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G.P.Korchemsky, A.V.Radyushkin

INFRARED ASYMPTOTICS OF PERTURBATIVE QCD. QUARK AND GLUON PROPAGATORS

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1. Introduction

Analysis of the behaviour of quark and gluon propagators in the infrared (IR) region represents an essential element of the general study of IR asymptotics of the perturbative QCD. One should distinguish between IR singularities inherent in nonrenormalised amplitudes and the ones that arise just from the renormalization procedure (for instance, in QED such singularities appear in the course of subtraction on the electron mass shell). An effective method for analysing the IR singularities of the second type is the renormalization group/1/. The applicability of the renormalization-group analysis in the given case results directly from a one-to-one correspondence of the IR singularities with the ultraviolet behaviour of the theory /2,3/. When the subtraction point is taken in a deep Euclidean region (just this choice is typical for QCD), IR singularities of the second type do not arise. Besides, they do not manifest themselves in expressions for gauge-invariant quantities. Much more complicated and interesting from a physical point of view is the IR asymptotics of initial nonrenormalised amplitudes to be studied in this paper.

In sect. 2 we identify the region of phase space of gluons that gives a dominant contribution to the IR asymptotics of the quark propagator and separate it into a universal IR factor. Its properties and dependence on the gauge of gluon potentials considered in sect. 3 allow us to formulate a renormalization-group equation for the IR behaviour of the quark propagator in sect. 4. Boundary conditions for the obtained renormalization-group (RG) equation are found in sect.5 and a general form of its solution is compared with analogous expressions established earlier. The IR asymptotics of the gluon propagator is studied in sect. 6.

2. Power Counting

Consider the quark propagator whose diagram is drawn in Fig. 1. In what follows by the infrared asymptotics we shall mean the asymptotics of the contribution of the phase space of gluons with small momenta of each component $\{k^i_\mu\}$. The upper limit of "softness" of gluons of this set is defined by the kinematic invariants of the process under study (i.e. quark momentum squared, virtuality, various scales gauges of not yet specified, etc.), whereas the lower limit is defined by the range of applicability of a perturbative approach.

To determine the structure of subgraphs that essentially contribute to the quark propagator, we shall use the method of analysing asymptotics of Feynman integrals developed in /4/; we shall call it the power counting. According to /4/ a main contribution comes from that region of phase space in which the amplitude drawn in Fig. 1 has singularities (i.e. in the momentum representation some set of denominators of the quark and gluon propagators disappears and in the α -representation a quadratic form of kinematic variables under exponential vanishes /5/) the smallness of phase space of which is compensated by a power of the corresponding singularity.

Besides conventional RG ultraviolet singularities corresponding to a hard subprocess H drawn in Fig. 1 the quark propagator contains collinear singularities (logarithms of quark masses) determined by jet subgraphs J of particles collinear to the quark momentum and infrared singularities (logarithms of quark virtuality) from subprocesses S of emission of soft gluons by jet particles; the latter is the object of our study. As follows from the power counting, the emission of soft gluons by collinear quarks and gluons in vertices of four-gluon self-interaction is suppressed, at least, by a linear power of the momentum of a soft gluon as compared to their emission in three-gluon and quark-gluon vertices, and thus there are no IR singularities. Besides, the emission of soft gluons by collinear quarks and gluons of jet subprocesses J cannot change the spin state of the latter, and as a result, each of such vertices of triple interaction transforms into the well-known scalar factor $2p_\mu$ (p is the momentum of a collinear quark/gluon, μ is the polarization of a soft gluon) thus reproducing the Dirac and Lorentz structure of the initial collinear subprocess. Consequently, a soft gluon to make a singular contribution to the IR asymptotics of the whole diagram should be polarized longitudinally to the momentum p_μ . Propagation of such polarisations may be suppressed if the corresponding gluon potentials are subjected to the condition of axial gauge:

$$p_\mu A^a_\mu(x) = 0. \tag{1}$$

This means that if we express the propagators of particles of jet subgraphs J propagating in an "external" field A^a_μ of soft gluons taken in an arbitrary gauge in terms of the corresponding propagators in the axial gauge (1) for soft gluons of the block S :

$$S(x,y;A) = U^+(x)\, S(x,y;A^{axial})\, U(y) \tag{2}$$

for the jet quark propagator, and

$$D^{tr}_{\mu\nu}(x,y;A) = \tilde U^+(x)\, D^{tr}_{\mu\nu}(x,y;A^{axial})\, \tilde U(y) \tag{3}$$

for the transversal part of the jet gluon propagator*) (contributions of the longitudinal part of the jet gluon propagator and ghosts are cancelled in the total amplitude), we single out the IR singularities of the initial amplitude into operators of the gauge transformation $U(x)$. Axial gauge belongs to the class of so-called contour gauges /6/ for which the operator $U(x)$ is an exponential ordered along a ray parallel to the vector p_μ :

$$\hat E_p(x,\infty) = P\exp\Big(ig\int_0^\infty ds\, p_\mu \hat A_\mu(x+ps)\, e^{-\epsilon s}\Big), \quad \epsilon \to 0 \tag{4}$$

and the transformed field $\tilde A^a_\mu(x)$ satisfying eq.(1) is connected with the initial one by

$$\tilde A^a_\mu(x) = p_\nu \int_0^\infty ds\, e^{-\epsilon s}\, G^{\ell}_{\mu\nu}(x+ps;A)\, \tilde E^{a\ell}_p(x+ps,\infty). \tag{5}$$

The gauge transformations performed have changed not only the IR asymptotics of various multipliers of eqs.(2) and (3), but also the UV properties of propagator. Indeed, the renormalization constants of gluon and quark fields are gauge-variant and according to eqs.(2), (3) the difference between them in the original gauge ($\hat A_\mu$ -field) and in the axial gauge ($\hat A^{axial}_\mu$ - field) is generated by the UV singularities related to the gluons emitted from vertices of the P -ordered exponentials. There are two sources of extra UV singularities: subgraphs of emission and absorption of gluons in vertices of P -exponentials (Fig. 2a) containing also IR singularities and subgraphs of emission of gluons in vertices of P -exponentials and absorption by collinear quarks (gluons) (Fig. 2b) having no IR singularities by construction. By using a general method of ref. /5/ it may be shown that upon commutation of the operators $\hat E_p$ and $\tilde E_p$ through vertices of collinear and hard subprocesses there remain only two P -ordered exponentials at ends of the quark propagator. In the momentum representation it means:

*) Hereafter, the wavy line $\tilde U$ stands for the adjoint representation of the gauge group.

$$S(p) = \int d^4x\, e^{ipx} S(x) = \int d^4x\, e^{ipx} \langle 0| T\, \hat{E}^+_p(x,\infty)\, \hat{E}_p(0,\infty)|0\rangle\, \mathcal{M}_{hard}(x).$$

After separating from the factorized amplitude of hard and collinear subprocesses $\mathcal{M}_{hard}$ the free quark propagator: $\mathcal{M}_{hard}(x) = \int d^4y\, S_0(x-y)\, \overline{\mathcal{M}}_{hard}(y)$ and passing to α -representation for $S_0(x-y)$ one obtains /7/:

$$S(p) = -(\hat{p}+m)\int_0^\infty d\sigma\, e^{i\sigma(p^2-m^2+i0)} \langle 0|T\, \hat{E}^+_p(2\sigma p,\infty)\, \hat{E}_p(0,\infty)|0\rangle\, \overline{\mathcal{M}}_{hard}(p). \qquad (6)$$

Thus, the IR asymptotics of the quark propagator is determined by the contour integral

$$\mathcal{M}_{IR} = \langle 0|T\, \hat{E}^+_p(2\sigma p,\infty)\, \hat{E}_p(0,\infty)|0\rangle = \langle 0|T P \exp\Big(ig\int_0^{2\sigma\sqrt{p^2}} ds\, \frac{p_\mu}{\sqrt{p^2}}\, \hat{A}_\mu\big(s\frac{p}{\sqrt{p^2}}\big)\Big)|0\rangle \qquad (7)$$

along the straight line segment with length $2\sigma\sqrt{p^2}$ determined by the quark virtuality because a dominant contribution to the integral over σ in eq.(6) comes from a vicinity of the point $\sigma_0 = |p^2-m^2|^{-1}$.

3. Renormalization of the Infrared Factor

The IR asymptotics of the quark propagator enters completely into the P -ordered exponential (7), averaged over the QCD vacuum. The gauge dependence of $\mathcal{M}_{IR}$ results from the gauge noninvariance of contour integrals, and the contour length (fig. 3a) determines the scale of wave lengths of soft gluons.

Besides IR singularities displayed in the logarithmic dependence on σ_0 the contour average (7) contains also ultraviolet divergences studied in refs. /8-12/. All the further analysis reduces to establishing the connection between IR and UV asymptotics of eq.(7), and the corresponding consequences. It is well known that the IR asymptotics of the quark propagator changes qualitatively when one passes from a covariant (e.g., Feynman) gauge to a physical (e.g., axial) gauge of the gluon field, and for this reason it is just these two gauge conditions we shall be restricted in what follows. In the axial gauge given by the condition $nA(x)=0$ we perform an identical transformation of eq.(7), i.e. add to the contour drawn in Fig. 3a two rays parallel to the vector n_μ , on which, as is easily seen,

$$P\exp\Big(ig\, n_\mu \int_0^\infty ds\, e^{-\epsilon s} A_\mu(x+ns)\Big) = \hat{1}\,.$$

As a result, eq.(7) may be rewritten as the Wilson loop (Fig. 3b) closed at infinity:

$$\mathcal{M}_{IR,axial} = \frac{1}{N_c}\, Tr \langle 0|T\, \hat{E}_n(2\sigma p,\infty)\, \hat{E}^+_p(2\sigma p,\infty)\, \hat{E}_p(0,\infty)\, \hat{E}^+_n(0,\infty)|0\rangle\,. \qquad (8)$$

If the momentum of a quark tends to the mass shell ($\sigma_0 \to \infty$), eq.

(7) describes the IR asymptotics of self-energy corrections to the quark field $\frac{1}{\hat{p}-m}\sum(\hat{p})\big|_{\hat{p}=m}$:

$$\mathcal{M}^{on-shell}_{IR} = \langle 0|T P\exp\Big(ig\int_0^\infty ds\, e^{-\epsilon s} p_\mu\, \hat{A}_\mu(ps)\Big)|0\rangle \qquad (9)$$

and in the case of axial gauge

$$\mathcal{M}^{on-shell}_{IR,axial} = \langle 0|T\, \hat{E}^+_n(0,\infty)\, \hat{E}_p(0,\infty)|0\rangle \qquad (10)$$

which corresponds to contours of Fig. 3d and 3c, respectively.

Equations (7)-(10) contain extra UV singularities due to the cusp and end points on contours of Fig. 3 /8,9/. Renormalization properties of an arbitrary contour average $E_c = \langle 0|TP\exp(ig\int_c dz_\mu A_\mu)|0\rangle$ are described by the following RG equation /10/:

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + N\,\Gamma_{end}(g) + \sum_{i=1}^{I} \Gamma_{cusp}(g,\gamma_i)\Big) E_c(\mu L, g, \{\eta\}) = 0, \qquad (11)$$

where N and I represent the number of end points and cusps, respectively, of contour C with angles $\{\gamma_i\}$, Γ_{end} and Γ_{cusp} are the corresponding anomalous dimensions. From arguments of the P -exponential we have explicitly separated the length of contour L and the set of dimensionless parameters $\{\eta\}$. To subtract end-point singularities, we shall use the procedure of introduction of one-dimensional fermions lived on the contour C /8/, and to subtract cusp singularities $K^{MS}_\gamma R_{MS}$ - subtraction scheme of ref. /11/. The end-point gauge-dependent anomalous dimension is known in the 2-loop approximation in Feynman gauge in the MS scheme /12/:

$$\Gamma_{end}(g) = -\frac{\alpha_s}{2\pi} C_F - \frac{7}{48}\Big(\frac{\alpha_s}{\pi}\Big)^2 C_F N_c \qquad (12)$$

and the cusp gauge-invariant anomalous dimension was calculated by us /11,13/ in the same approximation:

$$\begin{aligned}\Gamma_{cusp}(g,\gamma) = \frac{\alpha_s}{\pi} C_F(\gamma\, cth\gamma - 1) + 2\Big(\frac{\alpha_s}{\pi}\Big)^2 C_F N_c \Big[\frac{67}{72}(\gamma\, cth\gamma - 1) + \frac{1}{4} \\ -\frac{\pi^2}{48}(\gamma\, cth\gamma - 1) - \frac{1}{2} cth\gamma \int_0^\gamma dx\, x\, cth x + \frac{1}{2} cth^2\gamma \int_0^\gamma dx\, x(\gamma - x)\, cth x \\ - \frac{1}{4} sh\, 2\gamma \int_0^\gamma dx\, \frac{x\, cth x - 1}{sh^2\gamma - sh^2 x}\, \ln\frac{sh\gamma}{sh x}\Big],\end{aligned} \qquad (13)$$

where γ is the cusp angle of a contour in the Minkowski space. Determine now the UV asymptotics of contour integrals in eqs.(7)-(10) corresponding to contours in Fig. 3. The total anomalous dimension, the number of end points and cusps equal respectively: contour in Fig. 3a -

$$\Gamma_a = 2\,\Gamma_{end}(g)\,, \quad N = 2\,, \quad I = 0 \qquad (14)$$

contour in Fig. 3b -

$$\Gamma_b = 2\,\Gamma_{cusp}(g,\gamma)\,, \quad N = 0\,, \quad I = 2\,, \qquad (15)$$

where γ is the angle between the vector n_μ fixing the gauge condition and the quark momentum in the Minkowski space ($ch\gamma = \frac{(pn)}{\sqrt{p^2 n^2}}$); contour in Fig. 3c -

$$\Gamma_c = \Gamma_{cusp}(g,\gamma)\ ,\ N = 0\ ,\ I = 1 \tag{16}$$

contour in Fig. 3d -

$$\Gamma_d = \Gamma_{end}(g)\ ,\quad N = 1\ ,\ I = 0\ . \tag{17}$$

Now let us establish the connection between IR and UV asymptotics of eqs.(9) and (10). An infinite length of the contour in Figs. 3c,d, a source of IR singularities, requires the introduction of infrared-regularization scheme. Since the integrals obtained over contour parameters are simultaneously divergent both in UV and IR regions, we supplement the dimensional regularization with a fictitious gluon mass Λ . Within the subtraction procedures we make use of the calculations to order α_s gives

$$m^{on-shell}_{IR,Feyn.} = 1 + \frac{\alpha_s}{4\pi} C_F \ln \frac{\mu^2}{\Lambda^2} \tag{18}$$

and in the case of axial gauge

$$m^{on-shell}_{IR,axial} = 1 - \frac{\alpha_s}{2\pi} C_F (\gamma cth\gamma - 1) \ln \frac{\mu^2}{\Lambda^2} \tag{19}$$

in a complete agreement with eqs.(16) and (17). That these equations contain renormalization parameters μ and Λ in ratio is a consequence of the invariance of nonrenormalized eqs.(9) and (10) under scale transformations of parameters of the integration over contour /14/ or, equivalently, the length of contour whose analog is an inverse of the infrared regulator:

$$m^{on-shell}_{IR} \equiv m^{on-shell}_{IR}(\mu L, g)\ ,\ L \sim 1/\Lambda\ . \tag{20}$$

This fact will allow us, in the next section, to generalize eq.(11) to take account of IR asymptotics. The above invariance and the general form of dependence of contour averages on μ and quark virtuality $1/\sigma_o$, the infrared cut-off, break for the contours of Figs. 3a and 3b. Here we present the resulting expressions in the one-loop approximation for eq.(7):

$$m^{off-shell}_{IR,Feyn.} = 1 + \frac{\alpha_s}{2\pi} C_F \left[\ln(\mu^2 p^2 \sigma_o^2) + 2\right] \tag{21}$$

and for eq.(8)*):

*) Calculation of contour integrals is essentially simplified in terms of angular variables /11/:

$$\frac{x\sqrt{p^2} + \bar{x} e^{\gamma}\sqrt{n^2}}{x\sqrt{p^2} + \bar{x} e^{-\gamma}\sqrt{n^2}} = e^{2\psi}\ ,\quad \bar{x} = 1 - x \qquad \text{or} \qquad \int_0^1 \frac{dx}{(xp + \bar{x}n)^2} = \frac{1}{\sqrt{p^2 n^2}\, sh\gamma} \int_0^{\gamma} d\psi\ ,$$

$$m^{off-shell}_{IR,axial} = 1 - \frac{\alpha_s}{\pi} C_F \left[(\gamma cth\gamma - 1)\ln(\mu^2 p^2 \sigma_o^2 - 1 - \gamma cth\gamma + 2 cth\gamma \int_0^{\gamma} dx\, x\, cth\, x\right]\ . \tag{22}$$

The UV asymptotics of eqs.(21) and (22) correspond to anomalous dimensions of eq.(14) and (15). Despite the fact that in the considered kinematics the functional form of eq.(20) is not valid, we can here introduce the concept of an effective length of a contour as an inverse of the value of renormalization parameter μ at which the contour average corresponding to Fig. 3a,b turns into unity:

$$m^{off-shell}_{IR} \equiv m^{off-shell}_{IR}(\mu L_{eff.}, g) = 1 \qquad \mu = 1/L_{eff.}\ . \tag{23}$$

From eqs.(21) and (22) we may easily derive the one-loop expression for the effective length of contour in Fig. 3a:

$$L^{Feyn}_{eff.} = \sigma_o \sqrt{p^2}\, C_1\ , \tag{24}$$

where $C_1 = \exp(1)$. The expression for the effective length of contour in Fig. 3b is rather cumbersome, and it is simplified in the limit of a large cusp angle. If we denote

$$\zeta = 4(pn)^2/(p^2 n^2)$$

then for $\zeta \gg 1$ we get

$$L^{axial}_{eff.} = \sigma_o \sqrt{p^2}\, \zeta^{1/4}\, C_2\ , \tag{25}$$

where $C_2 = \exp\left(\frac{\pi^2/6 - 1}{\ln\zeta} + O\left(\frac{1}{\zeta}\right)\right)$. The expressions obtained for the effective length, eqs.(24) and (25), are not exact and are modified by higher-order perturbative corrections whose general structure may be studied with the use of exponentiation theorem/15/. In particular, as has been shown in /11/, the effective length of a contour is determined not by a complete set of contour integrals but only by a subset of strongly connected diagrams with a "maximally non-Abelian" part of the colour factor/15/. It is to be noted that the effective length in any perturbation order will linearly depend on $\sigma_o\sqrt{p^2}$, the only dimensional parameter of contour averages in eqs.(7) and (8). In higher orders of PT corrections will change the constants C_1, C_2 in eqs.(24) and (25) and the power of ζ in eq.(25).

4. Renormalization-Group Equation for IR Asymptotics

Introducing the effective length inversely proportional to the quark virtuality and using eqs.(20), (23) we may generalize the RG equation (11) to describe the IR asymptotics of the quark propagator:

$$\left(L_{eff.}\frac{\partial}{\partial L_{eff.}} + \beta(g)\frac{\partial}{\partial g} + N\,\Gamma_{end}(g) + I\,\Gamma_{cusp}(g,\gamma)\right) m_{IR}(\mu L_{eff.}, g) = 0\ , \tag{26}$$

where for the on-shell kinematics $L_{eff.} \equiv 1/\Lambda$, the parameters N and I for contours in Fig. 3 are given by eqs.(14)-(17). The general solution to eq.(26) is as follows:

$$m_{IR}(\mu L_{eff}, g) = \exp\left(-\int_{g(1/L_{eff})}^{g(\mu)} \frac{dg}{\beta(g)} \left[N \Gamma_{end}(g) + I \Gamma_{cusp}(g, \gamma) \right] \right). \tag{27}$$

It describes the whole IR asymptotics of the quark propagator and contains extra UV singularities. It results in a number of important consequences. According to eqs.(14)-(17),(27) the difference between asymptotics in the Feynman and axial gauge reduces to a different behaviour of the end anomalous dimension of eq.(12) and cusp anomalous dimension that in the limit $\xi \gg 1$ has the logarithmic behaviour /11/:

$$\Gamma_{cusp}(g, \gamma) \underset{\gamma \to \infty}{=} \frac{\alpha_s}{2\pi} c_F \ln \xi + \left(\frac{\alpha_s}{\pi}\right)^2 c_F N_c \left(\frac{67}{72} - \frac{\pi^2}{24}\right) \ln \xi .$$

Besides, the shift of a quark from the mass shell doubles the number of singularities on the contour and, consequently, the IR asymptotics. Note that in the obtained equation (26) universal, i.e. independent of a particular form of the contour, are anomalous dimensions Γ_{end} and Γ_{cusp} /9,10/, while a specific feature of the process under investigation manifests itself in a definite value of N, I and the effective length $L_{eff.}$.

To compare eq.(26) with equations obtained earlier by counting in the lowest perturbation orders /16,17/, we differentiate both sides of eq.(6) with respect to the quark virtuality $1/\sigma_o$. Taking account of eq.(27) and linear dependence of the effective length L_{eff} on σ_o in all perturbation orders we get

$$\frac{d}{d \ln \sigma_o}\left(S_o^{-1}(p) S(p)\right) = \frac{d \ln m_{IR}}{d \ln \sigma_o} = N \Gamma_{end}(g(1/L_{eff})) + I \Gamma_{cusp}(g(1/L_{eff}), \gamma) . \tag{28}$$

The right-hand side of this equation represents a perturbation series in the coupling constant α_s . Starting from the second expansion term it describes a nonleading IR asymptotics of the quark propagator, and therefore a complete solution of the problem requires us to estimate corrections we have neglected by truncating the perturbation series. Earlier /11/, we have shown that in the limit of large cusp angles of the contour $\gamma \gg 1$ the angular anomalous dimension has the following asymptotic behaviour

$$\Gamma_{cusp}(g, \gamma) = \sum_{n=1}^{\infty} \alpha_s^n c_F N_c^{n-1} (a_n \ln \xi + b_n)$$

which signifies that in the r.h.s. of eq.(28) $(\alpha_s(1/L_{eff}) N_c)$ is the expansion parameter. This fact produces a strong constraint on the non-

leading IR asymptotics. Specifically, the appearance of terms of the form $\frac{11}{48}\left(\frac{\alpha_s}{\pi}\right)^2 c_F N_c \ln^2 \xi$ in the r.h.s. of eq.(28) (and, consequently, of the expansion parameter $\alpha_s N_c \ln \xi$) in calculations of refs. /18,19/ results from nontriviality of the argument of the coupling constant in eq. (28). From eq.(28) it also follows that in an n-loop approximation the IR asymptotics is determined by an n-loop expression for anomalous dimensions, an ($n-1$)-loop contribution to the effective length and β -function of QCD. In particular, eqs. (12),(13), (24), (25), (28) describe the whole IR asymptotics of the quark propagator in 2-loop approximation.

5. Extra UV Divergences

The obtained eq.(27), a cofactor to eq.(6), contains UV poles coefficients of which depend on kinematic invariants. As mentioned, this dependence vanishes when account is taken of extra UV divergences produced by the hard part of amplitude $\overline{m}_{hard}(p)$ (Fig. 2b). In order α_s it is equal to the sum of three diagrams (Fig. 4). Unusualness of the diagrams of Figs. 4a,b consists in that they contain the pairing of gluons A_μ^a emitted from vertices of the P -exponential and potential $\tilde{A}_\mu^a(x)$ transformed to the axial gauge. In order α_s they are connected by the condition

$$\tilde{A}_\mu^a(k) = \left(g_{\mu\nu} - \frac{k_\mu p_\nu}{kp}\right) A_\nu^a(k) .$$

The diagram of Fig. 4c is the quark propagator in axial gauge $p_\mu \tilde{A}_\mu^a(x) = 0$, where p_μ is the quark momentum. The contribution of diagram of Fig. 4a is given by *)

$$i g^2 c_F \int \frac{d^n k}{(2\pi)^n} \gamma_\mu \frac{\hat{p} + \hat{k} + m}{(p+k)^2 - m^2} \left(g_{\mu\nu} - \frac{k_\mu p_\nu}{kp}\right) D_{\nu\rho}(k) \frac{p_\rho}{(kp)} , \tag{29}$$

where $D_{\mu\nu}(k)$ is the gluon propagator in the gauge of field A_μ^a . Calculation of eq.(29) and the renormalized contribution of a mirror-symmetric diagram of Fig. 4b gives the following result: for $\sigma_o = |p^2 - m^2|^{-1} \gg$ $\gg m^{-2}$ in the axial gauge

$$\overline{m}^{\,off-shell}_{hard, a+b} = \frac{\alpha_s}{\pi} c_F \left[(\gamma \operatorname{cth} \gamma - 1) \ln \frac{\mu^2}{p^2} - 2 \operatorname{cth} \gamma \int_0^\gamma dx\, x \operatorname{cth} x + 2 + \gamma \operatorname{th} \frac{\gamma}{2} \right] \tag{30}$$

in the Feynman gauge

$$\overline{m}^{\,off-shell}_{hard, a+b} = - \frac{\alpha_s}{\pi} c_F \left[\frac{3}{2} \ln \frac{\mu^2}{p^2} - 1 \right] \tag{31}$$

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*) Here we make use of the momentum representation for P-ordered exponentials /11/.

At $p^2 = m^2$ there is the following connection with the previous case:

$$\overline{m}^{\text{on-shell}}_{\text{hard},a+b} = \frac{1}{2}\,\overline{m}^{\text{off-shell}}_{\text{hard},a+b}\,. \tag{32}$$

A renormalized amplitude of the diagram of Fig. 4c is independent of the gauge of field A^a_μ, and for $\sigma_0 \gg m^{-2}$ it is given by

$$\overline{m}^{\text{off-shell}}_{\text{hard},c} = \frac{\alpha_s}{\pi} C_F \left[\frac{3}{4}\ln\frac{\mu^2}{p^2} - 2\right] \tag{33}$$

and for $p^2 = m^2$ by

$$\overline{m}^{\text{on-shell}}_{\text{hard},c} = \frac{1}{2}\,\overline{m}^{\text{off-shell}}_{\text{hard},c}\,. \tag{34}$$

Now let us carefully analyse the properties of eqs.(30)-(34). That there are no IR divergences is a result of the factorization of the IR asymptotics in eq.(6). The amplitudes (33) and (34) obey the RG equation for the propagator in axial gauge in which the anomalous dimension of the fermion field equals

$$\Gamma^{(\text{axial})}_F = \frac{3}{4} C_F \frac{\alpha_s}{\pi}\,.$$

As UV divergences of the quark propagator and infrared factor of eq.(7) are renormalized multiplicatively, the hard amplitude $\overline{m}_{\text{hard}}$ satisfies the conventional RG equation (in the notation of eqs.(11)-(17)):

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - N\left(\Gamma_{\text{end}}(g) - \Gamma_F(g)\right) - I\,\Gamma_{\text{cusp}}(g,\gamma)\right)\overline{m}_{\text{hard}} = 0\,, \tag{35}$$

where Γ_F is the anomalous dimension of the fermion field in the gauge of field A^a_μ. It is easily verified that this equation is fulfilled for the 1-loop expression $\overline{m}_{\text{hard}} = 1 + \sum_{i=a,b,c}\overline{m}^{i}_{\text{hard}}$. From eqs.(6), (18),(19),(22),(22), (30)-(34) we get in the axial gauge:

$$\begin{aligned} \overline{m}^{\text{on-shell}}_{\text{hard}} &= 1 - \frac{\alpha_s}{2\pi} C_F\left[\left(\frac{1}{4} - \gamma\,\mathrm{cth}\,\gamma\right)\ln\frac{\mu^2}{p^2} + 2\,\mathrm{cth}\,\gamma\int_0^\gamma dx\,x\,\mathrm{cth}\,x - \gamma\,\mathrm{th}\frac{\gamma}{2}\right] \\ \overline{m}^{\text{off-shell}}_{\text{hard}} &= \left(\overline{m}^{\text{on-shell}}_{\text{hard}}\right)^2 \end{aligned} \tag{36}$$

and in the Feynman gauge:

$$\begin{aligned} \overline{m}^{\text{on-shell}}_{\text{hard}} &= 1 - \frac{\alpha_s}{2\pi} C_F\left[\frac{3}{4}\ln\frac{\mu^2}{p^2} + 1\right] \\ \overline{m}^{\text{off-shell}}_{\text{hard}} &= \left(\overline{m}^{\text{on-shell}}_{\text{hard}}\right)^2\,. \end{aligned} \tag{37}$$

The general form of solution to eq.(35)

$$\overline{m}_{\text{hard}}(\mu) = \exp\left(\int_{g(\bar\mu)}^{g(\mu)}\frac{dg}{\beta(g)}\left[N\left(\Gamma_{\text{end}}(g) - \Gamma_F(g)\right) + I\,\Gamma_{\text{cusp}}(g,\gamma)\right]\right)\overline{m}_{\text{hard}}(\bar\mu) \tag{38}$$

depends on the renormalization point $\bar\mu$ whose choice is RG arbitrary. We will require that the amplitude $\overline{m}_{\text{hard}}(\bar\mu)$ be minimized by the value of $\bar\mu$. Then, for $\zeta \gg 1$ eqs.(36) and (37) lead to the one-loop expression for $\bar\mu$ in tha Feynman gauge:

$$\bar\mu^2 = p^2 C_3 \tag{39}$$

and in the axial gauge:

$$\bar\mu^2 = p^2\sqrt{\zeta}\, C_4\,, \tag{40}$$

where $C_3 = \exp(-4/3)$, $C_4 = \exp\left(-\frac{3}{4} + O\left(\frac{1}{\ln\zeta}\right)\right)$ for both kinematics. Having determined the asymptotics of cofactors of eq.(6) in eqs.(27), (38) we arrive at the final expression for the quark propagator:

$$S(p) = \exp\left(-\int_{g(1/L_{\text{eff}})}^{g(\bar\mu)}\frac{dg}{\beta(g)}\left[N\Gamma_{\text{end}}(g) + I\,\Gamma_{\text{cusp}}(g,\gamma)\right]\right)\exp\left(-\int_{g(\bar\mu)}^{g(\mu)}\frac{dg}{\beta(g)}N\Gamma_F(g)\right) \times S_0(p)\,\overline{m}_{\text{hard}}(\bar\mu)\,. \tag{41}$$

The first multiplier of (41) contains all the IR singularities and depends on the effective length L_{eff} and renormalization parameter $\bar\mu$ minimizing the hard subprocess. The one-loop values for $L_{\text{eff}}, \bar\mu$ are defined by eqs.(24),(25),(39),(40) in the Feynman and axial gauge. The dependence of $S(p)$ on the quark virtuality is described by eq.(28). Till now we have dealt with the case when the quark is on the mass shell or near by it. The developed approach may naturally be generalized to study the quark propagator with $|p^2 - m^2| \gg m^2$. There are valid all the results obtained in sects. 1-4 but with $\sigma_0 = 1/|p^2|$; only finite parts of UV divergences will be changed. However, the performed calculation shows that in the considered kinematics minimizing values of $\bar\mu$ differ from those given by eqs.(39),(40) by a small change in the numerical factors C_3 and C_4. Substituting into (41) the explicit form of quantities $\bar\mu, L_{\text{eff}}$ obtained earlier one may easily verify that first expansion terms in the exponent of the first exponential of eq.(41) in the axial gauge coincide with those found in works /18/ and /19/ *).

*) In the studied limit of the quark momentum much shifted off the mass shell the expressions obtained for $\bar\mu, L_{\text{eff}}$, Γ_{end} and Γ_{cusp} allow us to take into account, in the exponent of eq.(41), terms $O(\alpha_s^n \ln^{n+1}\zeta)$ (for $n \geq 2$) including $O(\alpha_s^2 \ln^3\zeta)$ terms of refs. /18,19/.

6. The Gluon Propagator

Let us proceed to consider the IR asymptotics of the gluon propagator. Let $1/\sigma_0$ be the gluon virtuality defining the scale of softness of IR gluons. We shall now apply to the property well known from the factorization theory /4,5/: the Feynman diagrams for components of the total gluon propagator corresponding to physical polarizations have the same structure as the diagram for the quark propagator grawn in Fig. 1 provided the initial quark is replaced by a gluon of the corresponding polarization. There exists a class of so-called physical gauge conditions (for instance, the axial gauge and the Fock--Schwinger gauge, particular cases of the contour gauge /6/) admitting the propagation only of transversely polarized gluons. In a class of covariant gauges (α -gauges) the requirement of transversality is achieved by introducing ghosts into closed gluon cycles and by averaging finite states with a physical matrix of density of gluons. Notice that the IR factors singled out of each quark and gluon jet propagator in eqs.(2),(3) differ only in the representation of the gauge group. Following all the arguments of sect. 2 we arrive at the following conclusion: the IR asymptotics of the propagator of physical polarizations of gluons enters into the contour integral

$$m_{IR}(x) = \langle 0|T\,\tilde{E}^{+}_{\rho}(x,\infty)\,\tilde{E}_{\rho}(0,\infty)|0\rangle \tag{42}$$

whose properties in the fundamental representation have been studied in sect. 3,4. The gluon propagator in axial gauge in the momentum representation may be written analogously to eq.(6):

$$D^{\mu\nu}(p) = \int d^4x\, e^{ipx} D^{\mu\nu}(x) = \int d^4x\, e^{ipx}\, m_{IR}(x)\, m^{\mu\nu}_{hard}(x)\,.$$

Upon expanding $m^{\mu\nu}_{hard}$ over independent tensor structures, separating the free gluon propagator, and passing to α -representation, we get

$$D_{\mu\nu}(p) = \Big(g_{\mu\nu} - \frac{n_\mu p_\nu + n_\nu p_\mu}{(np)} + \frac{p_\mu p_\nu}{(np)^2}n^2\Big)\int_0^\infty d\sigma\, e^{i\sigma(p^2+i0)}\, m_{IR}(\sigma)\,\overline{m}_{hard} \tag{43}$$

$$+ \Big(\frac{n_\mu n_\nu}{n^2} - \frac{n_\mu p_\nu + n_\nu p_\mu}{(np)} + \frac{p_\mu p_\nu}{(np)^2}n^2\Big)\, O\Big(\frac{1}{\ln\zeta}\Big)\,,$$

where n_μ is the vector fixing axial gauge, and $m_{IR}(\sigma) = \langle 0|T\,\tilde{E}^{+}_{\rho}(2\sigma p,\infty)\tilde{E}_{\rho}(0,\infty$

Upon an obvious change of colour factor and substitution of σ_0 :

$$C_F \to C_A = N_c \qquad \sigma_0 = 1/|p^2| \tag{44}$$

we find that all the results of sects. 3 and 4 are valid for the IR factor m_{IR} of the gluon propagator.

The hard part of amplitude of the gluon propagator given by eq. (43) in order α_s is contributed by four diagrams drawn in Fig. 5. We have calculated it with the use of the following expression for the gluon polarization operator in axial gauge /20/:

$$\Pi_{\mu\nu}(p) = -i\,\big[\,\Pi_0(p)\,P_{\mu\nu} + \Pi_1(p)\,N_{\mu\nu}\big] \tag{45}$$

with the tensor structures

$$P_{\mu\nu} = p^2 g_{\mu\nu} - p_\mu p_\nu\,, \quad N_{\mu\nu} = p_\mu p_\nu - \frac{p_\mu n_\nu + p_\nu n_\mu}{(pn)}p^2 + \frac{n_\mu n_\nu}{(pn)^2}p^4$$

and the notation $\mathrm{ch}^2\gamma = \frac{(pn)^2}{p^2n^2} = \frac{\zeta}{4}$

$$\Pi_0(p) = \frac{g^2N_c}{32\pi^2}\Big[-\frac{22}{3}\ln\frac{\mu^2}{p^2} + \mathrm{cth}^2\gamma\Big\{-\ln(4\mathrm{ch}^2\gamma)(8 - 6\,\mathrm{ch}^{-2}\gamma + \mathrm{ch}^{-4}\gamma) - \frac{62}{9} + \frac{44}{9}\mathrm{ch}^{-2}\gamma$$
$$+ 2\,\mathrm{ch}^{-4}\gamma + \big(8\,\mathrm{ch}^2\gamma - 8 + 2\,\mathrm{ch}^{-2}\gamma - \tfrac{1}{2}\mathrm{ch}^{-4}\gamma\big)Z\Big\}\Big]$$

$$\Pi_1(p) = \frac{g^2N_c}{32\pi^2}\mathrm{cth}^2\gamma\Big[-\frac{10}{3} + 2\,\mathrm{ch}^{-2}\gamma + \ln(4\mathrm{ch}^2\gamma)(7 - \mathrm{ch}^{-2}\gamma - 9\,\mathrm{cth}^2\gamma) - \tfrac{1}{2}(16\,\mathrm{ch}^2\gamma - 5 + \mathrm{ch}^{-2}\gamma$$
$$- 9\,\mathrm{cth}^2\gamma)Z\Big]$$

$$Z = 16i\pi^2\frac{p^2n^2}{(pn)}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2(kn)(k+p)^2} = \frac{4}{\mathrm{sh}\,2\gamma}\Big(\gamma\ln(2\mathrm{ch}\gamma) + \mathrm{sh}\gamma\int_0^\gamma\frac{dx\,x}{\mathrm{sh}x\,\mathrm{sh}(\gamma+x)}\Big)\,.$$

In the limit $\zeta \gg 1$ the contribution of $\Pi_1(p)$ to the gluon propagator is suppressed by a power of ζ :

$$D^{\mu\nu}(p) = D_0^{\mu\nu}(p)\Big[1 - \frac{\alpha_s}{2\pi}N_c\Big(-\frac{11}{6}\ln\frac{\mu^2}{p^2} + \ln^2\zeta - 2\ln\zeta + \frac{\pi^2}{3} - \frac{31}{18}\Big)\Big]\,. \tag{46}$$

The resulting expression for $\overline{m}_{hard}$ is

$$\overline{m}_{hard} = 1 - \frac{\alpha_s}{2\pi}N_c\Big[\Big(-\ln\zeta + \frac{1}{6}\Big)\ln\frac{\mu^2}{p^2} + \frac{1}{2}\ln^2\zeta - \ln\zeta + \frac{5}{18}\Big] \tag{47}$$

from which we obtain the minimizing renormalization parameter $\bar{\mu}$ in the one-loop approximation in axial gauge:

$$\bar{\mu}^2 = p^2\sqrt{\zeta}\,c_5\,, \quad c_5 = \exp\Big(-\frac{11}{12} + O\Big(\frac{1}{\ln\zeta}\Big)\Big)\,. \tag{48}$$

Therefore, ultimately, the gluon propagator in axial gauge in the limit $\gamma,\zeta \gg 1$ is defined by the expression (cf. eq.(41)):

$$D_{\mu\nu}(p) = D^{(0)}_{\mu\nu}(p)\exp\Big(-2\int_{g(1/L_{eff})}^{g(\bar{\mu})}\frac{dg}{\beta(g)}\Gamma_{cusp}(g,\gamma)\Big)\Big(\frac{g^2(\bar{\mu})}{g^2(\mu)}\Big)\,\overline{m}_{hard}(\bar{\mu})\,, \tag{49}$$

where L_{eff} and Γ_{cusp} are given by eqs.(25) and (13) with a subsequent transformation (44), $\bar{\mu}$ is defined by eq.(48) and account is taken of the connection between the anomalous dimension of gluon fields in axial gauge and the QCD β -function. The first expansion term in the exponent of exponential (49) in powers of $\alpha_s(\frac{4(pn)^2}{n^2})$ was calculated in refs. /16,21/, the second (partially) in ref. /19/.

7. Conclusion

In the present paper we have studied the infrared asymptotics of the quark and gluon propagators within the framework of perturbative QCD. Using gauge properties of underline theory we have demonstrated that the IR asymptotics of the propagators are accumulated by new nonlocal objects, namely, the P-exponentials, averaged over the PT vacuum and ordered along of contours of the type of Fig. 3. The form of the contours is uniquely fixed by kinematics of the process under consideration and by the gauge condition used. The relative simplicity of the contour structure had allowed us to establish a one-to-one correspondence of the UV and IR singularities of corresponding contour integrals and to formulate then the renormalization group equation for the IR asymptotics of the quark and gluon propagators. Its solution for different values of the external quark (gluon) momenta squared in the axial and Feynman gauges depends on certain quantities which we determined to one-loop and contains the nonleading IR asymptotics known from refs. /18,19/. Using the renormalization properties of contour averages we have found a general functional form of the anomalous dimension appeared in the RG equation. We succeded to sum up the large corrections obtained earlier in refs. /18,19/ absorbing them into the effective argument of the running coupling constant which is the parameter of perturbation theory expansion for the anomalous dimension.

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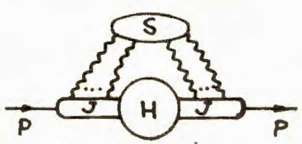

Fig. 1. General structure of the diagram for the quark propagator with momenta p_μ contained hard (H), collinear (J) and soft (S) subprocesses.

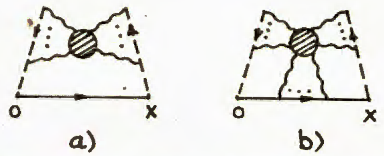

Fig. 2. Diagrams contributed to infrared asymptotics (a) and hard part (b) of the quark propagator. Here - after the dashed line with n connected gluons denotes an n-th term of the perturbative expansion of the P-exponential /11/. The blob denotes an arbitrary gluon subprocess.

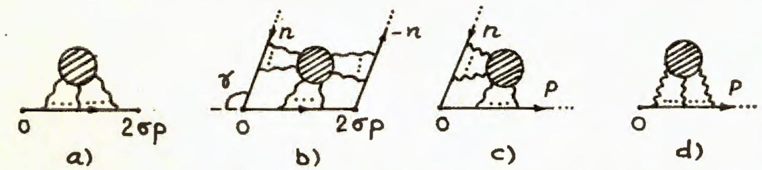

Fig. 3. The integration contours determined the infrared asymptotics of the quark and gluon propagators in the Feynman gauge for off-shell (a) or on-shell (d) external momenta and in the axial gauge for off-shell (b) or on-shell (c) kinematics.

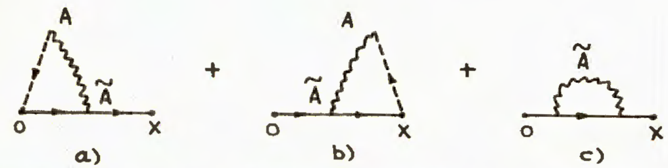

Fig. 4. Total set of the diagrams contributing to the hard part of the quark propagator in α_s order. The type of the gauge potential of the emitted gluons is pointed out in the interaction vertices.

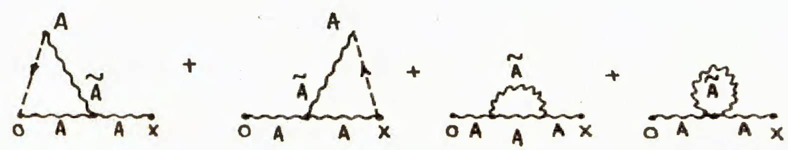

Fig. 5. Total set of the diagrams contributing to the hard pard of the gluon propagator in α_s order.

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Корчемский Г.П., Радюшкин А.В.

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Инфракрасная асимптотика пертурбативной КХД.
Кварковый и глюонный пропагаторы

Изучена инфракрасная асимптотика кваркового и глюонного пропагаторов в рамках пертурбативной КХД. Найдена ее связь с ренормализационными свойствами контурных интегралов. Показано, что вся инфракрасная асимптотика пропагаторов поглощается новыми нелокальными обьектами - экспонентами, упорядоченными вдоль контура и усредненными по вакууму теории возмущений. Вид контура однозначно фиксируется кинематикой процесса и налагаемым калибровочным условием. Установлена связь ультрафиолетовых и инфракрасных особенностей соответствующих контурных интегралов и на ее основе сформулировано ренормгрупповое уравнение для инфракрасной асимптотики кваркового и глюонного пропагаторов. Решение этого уравнения сравнивается с результатами вычислений в низших порядках по константе сильного взаимодействия.

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Korchemsky G.P., Radyushkin A.V.

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Infrared Asymptotics of Perturbative QCD.
Quark and Gluon Propagators

The infrared asymptotics of the quark and gluon propagators are investigated within the framework of perturbative QCD. The deep connection between infrared problem and renormalization properties of contour averages is found. It is demonstrated that the infrared asymptotics of the propagators are accumulated by new nonlocal objects (the path ordered exponentials) averaged over the perturbation theory vacuum. The form of the contours is uniquely fixed by kinematics of the process under consideration and by the gauge condition used. The one-to-one correspondence of the ultraviolet and infrared singularities of corresponding contour integrals is established and the renormalization group equation for the infrared asymptotics of the quark and gluon propagators is formulated. The solution of the equation is compared with the results of the lowest perturbation theory calculations.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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