e. if $\tau(a)^{+} = -\tau(a)$ for all $a \in G$ .

<u>Note 1</u>: An involution on $W_{2N}$ together with the usual Hermitean adjoint in $Mat_M$ defines involution in $W_{2N,M}$ . In what follows, we consider such an involution with a special choice induced by

$$(p^{i})^{+} = -p^{i}, \quad q_{i}^{+} = q_{i}$$

on the algebra $W_{2N}$ .

<u>Definition 4</u>: A matrix canonical realization $\tau : G \to W_{2N,M}$ is called <u>Schur-realization</u> if all the central elements of the enveloping algebra $\mathcal{U}G$ of $G$ are realized by the multiples of the identity element.

For the classification of matrix canonical realizations we introduce the following last concept:

<u>Definition 5</u>: Matrix canonical realizations $\tau$ and $\tau'$ of the Lie algebra $G$ in $W_{2N,M}$ are called <u>related</u> if a conserving identity endomorphism $\vartheta$ of $W_{2N,M}$ exists so that either $\vartheta \circ \tau = \tau'$ or $\vartheta \circ \tau' = \tau$ .

## 4. Matrix canonical realizations of the Lie algebra $\sigma(m,n)$.

<u>Theorem 1</u>: Let skew-hermitean Schur-realization of the algebra $\sigma(m-1,n-1)$, $m+n \geq 2$ x/, in $W_{2N,M}$ be given and let $M_{ij}=-M_{ji} \in W_{2N,M}$ denote the realization of the basis elements of $\sigma(m-1,n-1)$ . Then

i. the following formulae define the skew-hermitean Schur-realization of $\sigma(m,n)$ in $W_{2(m+n-2+N),M}$ :

$$
\begin{aligned}
\tau(P_i) &= p_i \\
\tau(L_{ij}) &= q_i p_j - q_j p_i + M_{ij} , \\
\tau(R) &= -(qp) - \left[\tfrac{1}{2}(m+n-2) - i\alpha\right]\mathbb{1} , \ \alpha \in \mathbb{R} , \\
\tau(Q_i) &= -q^2 p_i - 2q_i \tau(R) - 2q^j M_{ji} , \ i,j = 1,2,\dots, m+n-2 ,
\end{aligned}
\tag{3}
$$

where

$$(qp) = q^i p_i , \ q^2 = q^i q_i , \ p_i = g_{ij} p^j , \ q^i = g^{ij} q_j ,$$

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x/ In order to reduce maximally the number of exceptional cases we consider also "Lie algebras" $\sigma(0)$ and $\sigma(1)$ when we define $M_{ij} = 0$ . In the first case the formulae (3) define the realization $\tau(R) = i\alpha\mathbb{1}$ of $\sigma(1,1)$ in $W_{0,1}$ , while in the second case the realization of $\sigma(2,1)$ in $W_{2,1}$ .

ii. two realizations (3) with different values of the parameter $\alpha$ are not related,

iii. two realizations (3) differing only in realization of $\sigma(m-1,n-1)$ are related if and only if the realizations of $\sigma(m-1,n-1)$ are related.

For the proof of this theorem we shall use the following two easily provable assertions:

(a) If an element $a \in W_{2N,M}$ commutes with canonical variable $p^i$ (or $q_i$), then $a$ does not depend on $q_i$ (or $p^i$).

(b) If for $a \in W_{2N,M}$ and $(qp)_{N'} \equiv q_1 p^1 + \dots + q_{N'} p^{N'}$, $N' \leq N$ the commutation relation

$$[(qp)_{N'}, a] = t \cdot a \;,$$

is valid for some t = 0,±1... , then

$$a = \sum_{\substack{r,s \\ r-s=t}} a_{r,s} \cdot q^r (p)^s$$

$(a_{r,s}\, q^r (p)^s \equiv a_{r_1 \dots r_{N'}, s_1 \dots s_{N'}} \cdot q_1^{r_1} \dots q_{N'}^{r_{N'}} \cdot (p^1)^{s_1} \dots (p^{N'})^{s_{N'}}$, $r - s = r_1 + r_2 + \dots + r_{N'} - s_1 - \dots - s_{N'}$) and $a_{r,s}$ does not depend on $q_1, \dots, q_{N'}, p^1, \dots, p^{N'}$.

<u>Proof</u>: One can by direct commutation verify that the expressions (3) form a realization $\tau$ of the basis $\sigma(m,n)$. With respect to involution we use (see Note 1), all the expressions (3) are skew-hermitean and therefore they generate (through real linear combinations) the skew-hermitean realization $\tau$ of $\sigma(m,n)$. We shall prove that $\tau$ is Schur-realization.

Let us take some arbitrary centre element $z$ of $U\sigma(m,n)$. Its realization $\tau(z)$ is a polynomial

$$\tau(z) = \sum_{r,s} \beta_{r,s}(M_{ij})\, q^r (p)^s$$

($r = (r_1,\dots,r_{m+n-2})$, $s = (s_1,\dots,s_{m+n-2})$) which commutes with all the expressions (3). The coefficients $\beta_{r,s}(M_{ij})$ depend polynomially on the basis elements $M_{ij}$ of the realization of $\sigma(m-1,n-1)$. Due to the commutation relations $[\tau(z), \tau(P_i)] = 0$, i.e. $[\tau(z), p_i] = 0$, in accord with the assertion (a), $\tau(z)$ does not depend on $q_1,\dots,q_{m+n-2}$, i.e. $\tau(z) = \sum_s \beta_{0,s}(M_{ij})(p)^s$. As $\tau(z)$ commutes also with $\tau(R) = -(qp) + const$, the assertion (b) can be applied, which gives

$$\tau(z) = \beta_{0,0}(M_{ij}).$$

We shall use once more the fact that $\tau(z)$ realizes a centre element of $U\sigma(m,n)$. It implies that $\tau(z)$ commutes with all $\tau(L_{ij}) = q_i p_j - q_j p_i + M_{ij}$, and consequently

$$[\tau(z), M_{ij}] = 0.$$

Due to the assumption of the theorem the realization of $\sigma(m-1,n-1)$ is Schur-realization, and therefore each polynomial in its basis elements commuting with all of them equals some multiple of the identity element, $\beta_{0,0}(M_{ij}) = \beta\cdot\mathbb{1}$, $\beta \in \mathbb{C}$, which proves the first statement of the theorem.

ii and iii. Let us consider two realizations $\tau$ and $\tau'$ of the type (3) and assume the existence of an endomorphism $\vartheta : W_{2(m+n-2+N),M} \to W_{2(m+n-2+N),M}$ such that $\vartheta(1) = 1$ and $\vartheta \circ \tau = \tau'$ (i.e. $\tau$ and $\tau'$ are related). We shall show that $\alpha = \alpha'$ and that the corresponding realizations of $\sigma(m-1, n-1)$ are also related. The equations $\vartheta \circ \tau(P_i) = \tau'(P_i)$ and $\vartheta \circ \tau(R) = \tau'(R)$ give immediatelly

$$\vartheta(p_i) = p_i \ , \quad i = 1,2,\dots, m+n-2 \tag{4}$$

and

$$\vartheta(qp) = (qp) + i(\alpha - \alpha')1 \ . \tag{5}$$

The element $\vartheta(q_i) \in W_{2(m+n-2+N),M}$ can be written in the form

$$\vartheta(q_i) = \sum_{r,s} \beta_{i,rs}\, q^r (p)^s$$

where $\beta_{i,rs} \equiv \beta_{i,r_1 \dots r_{m+n-2}, s_1 \dots s_{m+n-2}} \in W_{2N,M}$ .
The polynomial

$$\vartheta(qp) - (qp) = (\vartheta(q_i) - q_i)\cdot p^i$$

is either zero or its lowest degree in the "variables" $p_1, \dots, p_{m+n-2}$ is greater or equal to one . As this polynomial equals to $i(\alpha - \alpha')$ , the second possibility is excluded and the first one implies $\alpha = \alpha'$ , which proves the assertion ii.

For $m+n = 2,3$ the theorem is proved completely because only trivial realization of the "Lie algebras" $\sigma(0)$ and $\sigma(1)$ exists; we shall assume therefore further $m+n > 3$ .

The polynomial $\vartheta(q_i) - q_i$ commutes with all $p_j \equiv \vartheta(p_j)$ and therefore due to the assertion (a) does not depend on $q_i$, $i = 1,2,\dots,m+n-2$, i.e.

$$\vartheta(q_i) = q_i + \sum_{s} \beta_{i,\alpha s} (p)^s.$$

As $\vartheta$ is an endomorphism of $W_{2(m+n-2+N),M}$ and $\vartheta(qp) = (qp)$ (see eq. (5), $\alpha = \alpha'$), we know the commutation relations between $\vartheta(q_i)$ and $(qp)$,

$$[(qp), \vartheta(q_i)] = [\vartheta(qp), \vartheta(q_i)] = \vartheta([(qp), q_i]) = \vartheta(q_i),$$

so that we can apply the assertion (b) to obtain:

$$\vartheta(q_i) = q_i\ , \quad i = 1,2,\dots, m+n-2. \tag{6}$$

The images $\vartheta(q_{m+n-1})$, $\vartheta(p_{m+n-1}), \dots$ of the other canonical variables commute with all $\vartheta(q_i)$ ; $\vartheta(p_i)$ , $i = 1,2,\dots, m+n-2$ . From the assertion (a) the independence of $\vartheta(q_{m+n-1})$, $\vartheta(p_{m+n-1}), \dots$ of $q_i = \vartheta(q_i)$, $p_i = \vartheta(p_i)$ , $i = 1,2,\dots, m+n-2$ , follows. Therefore the restriction of $\vartheta$ to $W_{2N,M} \subset W_{2(m+n-2+N),M}$ is the endomorphism of $W_{2N,M}$ .

The equation $\vartheta \circ \tau(L_{ij}) = \tau'(L_{ij})$ together with eqs. (4), (6) lead immediately to $\vartheta(M_{ij}) = M_{ij}$ , which finishes the proof of the first part of the statement iii.

The proof of the last statement in the opposite direction is simple: If $\vartheta : W_{2N,M} \to W_{2N,M}$ is an endomorphism of $W_{2N,M}$ then we can easily extend it to $W_{2(m+n-2+N),M}$ putting

$$\vartheta(q_i) = q_i\ ,\ \vartheta(p_i) = p_i\ ,\ i = 1,2,\dots, m+n-2.$$

If $\tau$ and $\tau'$ are two realizations of the type (3) (with $a = \alpha'$) and $\vartheta$ is the endomorphism of $W_{2N,M}$ such that $\vartheta(M_{ij}) = M'_{ij}$ the mentioned extension gives $\vartheta\tau = \tau'$ and completes the proof.

<u>Lemma 1</u>: Let $\gamma_{r,\mu}$, $r = 0,1,\dots,R$, $\mu = 1,2,\dots,m$, be elements of $W_{2N,M}$ obeying the following system of equations

$$\sum_{\substack{r,s=0\\ r+s=t}}^{R} \sum_{\mu=1}^{m} (-1)^{r+1} \gamma^{+}_{r,\mu} \gamma_{s,\mu} = 0 , \tag{7}$$

$t = 0,1,\dots,2R$. Then $\gamma_{r,\mu} = 0$ for $r = 0,1,\dots,R$ and $\mu = 0,1,\dots,m$.

<u>Proof</u>: By induction: a) For $N=0$ the statement concerns $M\times M$ matrices; it can be proved easily that the matrix equation

$$\sum_{\mu=1}^{m} A^{+}_{\mu} A_{\mu} = 0$$

implies $A_\mu = 0$ for $\mu = 1,2,\dots,m$. As the first and the last equation of the system (7) are just of this form, we conclude $\gamma_{0,\mu} = \gamma_{R,\mu} = 0$. Substituting it into the remaining equations and repeating the procedure we obtain $\gamma_{1,\mu} = \gamma_{R-1,\mu} = 0$, etc. b) Let us assume that the statement is valid for $N-1$ and not for $N$. It means that an index $r_0$ exists $(0 \le r_0 \le R)$ such that $\gamma_{0,\mu} = \gamma_{1,\mu} = \dots = \gamma_{r_0-1,\mu} = 0$ for all $\mu$ and $\gamma_{r_0,\mu} \neq 0$ for some $\mu$. Then the first $2r_0$ equations of the system (7) are fulfilled identically while the $(2r_0+1)$-st looks as follows

$$(-1)^{r_0+1} \cdot \sum_{\mu=1}^{n} \gamma'^{+}_{r_0,\mu} \gamma'_{r_0,\mu} = 0. \tag{8}$$

The elements $\gamma_{r_0,\mu}$ can be written in the form

$$\gamma_{r_0,\mu} = \sum_{k=0}^{K} \sigma_{k,\mu} \; q_N^{K-k} \cdot p_N^{k} + \gamma'_{r_0,\mu} \tag{9}$$

$\sigma_{k,\mu} \in W_{2(N-1),M}$ , $K=0,1,\ldots$ is the highest degree of the polynomials $\gamma_{r_0,\mu}$ , $\mu=1,2,\ldots,m$ , in the "variables" $q_N$ , $p_N$ , while the highest degree of the polynomial $\gamma'_{r_0,\mu}$ considered in the same way is less than $K$ .

As the consequence of our assumption ( $\gamma_{r_0,\mu} \neq 0$ for some $r_0$ and $\mu$ ) some of the coefficients $\sigma_{k,\mu}$ differ from zero. Substituting now $\gamma_{r_0,\mu}$ from eq. (9) into eq. (8), we obtain up to the lower order terms

$$\sum_{\mu=1}^{m} \sum_{k,\ell=0}^{K} \sigma^{+}_{k,\mu} \cdot \sigma_{\ell,\mu} (-1)^{k} \cdot q_N^{2K-\ell-k} \cdot p_N^{\ell+k} + \ldots = 0$$

from which we have

$$\sum_{\mu=1}^{r} \sum_{\substack{k,\ell=0 \\ k+\ell=s}}^{K} \sigma^{+}_{k,\mu} \; \sigma_{\ell,\mu} \cdot (-1)^{k} = 0 \,, \quad s=0,1,\ldots,2K.$$

However, according to the statement of the lemma for $N-1$ , this equation implies $\sigma_{k,\mu}=0$ for all $k$ and $\mu$ and therefore contradicts our assumption.

As it is shown in ref. /4/, no skew-hermitean Schur-realization of a compact Lie algebra could exist in $W_{2N}$ . By means of the proved lemma we shall generalize now this result to realizations in $W_{2N,M}$ , $M>1$ .

<u>Theorem 2</u>: 1. Any skew-hermitean Schur-realization of the compact Lie algebra $G$ in $W_{2N,M}$ does not depend

on $q_i, p_i$, $i = 1,2,\dots,N$ , i.e. $\tau$ is a usual matrix representation of $G$ in $Mat_N \simeq W_{0,N} \subset W_{2N,N}$ .

ii. Two such realizations $\tau$ and $\tau'$ are related if and only if they are equivalent in the usual matrix representation sense.

Proof i. For $N=0$ the assertion is trivially right, we assume therefore $N \geq 1$.

As the algebra $G$ is compact, a basis $X_1,\dots,X_m$ can be chosen such that $I_2 = X_1^2 + \dots + X_m^2$ belongs to a centre of $UG$. The realization $\tau$ is Schur-realization, and therefore $\tau(I_2) = \beta \cdot \mathbb{1}$ , $\beta \in R$ . As the polynomial in $q_N$ , $p_N$ , every $\tau(X_\mu)$ can be written in the form

$$\tau(X_\mu) = \sum_{r=0}^{R} \gamma_{r,\mu} \cdot q_N^{R-r} p_N^{r} + \dots ,$$

$\gamma_{r,\mu} \in W_{2(N-1),M}$ , where $R = 0,1,\dots$ denotes the highest degree in $q_N$ and $p_N$ of $\tau(X_1),\dots,\tau(X_m)$ and the dots stand for the lower order terms. Due to skew- hermiticity of $\tau(X_\mu)$ we have

$$\beta \mathbb{1} = \tau(I_2) = -\sum_{\mu=1}^{m} \tau(X_\mu)^{+} \cdot \tau(X_\mu) .$$

Substituting here $\tau(X_\mu)$ and $\tau(X_\mu)^{+}$ from the above equation we obtain

$$\tau(I_2) = \sum_{\mu=1}^{m} \sum_{r,s=0}^{R} (-1)^{r+1} \cdot \gamma_{r,\mu}^{+} \gamma_{s,\mu} \cdot q_N^{2R-r-s} \cdot p_N^{r+s} + \dots$$

The assumption that $R$ is the highest degree in "variables" $q_N, p_N$ in the set of all elements $\tau(X_1), \dots,$ $\tau(X_m)$ means that at least one $\gamma_{r\mu}$ differs from zero. If $R$ would be greater than zero, the last equation implies eq. (7) and lemma 1 gives immediately $\gamma_{r\mu} = 0$ for all $\mu$ and $r$. The only possibility is therefore $R = 0$, which proves the statement i.

ii. As the matrix algebra $Mat_M$ has no non-trivial two-sided ideals, every nonzero endomorphism of $Mat_M$ is an automorphism (see, e.g. /7/, p.48). As any automorphism $\vartheta$ of $Mat_M$ is the inner one, the regular matrix $S_\vartheta \in Mat_M$ exists so that $\vartheta(A) = S_\vartheta^{-1} A S_\vartheta$ for every $A \in Mat_M$ (see /7/, p.50), so the proof has been completed.

From this theorem it follows that with our involution on $W_{2N,M}$ (see Note 1) any skew-hermitean Schur-realization of compact Lie algebra is a usual matrix skew-hermitean representation in which all Casimir operators are multiples of the identity matrix. Such representations, however, are equivalent to a direct sum of irreducible mutually equivalent representations and without essential loss of generality we can limit ourselves to the irreducible ones only.

As we pointed out in Preliminaries, every irreducible skew-hermitean representation of the compact Lie algebra $\sigma(m)$ is uniquely, up to equivalence, determined by its signature. Now we shall generalize this concept in the way suitable for our further use:

**Definition 6**: Let $m \geqslant n > 0$ be pair of natural numbers and $d = 1, 2, \dots, n$. The finite sequence of the real numbers $\alpha_{m,n} = (d; \alpha_1, \dots, \alpha_{[\frac{m-n}{2}]}, \alpha_{[\frac{m-n}{2}]+1}, \dots, \alpha_{[\frac{m+n}{2}]})$ is called signature if

i. for $d < n$, $\alpha_1 = \dots = \alpha_{[\frac{m+n}{2}]-d} = 0$

ii. for $d = n$, $(\alpha_1, \dots, \alpha_{[\frac{m-n}{2}]})$ is a signature of irreducible skew-hermitean representation of the compact Lie algebra $\sigma(m-n)$.

Now we are in a position to formulate and prove our main theorem.

**Theorem 3**: i. To every signature $\alpha_{m,n} = (d; \alpha_1, \dots, \alpha_{[\frac{m+n}{2}]})$ the relations (3) define recurrently skew-hermitean Schur-realization $\tau \equiv \tau(\alpha_{m,n})$ of the Lie algebra $\sigma(m,n)$ in $W_{2N(d),M(d)}$. Here the number $M(d)$ is for $d = n$ and $m-n \geqslant 2$ the dimension of the irreducible skew-hermitean representation of the Lie algebra $\sigma(m-n)$ with the signature $(\alpha_1, \dots, \alpha_{[\frac{m-n}{2}]})$ and $M(d) = 1$ otherwise. The number $N(d)$ is given as $N(d) = d(m+n-d-1)$.

ii. Two such realizations are non-related if and only if their signatures are different.

**Proof**: By induction: a) Firstly we shall prove the theorem for $\sigma(m,1)$. For $m = 1,2$ the assertions are contained in theorem 1. Let us assume $m \geqslant 3$ and take a signature $\alpha_{m,1} = (1; \alpha_1, \dots, \alpha_{[\frac{m-1}{2}]}, \alpha_{[\frac{m+1}{2}]})$. The sequence

$(\alpha_1, \dots, \alpha_{[\frac{m-1}{2}]})$ determines the irreducible skew-hermitean representation of the compact Lie algebra $\sigma(m-1)$ (its dimension we denote as $M$ ), i.e. the skew-hermitean Schur-realization of $\sigma(m-1)$ in $W_{0,M}$. Using the formulae (3) with $\alpha = \alpha_{[\frac{m+1}{2}]}$ we can define the skew-hermitean Schur-realization of the Lie algebra $\sigma(m,1)$ in $W_{2(m-1),M}$. Using further the assertions ii. and iii. of theorem 1, ii. of theorem 2 and the part C. of Preliminaries, we have the assertion ii. of the theorem for $\sigma(m,1)$.

b) Assume further that the statements of the theorem are valid for the algebra $\sigma(m-1,n-1)$. Let us take the signature $\alpha_{m,n} = (d; \alpha_1, \dots, \alpha_{[\frac{m+n}{2}]})$. For $d > 1$ we shall use the realization of $\sigma(m-1,n-1)$ in $W_{2N(d-1),M(d-1)}$ corresponding to the signature $(d-1; \alpha_1, \dots \alpha_{[\frac{m+n}{2}]-1})$ to insert it in the formulae (3) with $\alpha = \alpha_{[\frac{m+n}{2}]}$. If $d = 1$ we shall use for the same purpose the trivial realization of $\sigma(m-1,n-1)$ in $W_{0,1}$. Due to the assertion i. of theorem 1 we obtain in this way the skew-hermitean Schur-realization $\tau \equiv \tau(\alpha_{m,n})$ of $\sigma(m,n)$ in $W_{2N,M}$, where $M = M(d-1) = M(d)$ and $N = m+n-2+N(d-1) = N(d)$, which proves the assertion i. of the theorem. The assertions ii., iii. of theorem 1 together with assumed validity of the theorem for $\sigma(m-1,n-1)$ imply the assertion ii.

5. <u>Conclusion</u>

All considered algebras $\sigma(m,n)$, $m+n = N$, $N = const$ are different real form of their common complexification $\sigma_{\mathbb{C}}(N)$ x/. It is not difficult to see that all results

x/ Note that in the Cartan classification of semi-simple Lie algebras $\sigma_{\mathbb{C}}(2n) \simeq D_n$ and $\sigma_{\mathbb{C}}(2r+1) \simeq B_n$.

contained in theorems 1 and 3 remain valid also for $\sigma_{\mathbb{C}}(N)$, if we ignore the skew-hermiteanproperty and its consequences. As we are not forced now to respect theorem 2 relations (3) define recourently a usual canonical Schur-realization of $\sigma_{\mathbb{C}}(N)$ in $W_{\frac{1}{2}(N-1)^2}$ for odd $N$ or in $W_{\frac{1}{2}N(N-2)}$ for even $N$ depending on the $[\frac{N}{2}]$ free parameters. As to real forms, the same situation arises for algebras $\sigma(n+2,n)$, $\sigma(n+1,n)$and $\sigma(n,n)$ while in the remaining cases the described canonical realizations depend at most on $[\frac{m+n}{2}]-$ $-[\frac{m-n}{2}]=n$ free parameters only. To obtain in these cases the "full" number $[\frac{m+n}{2}]$ of the parameters in realizations described, we have to use the signature $\alpha_{m,n}=(n|\alpha_1,\dots,\alpha_{[\frac{m+n}{2}]})$and realizations are right matrix canonical realizations.

It is also clear, that instead of realizations of the auxiliary Lie algebra $\sigma(m-d,n-d)$, $d<n$ that we used in reduction of formulae (3), any other realization of $\sigma(m-d,n-d)$ can be taken. So, the possibility of deriving further, new, realizations of Lie algebra $\sigma(m,n)$ may arise.

The concept of matrix canonical realization, especially the realizations of the Lie algebras $\sigma(m,n)$ described in this paper, have the direct application in the representation theory.Replacing here $p_i$ and $q_i$ by their Schrödinger representations we obtain immediately a skew-symmetric representation of $\sigma(m,n)$ with "constant" Casimir operators. It has to be stressed that these representations were obtained purely in the algebraical way. As the second advantage of this approach, one can consider the fact that the analytical properties of representations can be investigated separately as the second step.

We often work, for example, with skew-symmetric representations of Lie algebras which are differentials of some unitary representation of the corresponding (connected, simple connected) Lie group, i.e. with the integrable ones. By means of some known methods /8/, we can try to solve the integrability for the above described representations too. It may happen that some meaningful part of the set of all integrable representations of the Lie algebra $\sigma(m,n)$ would be obtained in this way. The other possibility of obtaining representations both integrable or not arises, when we replace the Schrödinger representation of $p^i$, $q_i$ by some other, e.g., by representation on the space of analytical functions.

As was pointed out by Doebner and Melsheimer /9/, the integrability condition on representation of Lie algebra is often from the physical point of view not necessary. So, some classes of non-integrable representations of Lie algebras could also be interesting for physics, e.g. partly integrable representations with respect to chosen subalgebra or those in which some, physically interpreted generators are essentially self-adjoint, etc. In matrix canonical approach to the representation theory we are not limited by any sort of integrability conditions so that the wide class of representations could be obtained. This fact represents the third advantage of the described approach.

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