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CLASSICAL PARTICLE DYNAMICS IN THE QUANTUM SPACE

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1. Introduction

In previous papers (Namsrai /1/ ; Dineykhan and Namsrai /2/) we have introduced quantum space-time into physics and considered some of its interesting consequences. Our method of introducing quantum space-time may be regarded as a local general coordinate transformation:

$$x^{\mu} \Rightarrow \hat{x}^{\mu} = x^{\mu} + \ell \Gamma^{\mu}(x) , \tag{1}$$

where $\Gamma^{\mu}(x)$ are arbitrary noncommutative functions of the points x^{μ}, and ℓ represents a value of the fundamental length.

The attractivity of the approach based on the hypothesis of quantum space-time (1) is that it gives rise to the appearance of the space-time torsion and to the existence of magnetic monopoles. The latter two facts may be understood as follows; whole space-time at a large scale, continued from microscale and obtained by an averaging procedure, where its quantum character appears, differs from our usual space-time structure where there is no place for magnetic monopoles described by some regular potential $\vec{A}(\vec{x})$, since $\mathrm{div\,rot}\vec{A} \equiv 0$ in it. On the contrary, as we have shown (Dineykhan and Namsrai, /2/) in space-time with torsion, $\mathrm{div\,rot}\vec{A} \neq 0$ for regular magnetic monopole potential, say $\vec{A} = -ig\vec{\sigma}/r$, g is the magnetic charge value.

We assume that this structural difference of whole space-time at a large scale continued from microscale must have an influential effect on particle behaviour, even at classical level. In this paper, we consider this problem and study particle dynamics for the nonrelativistic case. It turns out that, in our scheme, the Lagrangian function of classical free particle is determined by usual kinetic part and additional term connected with rotation degree of freedom, corresponding to an inner angular momentum (or sector velocity $\vec{v}_s = \frac{1}{2}[\vec{r}, \vec{v}]$) caused by the quantum structure of space. Due to the latter term, dynamics of the particle is changed and determined by a nonlinear differential equation. In the simple case of two dimensional space and of free motion, the derived equation of motion is integrated completely. The initial value problem of this equation is investigated. Depending on initial conditions, a particle's trajectory is complicated and the particle makes a spiral-like motion along the direction of the rectilinear classical trajectory.

However, it is generally difficult to solve an equation of motion in quantum space by analytic methods and numerical integration is needed. The resulting particle's trajectory is very tortuous and, it seems, it behaves like a strange attractor at least in the domain determined by the parameter ℓ. We know that the strange attractor is the direct image of "finite" turbulence, characterized by continuous spectrum over time variable and it is also the mathematical image of stochastic autooscillation. So, there is hope that our approach may be useful to understand the origin of twisting, stochastic and turbulent-like processes in physics. However, in order to shed light on this problem, further careful study is needed.

In Section 2 we obtain the Lagrangian function for free particles in quantum space and the Euler-Lagrange equation by using the action principle for the particle trajectory taking place in large scale - nonquantum space. The concrete form of the motion equation is obtained in Section 3. Sections 4 and 5 are devoted to study of the Cauchy problem for the obtained equation of motion in two dimensional space. Here integration of the motion equation is carried out explicitly and some interesting possible types of particle motion due to quantum space are also considered. In Section 6 we discuss obtained results and their specific peculiarity and perspective in order to generalize the given scheme to the relativistic and quantum mechanical cases.

2. The Lagrangian Function and the Action Principle for the Nonrelativistic Particle in the Quantum Space

In the nonrelativistic case, we suggest that quantum character of space-time is manifested only through spatial variables, i.e., coordinates of quantum space at small distances consist of two parts [like (1)]

$$x^i \Rightarrow \hat{x}^i = x^i + \ell\,\Gamma^i(x^j), \quad i,j = 1,2,3 \tag{2}$$

and time is usual continuous c-number variable here.

Further, we assume that all physical quantities characterizing a particle's state depend on quantum variables $\hat{x}^i$, $\dot{\hat{x}}^i$ and t, in particular, the Lagrangian function of the free particle is constructed by the correspondence principle, as in the classical mechanics

$$\mathcal{L}(\dot{\hat{x}}^i) = \frac{1}{2}\, m\, (\dot{\hat{x}}^i)^2, \tag{3}$$

where $\dot{\hat{x}}^i = d\hat{x}^i/dt$ is velocity-like vector in quantum space, and m is mass of the particle. It should be noted that real observable particle motion over time t takes place in nonquantum space x^i at large scale, continued from microscale, where its quantum property is manifested. Thus, in order to go over to a large scale we must carry out some averaging procedure over microscale (for details, see Namsrai /1,3/). In the concrete case, where functions $\Gamma^i(x^j)$ in (2) are given by matrix form, averaging procedure is reduced to taking traces of matrices, for example, if $\Gamma^i(x) \sim \sigma^i$, (σ^i are the Pauli matrices)

$$\langle \hat{x}^i \rangle \stackrel{Df}{=} \frac{1}{d}\, Sp\, \hat{x}^i = x^i, \tag{4}$$

where the parameter d arises from the normalization condition, in given case $d=2$, since σ^i are two column matrices.

Now choose the following matrix form

$$\Gamma^i(x^j) = \sigma^a\, e^i_a(x^j) \tag{5}$$

in (2) [where $e^i_a(x^j)$ are the tetrad fields ($a,i,j = 1,2,3$)] and study expression (3) in whole space at large distances. For this, first we define generalized velocity of the particle by the formula

$$d\dot{\hat{x}}^i/dt = dx^i/dt + \ell\,\sigma^a\left[de^i_a(x)/dx^j\right](dx^j/dt) \tag{6}$$

in accordance with (2) and (5). To calculate Lagrangian (3) over large scale, expression (6) should be squared and averaged. As result, we have

$$\mathcal{L}(x^i, \dot{x}^i) = \langle \mathcal{L}(\dot{\hat{x}}^i)\rangle = \frac{1}{d}\, Sp\, \mathcal{L}(\dot{\hat{x}}^i) = \frac{1}{2}\, Sp\, \mathcal{L}(\dot{\hat{x}}^i) =$$

$$= \frac{1}{2}\, m (\dot{x}^i)^2 + \frac{1}{2}\, m \ell^2 \left(\frac{de^i_a}{dx^j}\, \dot{x}^j\right)^2. \tag{7}$$

As in the case of classical mechanics with Lagrangian function (7) we can formulate the law of motion of mechanical systems by using the action principle (or Hamilton's principle) (see, for example, Landau and Lifschitz /4/).

Let at time moments $t = t_1$ and $t = t_2$ mechanical system (in the given case, mechanical material point) occupies definite positions characterized by two sets of coordinate values $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$.

Then the system moves between them in that way that the action integral

$$S = \int_{t_1}^{t_2} dt\, \mathcal{L}(x^i, \dot{x}^i)$$

assumes the smallest possible value, where $\mathcal{L}(x^i, \dot{x}^i)$ is given by formula (7). As in the usual classical mechanical case, for our scheme the action principle tells us

$$\delta S = \delta \int_{t_1}^{t_2} dt\, \mathcal{L}(x^i, \dot{x}^i) = 0$$

and carring out variation, we obtain the Euler-Lagrange equation of nonrelativistic particle

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}^i} - \frac{\partial \mathcal{L}}{\partial x^i} = 0\,, \quad i = 1,2,3\,. \tag{8}$$

3. Equation of Motion for Free Particle in Quantum Space

Now our aim is to obtain the equation of motion of free nonrelativistic particle from the Euler-Lagrange equation (8) with the function (7). For this, a concrete form of the tetrad field $e^i_a(x)$ should be defined. As in a previous case (Dineykhan and Namsrai/2/), we choose the spherical frame of reference as the tetrad coordinate system and the Cartesian one for the world coordinate system. Then, in the nonrelativistic case the tetrad field $e^i_a = e^i_a(\vec{x}(t))$ has the form

$$e^i_a = \partial \xi^i / \partial x^a\,, \qquad e^a_i = \partial x^a / \partial \xi^i\,,$$

where

$$d\xi^1 = dr\,, \qquad d\xi^2 = r\,d\theta\,, \qquad d\xi^3 = \rho\, d\varphi$$

$$dx^1 = dx\,, \qquad dx^2 = dy\,, \qquad dx^3 = dz$$

and $r = (x^2+y^2+z^2)^{1/2}$, $\rho = (x^2+y^2)^{1/2}$. One can easily see that the field e^a_i is given by

$$e^a_i = \begin{pmatrix} x/r & y/r & z/r \\ zx/r\rho & zy/r\rho & -\rho/r \\ -y/\rho & x/\rho & 0 \end{pmatrix}. \tag{9}$$

In this case, the square of generalized velocity $\langle(\dot{\hat{x}}^i)^2\rangle = \langle\dot{\hat{x}}^2\rangle + \langle\dot{\hat{y}}^2\rangle + \langle\dot{\hat{z}}^2\rangle$ takes the form

$$\langle\dot{\hat{x}}^2\rangle = \dot{x}^2 + \frac{\ell^2}{r^4}\left[(x\dot{y}-y\dot{x})^2 + (y\dot{z}-z\dot{y})^2 + (z\dot{x}-x\dot{z})^2\right],$$

$$\langle\dot{\hat{y}}^2\rangle = \dot{y}^2 + \frac{\ell^2}{r^4}\left[(z\dot{x}-x\dot{z})^2 + (y\dot{z}-z\dot{y})^2 + \frac{z^4}{\rho^4}(x\dot{y}-y\dot{x})^2\right],$$

$$\langle\dot{\hat{z}}^2\rangle = \dot{z}^2 + \frac{\ell^2}{\rho^4}(x\dot{y}-y\dot{x})^2.$$

Therefore the averaged Lagrangian function (7) acquires the following form

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) + \frac{m\ell^2}{r^4}\left[(x\dot{y}-y\dot{x})^2 + (y\dot{z}-z\dot{y})^2 + (z\dot{x}-x\dot{z})^2\right] + \frac{m\ell^2}{\rho^4}\cdot\frac{z^2}{r^2}(x\dot{y}-y\dot{x})^2. \tag{10}$$

Last two terms may be rewritten in the form

$$R_\ell = \frac{\ell^2}{mr^4}\vec{M}^2 + \frac{\ell^2}{m\rho^4}(z/r)^2 M_z^2\,,$$

where

$$\vec{M}^2 = M_x^2 + M_y^2 + M_z^2\,,$$

$$M_z = xp_y - yp_x\,, \quad M_y = zp_x - xp_z\,, \quad M_x = yp_z - zp_y\,, \quad \vec{p} = m\vec{v}.$$

Thus,

$$\mathcal{L} = T + R_\ell = \frac{1}{2}m\vec{v}^2 + \frac{\ell^2}{mr^4}\vec{M}^2 + \frac{\ell^2}{m\rho^4}(z/r)^2 M_z^2\,, \tag{11}$$

where vectors $\vec{v}$ and $\vec{M}\,(M_z)$ are particle velocity and angular momentum (its third component). We see that in our case the Lagrangian function of the free particle has two parts: usual kinetic one T and additional rotation term R_ℓ due to quantum nature of space at small distances. On the other hand, function (11) does not depend on time explicitly and therefore energy of the particle

$$E = \dot{x}^i \frac{\partial \mathcal{L}}{\partial \dot{x}^i} - \mathcal{L} = T + R_\ell$$

is conserved.

It is clear that, because of last two terms in (11), equation of motion for free particle is complicated in our case and takes the form

$$m\ddot{x}^i + m\ell^2 \dot{x}^n \dot{x}^m \left[\frac{\partial e_a^\kappa(x)}{\partial x^i}\frac{\partial^2 e_a^m(x)}{\partial x^n \partial x^\kappa} - \frac{\partial^2 e_a^\kappa(x)}{\partial x^n \partial x^i}\frac{\partial e_a^\kappa(x)}{\partial x^m}\right] = 0, \qquad (12)$$

where the tetrad field $e_a^\kappa(x)$ is given by (9) ($i, n, m, \kappa, a = 1,2,3$). This equation of motion is obtained from Euler-Lagrange equation (8) and we write it in components

$$\begin{aligned}
& m\ddot{x} + \frac{2m\ell^2}{r^4}\left[y(\ddot{x}y - \ddot{y}x)\left(1+\frac{r^2z^2}{\rho^4}\right) + z(\ddot{x}z - \ddot{z}x)\right] + \\
&+ \frac{2m\ell^2}{r^4}(\dot{x}y - \dot{y}x)\left\{\left[x(\dot{x}y-\dot{y}x) + 2(z\dot{z} - \vec{r}\cdot\dot{\vec{r}})y\right]\left(\frac{z^2}{\rho^4} + 2\frac{z^2r^2}{\rho^6}\right) + \right. \\
&\left. + \frac{2z}{\rho^2}\left(y\dot{z} + \dot{y}z\frac{r^2}{\rho^2}\right)\right\} + \frac{4m\ell^2}{r^6}\left\{z\dot{z}\left[x(\vec{r}\cdot\dot{\vec{r}}) - \dot{x}\vec{r}^2\right] + \right. \\
&\left. + (x\dot{x} + y\dot{y})\left[z(x\dot{z} - z\dot{x}) - y(\dot{x}y - \dot{y}x)\right]\right\} = 0,
\end{aligned}$$

$$\begin{aligned}
& m\ddot{y} - \frac{2m\ell^2}{r^4}\left[x(\ddot{x}y - \ddot{y}x)\left(1+\frac{r^2z^2}{\rho^4}\right) - z(\ddot{y}z - \ddot{z}y)\right] + \\
&+ \frac{2m\ell^2}{r^4}(\dot{x}y - \dot{y}x)\left\{\left[y(\dot{x}y-\dot{y}x) + 2x(z\dot{z} - \vec{r}\cdot\dot{\vec{r}})\right]\left(\frac{z^2}{\rho^4} + \frac{2z^2r^2}{\rho^6}\right) + \right. \\
&\left. - \frac{2z}{\rho^2}\left(x\dot{z} + \dot{x}z\frac{r^2}{\rho^2}\right)\right\} + \frac{4m\ell^2}{r^6}\left\{z\dot{z}\left[y(\vec{r}\cdot\dot{\vec{r}}) - \dot{y}\vec{r}^2\right] + \right. \\
&\left. + (x\dot{x} + y\dot{y})\left[z(y\dot{z} - z\dot{y}) - x(\dot{y}x - \dot{x}y)\right]\right\} = 0,
\end{aligned}$$

$$\begin{aligned}
& m\ddot{z} - \frac{2m\ell^2}{r^4}\left[x(\ddot{x}z - \ddot{z}x) + y(\ddot{y}z - \ddot{z}y)\right] - \frac{2m\ell^2}{r^4}\frac{z}{\rho^2}(\dot{x}y - \dot{y}x)^2 + \\
&+ \frac{4m\ell^2}{r^6}\left\{(x\dot{x} + y\dot{y})\left[z(\vec{r}\cdot\dot{\vec{r}}) - \dot{z}\vec{r}^2\right] - z\dot{z}^2\rho^2\right\} = 0. \qquad (13)
\end{aligned}$$

We see that these equations of motion are too complicated in nature to be solved by analytic methods and will require numerical investigations. However, there is a concrete case for which the equation of motion in quantum space is integrated completely. That is the situation when along one of directions of coordinate system, particle moves rectilinearly and its motion becomes complicated along other two directions due to quantum structure of space, i.e., it is equivalent to two dimensional case. To prove this, we choose the cylindrical frame of reference as the tetrad coordinate system. In this case, instead of (9), we have

$$e_i^a = \begin{pmatrix} x/\rho & y/\rho & 0 \\ -y/\rho & x/\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By using this tetrad field, Lagrangian (11) and equation of motion (13) can be easily calculated to be of the form

$$\mathcal{L} = \frac{1}{2}m\vec{v}^2 + \frac{\ell^2}{m\rho^4}M_z^2,$$

and

$$m\ddot{z} = 0$$

$$\left.\begin{aligned}
& m\ddot{x} + \frac{2m\ell^2}{\rho^4}y(\ddot{x}y - \ddot{y}x) - \frac{4m\ell^2}{\rho^6}y(\dot{x}y - \dot{y}x)(x\dot{x} + y\dot{y}) = 0 \\
& m\ddot{y} - \frac{2m\ell^2}{\rho^4}x(\ddot{x}y - \ddot{y}x) + \frac{4m\ell^2}{\rho^6}x(\dot{x}y - \dot{y}x)(x\dot{x} + y\dot{y}) = 0
\end{aligned}\right\}. \qquad (14)$$

From these equations, we see that particle moves rectilinearly along the z-axis and its motion along x - and y -axis is complicated and twisted in accordance with (14). We now go to study a particle trajectory determined by equations (14).

4. Integration of the Motion Equation in Two Dimensional Case

It is convenient to study the equation of motion (14) in polar system coordinate (ρ, φ) in which the Lagrangian function has the simple form

$$\mathcal{L}=\frac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\varphi}^2+\dot{z}^2)+m\ell^2\dot{\varphi}^2. \tag{15}$$

This function does not contain coordinate φ explicitly. Any generalized coordinate q_i not entering explicitly into the Lagrangian function is called cyclic. Due to the Euler-Lagrange equation for such coordinate we have

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}=\frac{\partial\mathcal{L}}{\partial q_i}=0,$$

i.e., the corresponding generalized momentum $P_i=\partial\mathcal{L}/\partial\dot{q}_i$ is the integral of motion. This situation leads to essential simplification of the integration problem for the equation of motion in the presence of cyclic coordinates.

In a given case, the generalized momentum is

$$P_\varphi=(mr^2+2m\ell^2)\dot{\varphi}.$$

The first term in it coincides with the angular momentum $M_z=m(x\dot{y}-y\dot{x})=mr^2\dot{\varphi}$. Thus, in our scheme the general angular momentum of the type

$$M=M_z+M_\ell=(mr^2+2m\ell^2)\dot{\varphi}=const. \tag{16}$$

is conserved.

The equation of the motion obtained by using the Lagrangian function (15) takes the following form

$$\left.\begin{aligned} &m\ddot{z}=0, \\ &m\ddot{\rho}-m\rho\dot{\varphi}^2=0 \\ &m(\rho^2+2\ell^2)\ddot{\varphi}+2m\rho\dot{\rho}\dot{\varphi}=0 \end{aligned}\right\}. \tag{17}$$

From the second equation in (17), we have

$$\ddot{\varphi}/\dot{\varphi}=-2\rho\dot{\rho}/(\rho^2+2\ell^2)$$

or

$$\frac{\partial}{\partial t}\ln\dot{\varphi}=-\frac{\partial}{\partial t}\ln(\rho^2+2\ell^2).$$

Direct integration of the last equation gives

$$\dot{\varphi}=C_1/(\rho^2+2\ell^2), \tag{18}$$

where integration constant C_1 is determined by initial conditions

$$\rho(t)\big|_{t=0}=\rho_0=a\ell, \quad \varphi(t)\big|_{t=0}=0, \quad \dot{\varphi}\big|_{t=0}=\omega_0. \tag{19}$$

From which $C_1=(a^2+2)\omega_0\ell^2$, and expression (18) acquires the form

$$\dot{\varphi}=(2+a^2)\omega_0\ell^2/(\rho^2+2\ell^2). \tag{20}$$

It should be noted that in the usual case, when $\ell=0$, we obtain rectilinear trajectory given by a ray $\varphi=\varphi_0=$ const. and $\rho(t)=\rho_0+v_0t$ along which classical particles move. Here parameters ρ_0 and v_0 are given by initial conditions (19) and

$$\partial\rho/\partial t\big|_{t=0}=v_0. \tag{21}$$

Further, substituting expression (20) into first equation in (17), we get

$$\ddot{\rho}-\rho(2+a^2)^2\omega_0^2\ell^4/(\rho^2+2\ell^2)^2=0.$$

To integrate this equation, put $P(\rho)=\dot{\rho}$ and write new equation for $P(\rho)$:

$$P(\rho)\frac{\partial P(\rho)}{\partial\rho}=\rho(2+a^2)^2\omega_0^2\ell^4(\rho^2+2\ell^2)^{-2}.$$

Simple integration of it gives

$$\dot{\rho}^2=-(2+a^2)^2\omega_0^2\ell^4(\rho^2+2\ell^2)^{-1}+C_2, \tag{22}$$

where integration constant

$$C_2=\frac{1}{2}v_0^2+\frac{1}{2}(2+a^2)\omega_0^2\ell^2 \tag{23}$$

arises from initial conditions (19) and (21). After separating integration variables equation (22) with (23) may be rewritten in the form

$$\pm\frac{d\rho}{\sqrt{v_0^2+(2+a^2)\omega_0^2\ell^2-\dfrac{(2+a^2)^2\omega_0^2\ell^4}{\rho^2+2\ell^2}}}=dt. \tag{24}$$

We notice that plus and minus signs in (24) are not important and we choose a plus sign and integrate this equation. For this, we put

$$\rho = \sqrt{2}\,\ell\, ctg x, \quad d\rho = -\sqrt{2}\,\ell\, \sin^{-2}x\, dx, \quad t = arcctg(\rho/\sqrt{2}\,\ell).$$

As a result of this change of varieble and integratinn, we have

$$\frac{-\sqrt{2}\,\ell}{\sqrt{v_0^2 + (2+a^2)\omega_0^2 \ell^2}} \int \frac{dx\, \sin^{-2}x}{\sqrt{1-\kappa^2 \sin^2 x}} = t + C_3 \, . \tag{25}$$

where

$$\kappa = (2+a^2)\,\omega_0 \ell\, 2^{-1/2} \left[v_0^2 + (2+a^2)\omega_0^2 \ell^2\right]^{-1/2} . \tag{26}$$

It is easily verified that integral (25) is reduced to normal elliptic integrals of the first

$$F(\varphi, \kappa) = \int_0^{\varphi} \frac{dx}{\sqrt{1-\kappa^2 \sin^2 x}}$$

and second

$$E(\varphi, \kappa) = \int_0^{\varphi} dx\, \sqrt{1-\kappa^2 \sin^2 x}$$

kind, respectively.

Thus, integral (25) results in

$$\frac{\sqrt{2}\,\ell}{\sqrt{v_0^2 + (2+a^2)\omega_0^2 \ell^2}} \left\{ E\left(arcctg \frac{\rho}{\sqrt{2}\,\ell}, \kappa\right) - F\left(arcctg \frac{\rho}{\sqrt{2}\,\ell}, \kappa\right) + \right.$$

$$\left. + (\rho/\sqrt{2}\,\ell)\left(1 - \kappa^2 \frac{2\ell^2}{\rho^2 + 2\ell^2}\right)^{1/2} \right\} = t + C_3 \, , \tag{27}$$

where integration constant may be calculated by using (19), that is

$$C_3 = \frac{\sqrt{2}\,\ell}{\sqrt{v_0^2 + (2+a^2)\omega_0^2 \ell^2}} \left\{ E\left(arcctg \frac{a}{\sqrt{2}}, \kappa\right) - F\left(arcctg \frac{a}{\sqrt{2}}, \kappa\right) + \right.$$

$$\left. + (a/\sqrt{2})\left(1 - \kappa^2 \frac{2}{2+a^2}\right)^{1/2} \right\} .$$

Furthermore, rewriting (20) in the form

$$d\varphi = \omega_0 (2+a^2)\,\ell^2 \left(\rho^2 + 2\ell^2\right)^{-1} \cdot dt$$

and substituting dt from (24) with a plus sign, and integrating, we get

$$\varphi = \sqrt{2}\,\ell \cdot \kappa \int \frac{d\rho}{\sqrt{\rho^2 + 2\ell^2}\, \sqrt{\rho^2 + 2\ell^2(1-\kappa^2)}} + \varphi_0 \, ,$$

where parameter κ is given by (26). Making use of the change of the integration variable

$$\rho = \sqrt{2}\,\ell\, tg x, \quad d\rho = \sqrt{2}\,\ell\, \cos^{-2}x \cdot dx, \quad x = arctg(\rho/\sqrt{2}\,\ell)$$

and integrating resulting integral, we have finally

$$\varphi = \kappa\, F\left(\arcsin \frac{\rho}{\sqrt{\rho^2 + 2\ell^2(1-\kappa^2)}}, \kappa\right) + \varphi_0 \, . \tag{28}$$

Formulas (27) and (28) solve, in general, the stated problem. The second of them determines connectinn between ρ and φ, i.e., the equation of trajectory. Formula (27) defines, in inexplicit form, distance ρ of the moving point from the centre as a function of time. Notice that the angle φ is always changed over time in monotone form - from (16) it is seen that $\dot{\varphi}$ does not change sign.

5. The Type of Particle's Motion in Quantum Space

As in Newtonian mechanics, particle's trajectory given by (28) takes different forms depending on initial conditions (19) and (21), i.e., on parameter κ (26). We distinguish several possibilities of interest. From (28), it is easily seen that the connection between quantities φ and ρ has a definite physical meaning if $\rho\left[\rho^2 + 2\ell^2(1-\kappa^2)\right]^{-1/2} \leq 1$. This inequality imposes on the parameter κ the following restriction $\kappa \leq 1$. Thus, physical condition of the problem gives $0 \leq \kappa \leq 1$.

1. First, we consider the case, when $\kappa = 0$. Before discussion of this limit, we indicate one essential moment concerning the value ω_0. We assume that ω_0 should depend on ℓ, and may there exists some link between them. In other words, condition $\ell = 0$ gives $\omega_0 = 0$. On the contrary, if $\omega_0 \neq 0$ even at $\ell = 0$, i.e., for the usual classical case, then from (18) and initial condition $\rho(t)\big|_{t=0} = \rho_0$ it follows that

$$\dot{\varphi} = \omega_0(\rho_0^2+2\ell^2)(\rho^2+2\ell^2)^{-1}\Big|_{\ell\to 0} = (\rho_0/\rho)^2\omega_0$$

and it, in turn, involves complication of particle's trajectory analogous to the one obtained above.

Such situation is completely ruled out by classical mechanical principle. We present here simple connection between ω_0 and ℓ,

$$\omega_0 = \ell v_0 \lambda^{-2} \qquad \text{or even} \qquad \omega_0 = \ell^2 v_0 \lambda^{-3}, \tag{29}$$

where λ is some typical length which may be identified with the dimension of atom $\lambda = a \sim 10^{-8}$ cm and the Planck length $\lambda = \ell_{Pl} = (\hbar G/c^3)^{1/2} = 10^{-33}\,cm$ in classical and quantum physics, respectively. In the last case, the parameter ℓ is determined as a multiple of the Planck length, $\ell = n\,\ell_{Pl}$, $n = 1, 2, 3, \ldots$

Thus, by definition (26) and assumption (29), equality $\kappa = 0$ is achieved at $\ell = 0$ even for $\rho_0 \neq 0$. Notice that $a = \rho_0/\ell$ in (26) in accordance with initial condition (19). In the case $\kappa = 0$, from (28) it immediately follows that $\varphi = \mathrm{const} = \varphi_0$. At the same time, the equation (27) takes the form

$$\rho(t) = \rho_0 + v_0 t. \tag{30}$$

Since $E(\varphi, 0) = F(\varphi, 0) = \varphi$ and

$$\sqrt{2}\,\ell\left(v_0^2 + (2+a^2)\omega_0^2\ell^2\right)^{-1/2}\left(\rho(t)/\sqrt{2}\,\ell\right)\Big|_{\ell\to 0} = \rho(t)/v_0\,,$$

$$C_3\Big|_{\ell\to 0} = \lim_{\ell\to 0}\sqrt{2}\,\ell\left(v_0^2+(2+a^2)\omega_0^2\ell^2\right)^{-1/2}\left\{E\left(\mathrm{arcctg}\frac{\rho_0}{\sqrt{2}\,\ell},\kappa\right) - F\left(\mathrm{arcctg}\frac{\rho_0}{\sqrt{2}\,\ell},\kappa\right) + \left(\rho_0/\sqrt{2}\,\ell\right)\left[1-\kappa^2\frac{2\ell^2}{\rho_0^2+2\ell^2}\right]^{1/2}\right\} = \rho_0/v_0 .$$

So, we see that the case $\kappa = 0$ is just the classical situation where particle moves along a rectilinear trajectory given by a ray $\varphi = \varphi_0$ (Fig. 1a).

2. For the case $\kappa \ll 1$ in order to expose the general pattern of a particle's trajectory one can use approximate integration of the motion equation (17). Instead of (27) and (28), we have the following approximate equations

$$t = \frac{\sqrt{2}\,\kappa}{\omega_0\ell(2+a^2)}\left[\rho - \rho_0 + \frac{\ell\kappa^2}{\sqrt{2}}\,\mathrm{arctg}\left(\frac{\sqrt{2}\,\ell(\rho-\rho_0)}{2\ell^2+\rho\rho_0}\right)\right] \tag{31}$$

and

$$\varphi = \kappa\,\mathrm{arctg}\left(\rho/\sqrt{2}\,\ell\right) + \varphi_0$$

or

$$\rho(\varphi) = \sqrt{2}\,\ell\,\mathrm{tg}\left(\frac{\varphi-\varphi_0}{\kappa}\right). \tag{32}$$

Thus, we see that in given limit $\kappa \ll 1$ the type of the particle motinn differs slightly from rectilinear as in the classical mechanical case.

3. We observe that when value of κ is increased deviation of the particle's trajectory from rectilinear becomes more appreciable and it begins to whirl. For example, we show form of particle's trajectory (sketched in Fig. 1b-d) for $a = 30°$, $a = 88°$, $a = 89.99°$, where $\kappa = \sin a$. We see that for $a = 30°$, $88°$ and $89.99°$ the maximum value of the angle φ is achieved at $\varphi = 48°$, $\varphi = 271°$ and $\varphi \sim 540°$, respectively, which in turn corresponds to a quarter, a three quarter and more than one full turn, approximately.

4. In our opinion, from the physical point of view, a very interesting case is the limit $\kappa = 1$ or $a = 90°$. In this limiting case, number of twisted loops (or orbits) becomes infinite (Fig. 1e) and the particle is subject to rotation motion for any time. Moreover, type of this rotation motion does not depend on the value of ρ, i.e., for any distance ρ from centre particle moves along spiral like trajectory. However, spiral like behaviour of the particle takes place in the domain characterized by the parameter ℓ of the theory. In other words, amplitude ℓ of this twisted trajectory determines maximum deviation from rectilinear trajectory.

Finally, to present general pattern of the particle motion over time, we illustrate in the right-hand side of Fig. 1 possible type of particle trajectory corresponding to left parts of Fig. 1.

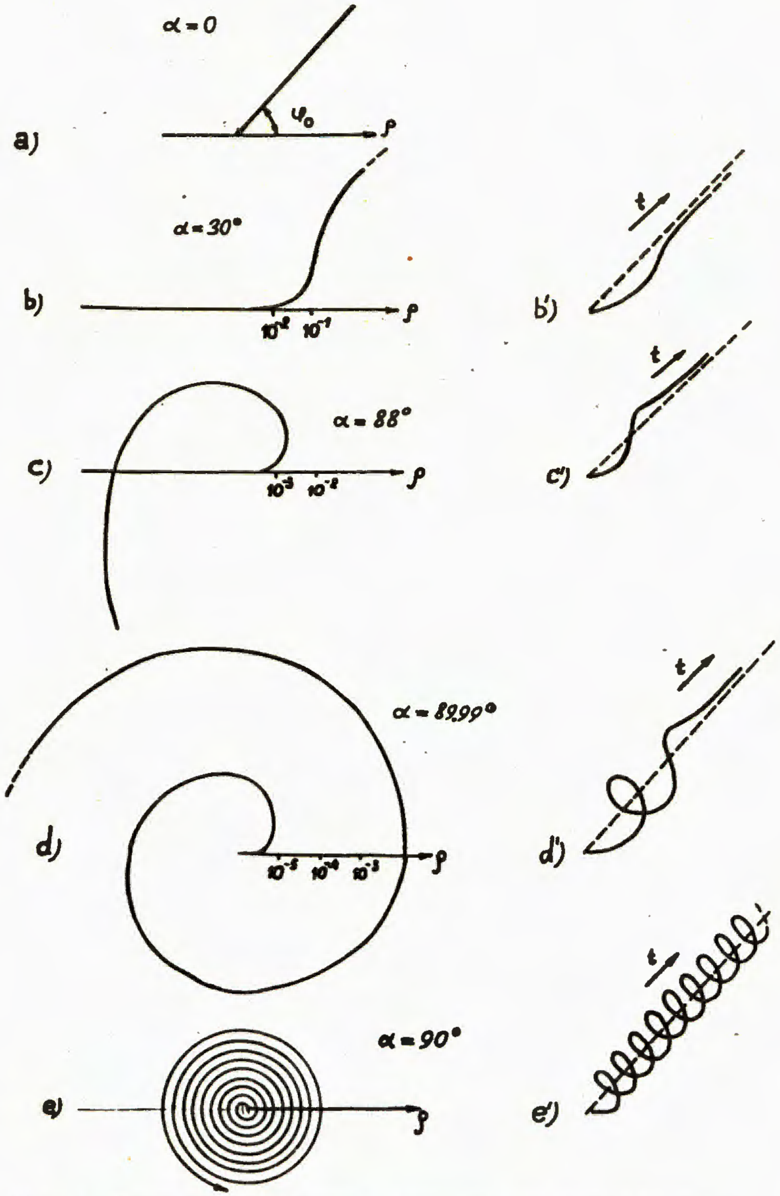

Fig. 1. There exist different types of the particle motion depending on initial conditions of the problem in quantum space.

6. Discussion of the Obtained Results

Thus, as shown above, in quantum space the Lagrangian function or energy of the particle is determined by two terms $E = T + R_\ell$, where

$$T = \frac{1}{2} m \vec{v}^{\,2}, \qquad R_\ell = \frac{\ell^2}{m r^4} \vec{M}^2 + \frac{\ell^2}{m \rho^4} (z/r)^2 M_z^2 .$$

In spherical system coordinate, they take the form

$$T = \frac{1}{2} m \vec{v}^{\,2} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\varphi}^2 \right),$$

$$R_\ell = m \ell^2 \left(\dot{\theta}^2 + \dot{\varphi}^2 \right)$$

or in two-dimensional case (in polar coordinates)

$$T = \frac{1}{2} m \left(\dot{\rho}^2 + \rho^2 \dot{\varphi}^2 \right), \qquad R_\ell^{(2)} = m \ell^2 \dot{\varphi}^2 .$$

Terms R_ℓ and $R_\ell^{(2)}$ we call torsion torque of the particle which appear due to quantum nature of space at small distances. In the presence of torsion torque the particle's trajectory is twisted and the particle moves along spiral-like lines.

Our hope is that if quantum nature of space does indeed exist at small distances, then its discovery may be made by the study of a particle's trajectory. In terms of the suitable choice of initial conditions, one can obtain the case of $\kappa = 1$ and at which the particle undergoes rotation motion at any time moment. It should be noted that it is quite possible to observe the physical effects caused by this twisted motion of the particle, especially for the motion of charged relativistic particle in the external electromagnetic field. We suggest that the influential effect due to quantum structure of space-time on the particle behaviour is crucial in the relativistic case. This problem requires separate investigation and is the subject of our future work.

The fact is that due to quantum structure of space there exists an effect of deviation of the particle trajectory from rectilinear at classical level, one can introduce the fundamental assumption that in quantum space micro-particle's positions are not definite and the particle cannot hit the definite place in space (of course for generalised complex case different from considered above). It occupies at least some domain characterised by the parameter λ or ℓ. We now find amplitude of this deviation from that point at which the particle would arrive at exactly, if space would possess nonquantum character. Let the particle moves with constant velocity v_0 along z-axis. If space is nonquantum, then after the time moment $t_0 = z_0/v_0$ the particle hits the target (slot) at the point $z = z_0$ exactly (Fig. 2a). Here there is no deviation along the x- and y-axis. However, in accordance with the assumption that space possesses quantum structure, the particle makes a deviation from the initial position and gets into the circle of the radius ρ determined by equation (27), there it should be put $t = t_0$ and $v_0 = 0$, since we have originally suggested that there was no motion along x- and y-axis. In this sense, deviation is a pure quantum-space effect (see Fig. 2b).

The amplitude of deviation we are interested in, is given by (27). In which we put $t = t_0$, $v_0 = 0$ and $\rho \gg \ell$, since in the classical mechanical case, influential effect due to quantum space may be observable if deviation is much larger than the value of ℓ.

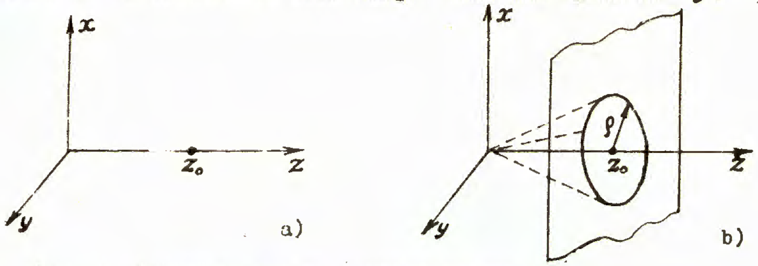

Fig. 2. Illustration of particle's positions according to the assumption of nonquantum (a)and quantum (b) space in which particle moves.

It should be noted that from high-energy experiments it follows that $\ell \lesssim 10^{-16}$ cm (see, for example, Namsrai /1,3/). Thus, assuming $\rho \gg \ell$ and $\rho_0 = a\ell = 0$ in (27), we have

$$t = (\sqrt{2}\ell/\sqrt{2}\omega_0\ell)\left[E(0,\tfrac{1}{2}) - F(0,\tfrac{1}{2}) + \rho/\sqrt{2}\ell\right] \approx \rho/\sqrt{2}\omega_0\ell$$

and

$$\rho(t_0) = \sqrt{2}\,\omega_0 t_0 \ell .$$

Let idealized classical object - small bullet with initial velocity $v_0 = 1000$m/sec = 10^5cm/sec - move along z-axis and hit the target after t_0=100 sec. Now the following question arises; How far does its rectilinear trajectory deviate after this time moment t_0=100 sec? According to (29) we get

$$\omega_0 = \ell\lambda^{-2} v_0 = 10^5/\text{sec} = 0{,}1\ \text{MHz}$$

and, therefore

$$\rho(t_0) = 1.4 \times 10^{-9}\ \text{cm}.$$

Thus, this value is completely negligible from the classical experimental point of view.

In conclusion, we notice that in the microworld where physical processes take place at small distances the effect analogous to the one discussed above should play an important role,and due to torsion torque,trajectories of microparticles become very tortuous.Moreover,it is quite possible that the essence of observable quantum process may be understood as Brownian-type stochastic motion taking place in quantum space-time at small distances. Thus, this way may be an open door for stochastic foundation of quantum mechanics (see, for example, Prugovečki /5/, Namsrai /3/) originally started by A.Einstein and L.de Broglie seeking to describe quantum processes by means of sub-quantum deterministic motions.

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