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ON A PROPERTY OF EUCLIDEAN GREEN FUNCTIONS OF SU(2) YANG-MILLS FIELDS IN BPST-INSTANTON BACKGROUND

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1. INTRODUCTION

Topological nontrivial solutions of nonlinear classical field equations continue to play an increasing role in particle physics. Unfortunately, the quantization of corresponding field theories carried out with the help of path integrals suffers from the fact that (presently) it is impossible to take into account the whole manifold of field configurations. The global evalution of the path integral is therefore approximated by means of the saddle point method: topological nontrivial configurations are considered at the classical level only, and the integration is performed over small fluctuations around them. According to this approximation the contributions to any physical quantity originate from sectors of different topological charge. Within each sector the renormalized perturbation theory works consistently as has been proved by different methods for various cases. In the case of instantons this has been shown by using the formalism of quantum action principle and solving some associated cohomology problems [1].

For conceptional reasons and also to facilitate the practical evaluation of physical effects it seems to be useful to have statements concerning the relation between different topological sectors. Here, we show that the Euclidean Green functions in the BPST-instanton background [5] - if restricted to gauge fields transverse to the zero modes and to vanishing ghost sources - are obtained from appropriately determined Green functions in the perturbative vacuum, i.e., the sector of zero topological charge, by a mere translation. The method to obtain that result is the same as has been introduced by A. Rouet [2] to study the dependence of physical Green functions on arbitrary background fields in the perturbative vacuum for Yang-Mills theory in Minkowski space.

2. THE METHOD OF A. ROUET

In this preparatory section we shortly review Rouet's method which in a natural way may be extented to a more generel case of the 1-instanton sector. For the sake of conceptional clarity we do not specify explicitly either the underlying gauge theory or the corresponding sector. Thus only the general structure of the theory is characterized in an appropriate manner, and then the dependence of the theory on the background field $\mathcal{A}_\mu$ is studied in general terms. The full contens of this general description becomes more obvious in section 3 where the SU(2) Yang-Mills theory in the BPST-instanton background is formulated, and in section 4 where their $\mathcal{A}_\mu$-dependence is studied.

A perturbatively renormalized, anomaly free gauge theory originating from a local gauge invariant classical Lagrangian with semisimple gauge group is properly defined by requiring the Slavnov identity

$$\mathcal{S} Z = 0 , \tag{2.1}$$

which expresses the invariance of the effective (tree) Lagrangian $\mathcal{L}$ under the global BRS-transformation of the gauge field Q_μ, the ghost fields $\{c\}$ and the antighost fields $\{\bar{c}\}$; here, the renormalized generating functional of the complete Green functions $Z = Z[\mathcal{A}_\mu; j_\mu, \{\bar{\xi}\}, \{\xi\}, \{\eta\}]$ depends on the sources j_μ, $\{\bar{\xi}\}$ and $\{\xi\}$ of the gauge, ghost and antighost fields respectively, some external sources $\{\eta\}$ which are coupled to the nonlinear BRS-transforms of these fields, as well as on the background field $\mathcal{A}_\mu$. The nilpotent Slavnov operator $\mathcal{S}$ acting on Z is linear in the derivatives with respect to these sources.

In principle every such statement about the structure of the renormalized theory can be drawn from the Slavnov operator. Thereby, to be true in every order of the perturbation theory any such assertion must be formulated like (2.1) as an operator identity with respect to Z. We shall keep this in mind in the following.

Of course, as a guide to find these structural identities the knowledge of the most general tree Lagrangian $\mathcal{L}$ satisfying (2.1) will be useful. If $\mathcal{S}$ is known, the Lagrangian $\mathcal{L}$ may be constructed according to the methods explained in Ref. /3/. It depends on a set of free parameters $\{\kappa^i\}$ which indicate different BRS-invariant parts of $\mathcal{L}$. Through the fulfillment of suitable normalization conditions some of these parameters related to Faddeev-Popov (FP) ghost equations of motion, which follow from the nilpotency $\mathcal{S}^2 = 0$, can be fixed. With respect to the remaining parameters there corresponds a linearly independent set of BRS-symmetrical insertions Δ^i, i.e., insertions satisfying

$$[\mathcal{S}, \Delta^i] Z = 0 . \tag{2.2}$$

In the following these insertions, which are also linear operators, play a crucial role.

As is well known the local gauge invariance of the classical Lagrangian is missing in the tree Lagrangian $\mathcal{L}$ due to the gauge fixing. But, in the presence of a background field if the field strength depends on the sum $\mathcal{A}_\mu + Q_\mu$ and the gauge is fixed covariantly according to $\mathcal{L}_{fix} = -(1/2\alpha)(\mathcal{D}_\mu(\mathcal{A}) Q_\mu)^2$ the Lagrangian is invariant under local type-I transformations /4/. However, since the most general Lagrangian $\mathcal{L}$ satisfying (2.1) is not type-I invariant, the action of the type-I operator $\mathcal{R}_0^a(x)$ on Z does not vanish, but is given by the divergence of a local BRS-symmetrical insertion $\Delta_\mu^a(x)$ which is uniquely determined. Consequently, the local symmetry behaviour of the theory is characterized by the following identity

$$\mathcal{R}^a(x) Z = 0 \tag{2.3}$$

with the BRS-symmetrical operator

$$\mathcal{R}^a(x) = \mathcal{R}_0^a(x) + \partial_\mu \Delta_\mu^a(x). \tag{2.4}$$

If the Eq. (2.4) is integrated over the space-time, the corresponding identity is nothing else than the global isospin symmetry of the Green functional generates by the operator $\int d^4x\, \mathcal{R}_0^a(x)$.

Information about the dependence of the generating functional on the background field Z is obtained from the treatment of the connection of the insertion $\int d^4x\, \mathcal{A}_\mu^a\, \delta/\delta \mathcal{A}_\mu^a$ to other independent insertions which are also proportional to $\mathcal{A}_\mu^a$. In fact, guided by the structure of $\mathcal{L}$ it can be shown that the following relation holds

$$\left(\int d^4x \, \mathcal{A}^a_\mu \frac{\delta}{\delta \mathcal{A}^a_\mu} \right) Z = (\Omega + \Delta) \, Z , \qquad (2.5)$$

where Ω is a (finite) non-symmetrical insertion related to the special choice of the gauge, and Δ is a linear combination of symmetrical insertions which are also proportional to $\mathcal{A}^a_\mu$.

Unfortunately, taking into consideration the actual form of these insertions (see Eqs. (3.14) - (3.16) and (4.2) below) it seems to be almost impossible to solve Eq. (2.5) without any loss of generality with regard to Z. Indeed, if we restrict ourselves to the consideration of physical Green functions it is easy to see that the action of the symmetrical insertion appearing in Eq. (2.5) on Z_{phys} reduces solely to a multiplication with a factor $b \int d^4x \, \mathcal{A}^a_\mu J^a_\mu$. Furthermore, physical Green functions with insertions Ω can be shown to vanish. From this it follows (for a precise definition of Z_{phys} see Eq. (4.4) below):

$$Z_{phys} [\mathcal{A}, J] = \exp (\bar{b} \int d^4x \, \mathcal{A}^a_\mu J^a_\mu) \, Z_{phys} [0, J] \qquad (2.6)$$

where $\bar{b}$ is a (finite) parameter.

The normalization condition $\langle Q_\mu \rangle = \mathcal{A}_\mu$ for the renormalized fields corresponding to b = 1 is consistent with the type-I invariance. Using Eq. (2.3), one can prove that gauge invariance of $Z_{phys}[0]$ implies type-I invariance of $Z_{phys}[\mathcal{A}]$. In this way it is shown that physical Green functions in a background field are obtained from corresponding Green functions in the perturbative vacuum by a mere translation, thereby the correct symmetry behaviour is manifest.

3. PURE SU(2) GAUGE THEORY IN THE BPST-INSTANTON BACKGROUND

Accordingly, on the basis of the analysis of the general structure of non-abelian gauge theory in a background field in sect. 2, this formalism also enables us to discuss the more general case of a pure Euclidean SU(2) Yang-Mills theory in the BPST-instanton background-field.

The BPST-instanton configuration which we shall use is given by /5/

$$\mathcal{A}^a_\mu (x) = \frac{1}{g_0} \, \frac{2 R^{ab} \eta_{b\mu\nu} (x_\nu - a_\nu)}{((x - a)^2 + \rho^2)^2} , \qquad (3.1)$$

where g_0 is the coupling constant and $\eta_{a\mu\nu}$ is the well-known t'Hooft tensor /6/; i.e., the instanton with global gauge orientation R^{ab} and scale ρ is located at a_μ. For later convenience we denote the field configuration (3.1) multiplied with the coupling constant as $\mathcal{O}\!\iota^a_\mu$. In a covariant basis the (not normalized) gauge, translation and dilation zero modes corresponding to the quadratic part of the classical Lagrangian

$$\mathcal{L}^{cl} = - (1/4 g_0^2) \, \mathcal{F}^a_{\mu\nu} (\mathcal{O}\!\iota + g_0 Q) \, \mathcal{F}^a_{\mu\nu} (\mathcal{O}\!\iota + g_0 Q) \qquad (3.2)$$

are given by /1/

$$\mathcal{D}^{ab}_\mu (\mathcal{O}\!\iota) \, \delta(x-y) , \quad \mathcal{F}^a_{\mu\nu} (\mathcal{O}\!\iota) \quad \text{and} \quad x_\nu \mathcal{F}^a_{\mu\nu} (\mathcal{O}\!\iota) \qquad (3.3)$$

respectively, where the following convention has been used:

$$\mathcal{D}^{ab}_\mu (\mathcal{O}\!\iota) = \delta^{ab} \partial_\mu + \varepsilon^{abc} \mathcal{O}\!\iota^c_\mu$$

and

$$\mathcal{F}^a_{\mu\nu} (\mathcal{O}\!\iota) = \partial_\mu \mathcal{O}\!\iota^a_\nu - \partial_\nu \mathcal{O}\!\iota^a_\mu + (\mathcal{O}\!\iota_\mu \times \mathcal{O}\!\iota_\nu)^a .$$

Now, the Slavnov identity with respect to the Green functional Z, which defines the Yang-Mills theory around the instanton, is given by

$$\mathcal{S} Z \equiv \Big\{ \int d^4x \Big[J^a_\mu(x) \frac{\delta}{\delta \eta^a_\mu(x)} + \frac{1}{\alpha} \bar{\xi}^a(x) \, \mathcal{D}^{ab}_\mu (\mathcal{O}\!\iota) \frac{\delta}{\delta J^b_\mu(x)} +$$

$$+ \Big(\frac{1}{\beta} \bar{\xi}_\nu + \frac{1}{\gamma} x_\nu \bar{\xi} \Big) \mathcal{F}^a_{\mu\nu} (\mathcal{O}\!\iota) \frac{\delta}{\delta J^a_\mu(x)} - \xi^a(x) \frac{\delta}{\delta \eta^a(x)} \Big]$$

$$- \xi_\mu \frac{\partial}{\partial \eta_\mu} \Big\} Z = 0 , \qquad (3.4)$$

$\bar{\xi}^a(x)$, $\bar{\xi}_\mu$ and $\bar{\xi}$ are the sources of the gauge, translation and dilation ghost fields $c^a(x)$, c_μ and c respectively; $\xi^a(x)$, ξ_μ and ξ are the sources of the related anti-ghost fields $\bar{c}^a(x)$, $\bar{c}_\mu$ and $\bar{c}$, whereas $\eta^a_\mu(x)$, $\eta^a(x)$ and η_μ are external sources coupled to appropriate products of field operators (see Eq. (3.6) below). α, β, γ are the parameters which fix the gauge. Canonical dimension and FP charge of the quantum fields and sources are assigned according to the following table:

quantum field	dimension	FP charge
$Q^a_\mu(x)$	1	0
$c^a(x)$	2	-1
c_μ	1	-1
c	2	-1
$\bar{c}^a(x)$	0	+1
$\bar{c}_\mu$	-1	+1
$\bar{c}$	0	+1

external field	dimension	FP charge
$\mathcal{A}^a_\mu(x)$	1	0
$\xi^a_\mu(x)$	1	0
$\eta^a_\mu(x)$	3	-1
$\eta^a(x)$	4	-2
η_μ	1	-2

Of course, quantities with odd integer FP charge anticommute, all the others commute. The negative sign in front of the last two terms of (3.4) is due to our convention to write quantities with a bar always on the right of the corresponding quantities without a bar; furthermore, the derivatives $d/d\xi$ and $d/d\bar{\xi}$, etc., are understood as left derivatives. Finally we remark in passing that the fields and sources related to the translation and dilatation zero modes do not depend on space-time as we have indicated explicitly in Eq. (3.4).

In the tree approximation the most general action $S = \int d^4x\, \mathcal{L}(x)$ satisfying (3.4) is given by /7/

$$S = \bar{S} - (2\alpha)^{-1} \int d^4x \left(\mathcal{D}^{ab}_\mu(\mathcal{A})\, Q^b_\mu \right)^2 - \tag{3.5}$$
$$- (2\beta)^{-1} \left(\int d^4x\, \mathcal{F}^a_{\mu\nu}(\mathcal{A})\, Q^a_\mu \right)^2 - (2\gamma)^{-1} \left(\int d^4x\, x_\nu \mathcal{F}^a_{\mu\nu}(\mathcal{A})\, Q^a_\mu \right)^2$$

with

$$\bar{S} = \int d^4x \Big\{ -(A/4G^2)\, \mathcal{F}^a_{\mu\nu}(G\hat{Q})\, \mathcal{F}^a_{\mu\nu}(G\hat{Q}) +$$
$$+ \hat{\eta}^a_\mu \left[E_1 \mathcal{D}^{ab}_\mu(G\hat{Q})\, \hat{c}^b + E_2 \bar{c}_\nu \partial_\nu \hat{Q}^a_\mu + E_3 \bar{c}\,(1 + x_\nu \partial_\nu)\, \hat{Q}^a_\mu \right] +$$
$$+ \eta^a \left[-\frac{1}{2} E_1 G \left(\hat{c} \times \hat{c} \right)^a + \left(E_2 \bar{c}_\nu + E_3 \bar{c}\, x_\nu \right) \partial_\nu \hat{c}^a \right] +$$
$$+ E_3\, \hat{\eta}_\mu \bar{c}_\mu \bar{c} \Big\}, \tag{3.6}$$

where

$$G \hat{Q}^a_\mu(x) = B_1 \mathcal{A}^a_\mu(x) + G Q^a_\mu(x) \tag{3.7}$$

$$\hat{c}^a(x) = c^a(x) - \left(B_2 \bar{c}_\mu + B_3 \bar{c}\, x_\mu \right) \mathcal{A}^a_\mu(x) \tag{3.8}$$

$$\hat{\eta}^a_\mu(x) = \eta^a_\mu(x) - \mathcal{D}^{ab}_\mu(\mathcal{A})\, c^b(x) + (c_\nu + x_\nu c)\, \mathcal{F}^a_{\mu\nu}(\mathcal{A}), \tag{3.9}$$

$$\hat{\eta}_\mu(x) = \eta_\mu\, \delta^4(x) + B_2\, \eta^a(x)\, \mathcal{A}^a_\mu(x) \tag{3.10}$$

is introduced to simplify the notation. We are going to see immediately that the modified action $\bar{S}$ depends on the background field only via expressions (3.6) - (3.10). This indicates that the Green functions in the background field $\mathcal{A}^a_\mu$ could be obtained from the Green functions in the vacuum sector by an appropriate transformation of corresponding source terms - by restriction to Green function transverse to the zero modes.

The parameters A, B_i, E_i (i=1,2,3) and G may be chosen arbitrarily. But, whereas E_i and A can be absorbed by rescaling the quantum fields, the external sources (η) and the coupling constant G, and which therefore will be assumed to be one, this is not the case for B_i and G which have to be normalized otherwise.

Owing to the multiplicative renormalizability of the theory defined by (3.4) and the just defined action /7/ the corresponding parameters after renormalization - which will be denoted by small letters - can be fixed in the same manner. Then the (finite) constants b_1, b_2 and b_3 may be considered as determining the strength of switching on the background field $\mathcal{A}^a_\mu$ relative to the renormalized fields $g Q^a_\mu$, $\bar{c}_\mu$ and $\bar{c}$ respectively. Of course, if $B_i = 0, 1$, then $B_i \mathcal{A}^a_\mu$ is no more a solution of the field equation.

The nilpotency of $\mathcal{S}$ leads to the equation

$$\int d^4x \left\{ \frac{1}{\alpha} \bar{\xi}^a \mathcal{D}^{ab}_\mu(\mathcal{A}) + \left(\frac{1}{\beta} \bar{\xi}_\nu + \frac{1}{\gamma} x_\nu \bar{\xi} \right) \mathcal{F}^a_{\mu\nu}(\mathcal{A})\, \delta^{ab} \right\} \frac{dZ}{d\eta^b_\mu(x)} = 0$$

in every order of the perturbation theory. Owing to the orthogonality of the zero modes Eq. (3.3) this equation breaks up into three independent equations. Because of the anticommutativity of sources with odd integer FP charge their solution are given by

$$\left(\bar{\xi}^a(x) - \mathcal{D}^{ab}_\mu(\mathcal{A}) \frac{d}{d\eta^b_\mu(x)} \right) Z = 0,$$
$$\left(\bar{\xi}_\nu - \int d^4x\, \mathcal{F}^a_{\mu\nu}(\mathcal{A}) \frac{d}{d\eta^a_\mu(x)} \right) Z = 0 \tag{3.11}$$

$$(\bar\xi - \int d^4x\, x_\nu \mathcal{F}^a_{\mu\nu}(\alpha) \frac{\delta}{\delta \eta^a_\mu(x)}) Z = 0,$$

i. e., by the equation of motion for the ghost fields. Here we remark that we could have introduced three independent parameters F_i ($i=1,2,3$) in front of the three different gauge terms appearing simultaneously in (3.5) and (3.9). Through the requiring (3.11) these parameters are normalized to $F_i = 1$.

Now, to any free parameter which occurs in $\bar{\Sigma}$ is associated an independent BRS-symmetrical insertion

$$\Delta^A = -\frac{1}{4}\int d^4x\, \mathcal{F}^a_{\mu\nu}(G\hat Q)\mathcal{F}^a_{\mu\nu}(G\hat Q) + \text{ terms } O(\kappa) \tag{3.12}$$

$$\Delta^G = \int d^4x \{ J^a_\mu \frac{\delta}{\delta J^a_\mu} + \eta^a_\mu \frac{\delta}{\delta \eta^a_\mu} + \bar\xi^a \frac{\delta}{\delta \bar\xi^a} \} + \bar\xi_\mu \frac{\delta}{\delta \bar\xi_\mu} + \bar\xi \frac{\delta}{\delta \bar\xi}) \tag{3.13}$$

$$\Delta^{B_1} = \int d^4x \{ J^a_\mu \alpha^a_\mu - \frac{1}{\alpha}(\eta_\mu(\alpha)\alpha_\mu)^a \eta^{ab}_\nu(\alpha)\frac{\delta}{\delta J^b_\nu} - \frac{1}{\beta}(\mathcal{F}^a_{\mu\nu}(\alpha)\alpha^a_\mu)\int d^4y\, \mathcal{F}^b_{\lambda\nu}(\alpha)\frac{\delta}{\delta J^b_\lambda(y)} - \frac{1}{\gamma}(x_\nu \mathcal{F}^a_{\mu\nu}(\alpha)\alpha^a_\mu)\int d^4y\, y_\kappa \mathcal{F}^b_{\lambda\kappa}(\alpha)\frac{\delta}{\delta J^b_\lambda(y)} \}, \tag{3.14}$$

$$\Delta^{B_2} = \int d^4x\, \alpha^a_\mu(x) (\xi^a(x)\frac{\partial}{\partial \xi_\mu} + \eta^a(x)\frac{\partial}{\partial \eta_\mu}) \tag{3.15}$$

$$\Delta^{B_3} = \int d^4x\, \alpha^a_\mu(x)\, x_\mu\, \xi^a(x)\frac{\partial}{\partial \xi} \tag{3.16}$$

$$\Delta^{E_1} = \int d^4x (\xi^a(x)\frac{\delta}{\delta \xi^a(x)} + \eta^a(x)\frac{\delta}{\delta \eta^a(x)}) \tag{3.17}$$

$$\Delta^{E_2} = \xi_\mu \frac{\partial}{\partial \xi_\mu} + \eta_\mu \frac{\partial}{\partial \eta_\mu} \tag{3.18}$$

$$\Delta^{E_3} = \xi \frac{\partial}{\partial \xi}. \tag{3.19}$$

It is easily verified that these insertions satisfy (2.2). They may be obtained from $\langle \partial S \wedge \partial K' \rangle$ by using the equation of motion for the quantum fields expressed by derivatives with respect to the associated sources.

The symmetry behaviour of the theory is characterized by the identity (2.3) where $\mathcal{R}^a(x)$ is given by

$$\mathcal{R}^a(x) = [\eta^{ab}_\mu(\alpha)\frac{\delta}{\delta \alpha^b_\mu} + (J_\mu \times \frac{\delta}{\delta J_\mu})^a + (\eta_\mu \times \frac{\delta}{\delta \eta_\mu})^a + (\eta \times \frac{\delta}{\delta \eta})^a - (\bar\xi \times \frac{\delta}{\delta \bar\xi})^a - (\xi \times \frac{\delta}{\delta \xi})^a] + \partial_\mu [c_1 (J^a_\mu + \frac{1}{\alpha}\eta^{ab}_\mu(\alpha)\eta^{bc}_\nu(\alpha)\frac{\delta}{\delta J^c_\nu} - \frac{1}{\beta}\mathcal{F}^a_{\mu\nu}(\alpha)\int d^4y\, \mathcal{F}^b_{\lambda\nu}(\alpha)\frac{\delta}{\delta J^b_\lambda(y)} - \frac{1}{\gamma}x_\nu \mathcal{F}^a_{\mu\nu}(\alpha)\int d^4y\, y_\kappa \mathcal{F}^b_{\lambda\kappa}(\alpha)\frac{\delta}{\delta J^b_\lambda(y)}) + c_2 (\xi^a \frac{\partial}{\partial \xi_\mu} + \eta^a \frac{\partial}{\partial \eta_\mu}) + c_3\, \xi^a x_\mu \frac{\partial}{\partial \xi}]. \tag{3.20}$$

The first part is the type-I operator $\mathcal{R}^a_0(x)$ and in the second part the expressions accompanying the parameters c_i ($i=1,2,3$) are uniquely determined, linearly independent local BRS-symmetrical insertions which transform as Lorentz vectors. The parameters c_i may be computed; in the tree approximation Eqs. (3.5) - (3.10) they are given by

$$c_1 = \frac{1}{G}(1 - B_1)\ ; \quad c_i = (\frac{E_i}{E_1 G} - B_i)\ ,\ i = 2,3. \tag{3.21}$$

Accordingly, if we require conformity with respect to the type-I transformation, (3.21) implies $B_i = 1$. In the other case $B_i \neq 1$ the gauge covariance related to the background field is destroyed.

4. DEPENDENCE OF PHYSICAL GREEN FUNCTIONS ON THE INSTANTON FIELD

After we have characterized the properties of Z let us now study its $\mathcal{A}_\mu^a$ - dependence. For a moment we shall assume the background field is replaced by $\lambda\,\mathcal{A}_\mu^a$ in the gauge dependent terms of the tree action S , i.e., in Eqs. (3.5) and (3.9). Then, the operator $\mathcal{A}_\mu^a\, \delta/\delta \mathcal{A}_\mu^a$ acting on S is given by

$$\int d^4x\, \mathcal{A}_\mu^a \frac{\delta S}{\delta \mathcal{A}_\mu^a} = \lambda \frac{\delta S}{\delta \lambda} + \sum_{i=1}^{3} \beta_i \frac{\delta S}{\delta \beta_i} \,. \tag{4.1}$$

For the reason, related to the first term in Eq. (4.1), we consider the following non-symmetrical insertion :

$$\begin{aligned}\Omega^\varepsilon \equiv \int d^4x \Big[& \Big(\mathcal{A}_\mu \times \frac{\delta}{\delta \eta_\mu}\Big)^a_{(x-\varepsilon)} \Big(\frac{\delta}{\delta \bar{\xi}^a}\Big)_{(x+\varepsilon)} - \\ & - \frac{1}{\alpha} \Big(\mathcal{A}_\mu \times \frac{\delta}{\delta j_\mu}\Big)^a_{(x-\varepsilon)} \Big(\mathcal{D}_\nu^{ab}(\mathcal{A}) \frac{\delta}{\delta j_\nu^b}\Big)_{(x+\varepsilon)} \Big] + \\ + \int d^4x \,\big(& \mathcal{D}_\mu^{ab}(\mathcal{A})\, \mathcal{A}_\nu^b - \mathcal{D}_\nu^{ab}(\mathcal{A})\, \mathcal{A}_\mu^b \big) \cdot \\ \cdot \Big\{ \Big[& \frac{\delta}{\delta \eta_\mu^a(x)} \frac{\partial}{\partial \bar{\xi}_\nu} - \frac{1}{\beta} \frac{\delta}{\delta j_\mu^a(x)} \int d^4y \Big(\mathcal{F}_{\lambda\nu}^b(\mathcal{A}) \frac{\delta}{\delta j_\lambda^b(y)}\Big)\Big] + \\ + \Big[& \frac{\delta}{\delta \eta_\mu^a(x)} \frac{\partial}{\partial \bar{\xi}} - \frac{1}{\gamma} \frac{\delta}{\delta j_\mu^a(x)} \int d^4y\, y_x\, \mathcal{F}_{\lambda x}^b(\mathcal{A}) \frac{\delta}{\delta y_\lambda^b(y)} \Big]\Big\} .\end{aligned} \tag{4.2}$$

For the time being we have regularized this expression by point-splitting to avoid any possible ill-definiteness due to products of field operators at coinciding points; of course, this is not necessary for terms containing derivatives with respect to the space-time independent field sources $\bar{\xi}_\mu$ and $\bar{\xi}$.

By explicit use of the ghost equation of motion (3.11) it follows

$$\begin{aligned}[\mathcal{S}, \Omega^\varepsilon]\, Z = - \int d^4x \Big\{ & \frac{1}{\alpha} \bar{\xi}^a_{(x+\varepsilon)} \Big(\mathcal{A}_\mu \times \frac{\delta}{\delta j_\mu}\Big)^a_{(x-\varepsilon)} + \\ + \big(\mathcal{D}_\mu^{ab}(\mathcal{A})\, \mathcal{A}_\nu^b - \mathcal{D}_\nu^{ab}(\mathcal{A})\, \mathcal{A}_\mu^b\big) & \Big(\frac{1}{\beta} \bar{\xi}_\nu + \frac{1}{\gamma} x_\nu \bar{\xi}\Big) \frac{\delta}{\delta j_\mu^a(x)} \Big\} Z ,\end{aligned} \tag{4.3}$$

which is finite in the limit $\varepsilon \to 0$. Therefore, a possible divergent part of Ω^ε for $\varepsilon \to 0$ must be a symmetrical insertion. For Ω^ε being proportional to $\mathcal{A}_\mu^a$ this insertion must be a linear combination of insertions Δ^{β_i} (i=1,2,3).

To proceed further we introduce the notation of physical Green functions with insertion $\mathcal{O}(x)$ - the latter being defined via functional derivatives with respect to the corresponding sources - according to

$$\begin{aligned}G^{(j)}_{phys}([j], x; y_1, \dots, y_N) & \equiv \langle \mathcal{O}(x)\, Q^T(y_1) \dots Q^T(y_N) \rangle_{phys} \\ & = \mathcal{O}\Big[\frac{\delta}{\delta j(x)}, \frac{\delta}{\delta \xi(x)}, \dots\Big] \prod_{n=1}^{N} \frac{\delta}{\delta j^T(y_n)}\, Z \Big|_{\{\xi\}=\{\bar{\xi}\}=\{\eta\}=0} ,\end{aligned} \tag{4.4}$$

i.e., after carrying out all derivatives on Z the sources (except j_μ^a) are to set equal to zero; thereby the notation of "transversality" is defined as

$$\begin{aligned}\mathcal{D}_\mu(\mathcal{A}) \frac{\delta}{\delta j^T_\mu(x)} Z &= 0 \\ \int d^4x\, \mathcal{F}_{\mu\nu}(\mathcal{A}) \frac{\delta}{\delta j^T_\mu(x)} Z &= 0 \\ \int d^4x\, x_x \mathcal{F}_{\mu\nu}(\mathcal{A}) \frac{\delta}{\delta j^T_\mu(x)} Z &= 0 .\end{aligned} \tag{4.5}$$

In the following the r.h.s. of Eq. (4.4) will be symbolized by $\mathcal{O}(x)\, Z$ phys .

Next we introduce the following operators

$$\begin{aligned}\Pi^\varepsilon \equiv \int d^4x \Big\{ & \Big(\mathcal{A}_\mu \times \frac{\delta}{\delta j_\mu}\Big)^a_{(x-\varepsilon)} \Big(\frac{\delta}{\delta \bar{\xi}^a}\Big)_{(x+\varepsilon)} + \\ & + \big(\mathcal{D}_\mu^{ab}(\mathcal{A})\, \mathcal{A}_\nu^b - \mathcal{D}_\nu^{ab}(\mathcal{A})\, \mathcal{A}_\mu^b\big) \frac{\delta}{\delta j_\mu^a(x)} \Big(\frac{\delta}{\delta \bar{\xi}_\nu} + x_\nu \frac{\delta}{\delta \bar{\xi}}\Big)\Big\} \prod_{n=1}^{N} \frac{\delta}{\delta j^T(y_n)}\end{aligned} \tag{4.6}$$

$$\begin{aligned}\mathcal{P} \equiv \int d^4x \Big\{ & \big(\mathcal{D}_\mu(\mathcal{A})\, \mathcal{A}_\mu\big)^a \frac{\delta}{\delta \bar{\xi}^a(x)} + \big(\mathcal{F}^a_{\mu\nu}(\mathcal{A})\, \mathcal{A}_\mu^a\big) \frac{\delta}{\delta \bar{\xi}_\nu} + \\ & + \big(x_\nu \mathcal{F}^a_{\mu\nu}(\mathcal{A})\, \mathcal{A}_\mu^a\big) \frac{\delta}{\delta \bar{\xi}} \Big\} \prod_{n=1}^{N} \frac{\delta}{\delta j^T(y_n)}\end{aligned} \tag{4.7}$$

defined for non-coinciding points y_n. Because of (2.1) we obtain

$$0 = \Pi^{\varepsilon} \varphi Z \Big|_{\{\xi\}=\{\bar\xi\}=\{\eta\}=0} = \Omega^{\varepsilon} Z_{phys} \tag{4.8}$$

$$0 = \mathcal{P} \varphi Z \Big|_{\{\xi\}=\{\bar\xi\}=\{\eta\}=0} = \left(\Delta^{B_1} - \int d^4x\, \mathcal{A}_\mu J_\mu \right) Z_{phys}, \tag{4.9}$$

in addition to the trivial relations

$$\Delta^{B_2} Z_{phys} = 0, \tag{4.10}$$

$$\Delta^{B_3} Z_{phys} = 0. \tag{4.11}$$

Now we assume that the symmetrical part $\Omega^{\varepsilon}_{sym}$ of Ω^{ε} will be proportional to Δ^{B_1} only, then, as a consequence of the above results, it follows that $(\Omega^{\varepsilon}_{non-sym} + \Omega^{\varepsilon}_{sym})$ is finite in the limit $\varepsilon \to 0$, since otherwise a finite and infinite part in (4.8) had to cancel each other. If, contrary to this assumption, Ω^{ε} contains also the symmetrical insertions Δ^{B_2} and Δ^{B_3}, then the correctness of the considerations below is not destroyed; because of (4.10) and (4.11) the coefficient of Δ^{B_1} must be finite by the same argument just given, and the divergent part of Ω^{ε} could only be proportional to a linear combination of Δ^{B_2} and Δ^{B_3} which however may be subtracted from Ω^{ε} order by order; thereby, a new operator $\tilde\Omega^{\varepsilon}$ is defined which has the same non-symmetrical part as Ω^{ε}, and which is finite in the limit $\varepsilon \to 0$. So, finally it holds

$$\tilde\Omega\, Z_{phys} = 0. \tag{4.12}$$

Now we consider the insertion $\int d^4x\, \mathcal{A}_\mu^a\, \delta/\delta\mathcal{A}_\mu^a$ acting on Z. It is easily shown - as it is obvious from (4.1) - that the non-symmetrical part of $\int d^4x\, \mathcal{A}_\mu^a\, \delta/\delta\mathcal{A}_\mu^a$ is given by $\Omega_{non-sym} = \tilde\Omega_{non-sym}$, i.e.,

$$\left[\varphi, \int d^4x\, \mathcal{A}_\mu^a(x) \frac{\delta}{\delta \mathcal{A}_\mu^a(x)} \right] Z = [\varphi, \tilde\Omega]\, Z,$$

and therefore it holds

$$\int d^4x\, \mathcal{A}_\mu^a(x) \frac{\delta Z}{\delta \mathcal{A}_\mu^a(x)} = \left(\tilde\Omega + \sum_{i=1}^{3} b_i \frac{1}{g} \Delta^{B_i} \right) Z, \tag{4.13}$$

where the b_i are (finite) parameters which reduce to B_i in the tree approximation (cp. Eq. (4.1)).

Using Eqs. (4.9) - (4.12) we obtain

$$\int d^4x\, \mathcal{A}_\mu^a(x) \frac{\delta}{\delta \mathcal{A}_\mu^a(x)} Z_{phys} = b_1 \left(\int d^4x\, \frac{\mathcal{A}_\mu^a(x)}{g} j_\mu^a(x) \right) Z_{phys}$$

which is solved by

$$Z_{phys}[\mathcal{A}, j] = \exp\left(b_1 \int d^4x\, \frac{1}{g} \mathcal{A}_\mu^a j_\mu^a \right) Z_{phys}[0, j]. \tag{4.14}$$

Formally, this relation coincides with the corresponding relation in reference /2/. However, a few remarks are in order here:

(1) The r.h.s. of Eq. (4.14) corresponds to physical Green functions in the perturbative vacuum with insertions $\exp(b_1 \int \frac{1}{g} \mathcal{A}_\mu j_\mu)$. However, here the terminus "physical" is adapted from the 1-instanton sector, i.e., transversality is defined according to the rules (4.5) where instead of $\mathcal{A}_\mu^a$ any external field configuration ξ_μ^a of topological charge zero and with the same zero modes as $\mathcal{A}_\mu^a$ can be used; of course, then the same additional ghost fields, etc., as above come into play. But, because of (4.12) neither $\mathcal{A}_\mu^a$ nor ξ_μ^a appears explicitly in Z_{phys}; this is obvious from the fact that Z_{phys} cannot depend on a special gauge fixing.

(2) In agreement with Eq. (2.6) it holds

$$Z_{phys}[\mathcal{A}_\mu, j_\mu] = \exp\left(b_1 \int d^4x\, \frac{1}{g} \mathcal{A}_\mu^a j_\mu^a \right) Z_{phys}[0, j_\mu] \tag{4.15}$$

in the case of the BPST-instanton background; this proceeds from the necessity to renormalize also the coupling constant which appears in Eq. (3.1).

(3) To define the theory completely, the free parameters, especially b_1, have to be fixed by corresponding normalization conditions. The normalization $\langle Q_\mu^a \rangle^{G'} = 0$ implies $b_1 = 0$, and the formal appearance of $\mathcal{A}_\mu^a$ in Z_{phys} is only a gauge artefact; on the other hand, the normalization $\langle Q_\mu^a \rangle^{G'} = \mathcal{A}_\mu^a / g$ implies $b_1 = 1$ which corresponds to the physical interesting situation.

(4) In the latter case it must be shown that the Green functions in the 1-instanton background have the correct symmetry behaviour provided the Green functions in the perturbative vacuum from which they are determined according to Eq. (4.15) have the correct behaviour. The corresponding proof may be taken over from /2/ almost literally as a consequence of the Eqs. (4.9) - (4.11). The action of $\mathcal{R}^a(x)$ on $Z_{phys}[\mathcal{A}]$ is given by

$$\mathcal{R}^a(x)\, Z_{phys}[\mathcal{A}^a_\mu, j] = \left(+ c_1 \partial_\mu j^a_\mu + D^{ab}_\mu(\mathcal{A}) \frac{\delta}{\delta \mathcal{A}^b_\mu} + \left(j_\mu \times \frac{\delta}{\delta j_\mu} \right)^a \right).$$

$$Z_{phys}[\mathcal{A}^a_\mu, j] = \left((b_1/g + c_1)\, \partial_\mu j^a_\mu + \left(j_\mu \times \frac{\delta}{\delta j_\mu} \right)^a \right) Z_{phys}[0, j]$$

$$\equiv \left(+ c_1^o\, \partial_\mu j^a_\mu + \left(j_\mu \times \frac{\delta}{\delta j_\mu} \right)^a \right) Z_{phys}[0, j] = 0. \tag{4.16}$$

where we have used Eq. (4.14) for the second relation. $c_1{}^o$ is the value of c_1 for the theory with $b_1 = 0$ which is characterized by $Z[\xi^a_\mu, J^a_\mu, \{\bar{\xi}\}, \{\xi\}, \{\eta\}]$. Therefore, it holds $c_1 = c_1{}^o - b_1/g$ and the first relation in Eq. (4.16) can be written as

$$\left(\frac{c_1^o}{b_1} \partial_\mu \frac{\delta}{\delta \mathcal{A}^a_\mu} + \left(\mathcal{A}_\mu \times \frac{\delta}{\delta \mathcal{A}_\mu} \right)^a + \left(j_\mu \times \frac{\delta}{\delta j_\mu} \right)^a \right) Z_{phys}[\mathcal{A}^a_\mu, j] = 0.$$

This shows that for $b_1 = 1$ type-I invariance holds for the renormalized theory (of course, in the tree approximation it holds $c_1{}^o = \frac{1}{G}$).

5. CONCLUSIONS

In sect. 4 we have shown that the Euclidean physical Green functions in the background of a BPST-instanton configuration are obtained by a mere translation from the Euclidean physical Green functions in the perturbative vacuum which are transversal not only with respect to the gauge zero modes but also to the translation and dilation zero modes, and which are computed with the corresponding ghost fields and some insertions corresponding to external sources.

The complications which originate from the occurrence of additional zero modes are only of an algebraic nature; they are get over by introduction of additional ghost fields and external sources, which amount to an extra coupling of the background field with these auxiliary fields. However, by restriction our considerations to physical Green functions these couplings are not further relevant.

An another formal complication was the explicit dependence of the instanton configuration on the coupling constant which has to be renormalized. But, because the dependence of $\mathcal{A}^a_\mu$ on g is rather trivial the result (4.14) could be written down in terms of a^a_μ and $\mathcal{A}^a_\mu$, i.e., by the Eqs. (4.15) and (4.16).

This result refers only to the $\mathcal{A}^a_\mu$ -dependence of Z_{phys} mediated by b_1, and does not make any assertion concerning the parameters b_2 and b_3 which result from a change of the anti-ghost field $\bar{c}(x)$ and of the source η_μ according to the Eqs. (3.8) and (3.10); but, the physical on shell Green functions do not depend on them! To get an assertion concerning the dependence of the Green functions on b_2 and b_3 we have to go off shell. Then, Eq. (4.12) has to be solved without any restriction which will be a hard task. Yet, having in mind the result of Gauthier and Rouet [1] on the 1PI-vertex functions in the 1-instanton background a result analogous to Eq. (4.14) concerning the coupling of $\mathcal{A}^a_\mu$ to the ghost sources may be anticipated.

Finally, we comment on the physical relevance of the result Eq. (4.14). Problems do not occure from the Green functions being defined in Euclidean space-time but mainly from the fact that any computation of physical effects depending on topological configurations necessarily has to take into account all instanton configurations, i.e., contributions from all sectors of different topological charge. Thereby, integrations over the instanton parameters (like group orientation, instanton scale and location, etc.) must be carried out. Also in the dilute gas approximation where knowledge only of the 1-instanton sector is required this leads to the well-known difficulties steming from the divergence of integration over instanton scale (otherwise, there must be introduced a cut off essentially by hand).

On the other hand our result to be useful in applications requires the knowledge of more general Green functions in the vacuum sector as usual. The determination of these functions seems to be as complicated as a direct determination of instanton-related Green functions. Therefore, we conclude that additional work has to be done, to find a connection of (physical) Green functions in the instanton background with the usual Green functions. With respect to this both problems just mentioned are related to each other.

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Дёрфель Б., Гайер Б., Мюльш Д. E2-85-95
Об одном свойстве евклидовой функции Грина полей Янга-Миллса группы SU(2) в поле инстантона БПШТ-типа

Применяя методику, развитую А.Рюэ, к полям Янга-Миллса в поле инстантона БПШТ-типа показано, что /евклидовы/ функции Грина поперечных калибровочных полей на массовой поверхности в присутствии инстантона получаются простым сдвигом из соответственно определенных функций Грина, вычисленных в пертурбативном вакууме.

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Сообщение Объединенного института ядерных исследований. Дубна 1985

Dörfel B., Geyer B., Mülsch D. E2-85-95
On a Property of Euclidean Green Functions of SU(2) Yang-Mills Fields in BPST-Instanton Background

Applying a method of A.Rouet to pure SU(2) Yang-Mills theory in the background of the BPST-instanton configuration one shows that the (Euclidean) on-shell Green functions of transverse gauge fields in the instanton background are obtained from appropriately determined Green functions in the perturbative vacuum by a mere translation.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1985