сообщения
объединенного
института
ядерных
исследований
дубна

E2-85-899

R.Gielerak

THE VAN HOVE THEOREM FOR THE TWO-DIMENSIONAL SINE-GORDON MODEL

1985

1. INTRODUCTION

Let $S'(R^2)$ be the space of tempered distributions over two-dimensional space R^2 and let $\mathcal{B}$ be the Borel σ-algebra of $S'(R^2)$. By μ_0 let us denote a free field Gaussian measure. Any measure μ_{s-G} on $\{S'(R^2), \mathcal{B}\}$ is called tempered sine-Gordon measure iff

i) μ_{s-G} is locally absolutely continuous with respect to the measure μ_0 ;

ii) For any localized in the bounded region $\Lambda \subset R^2$ observable $F(\phi)$ the conditional expectation values with respect to the measure μ_{s-G} and the σ-algebra $\Sigma(\Lambda^c)$ are given by the following formula:

$$E_{\mu_{s-G}}\{F(\phi) \mid \Sigma(\Lambda^c)\} = E_{\mu_\Lambda}\{F(\phi) \mid \Sigma(\Lambda^c)\},$$

where

$$\mu_\Lambda(d\phi) = (Z^\circ_\Lambda(z))^{-1} \exp(z \int_\Lambda :\cos\epsilon\phi:(x)\, \mu_0(d\phi),$$

$$Z^\circ_\Lambda(z) = \int \exp(z \int_\Lambda :\cos\epsilon\phi:(x)\, dx)\, \mu_0(d\phi).$$

Actually it is known that $Z^\circ_\Lambda(z)$ is finite for every $z \in R^1$ and $|\epsilon| < 2\sqrt{\pi}$ [1,2]. The set of tempered sine-Gordon measures with fixed z and ϵ we denote by $\mathcal{G}^T(z)$. For a given sine-Gordon measure μ_{s-G} we say that it is regular iff there exists a constant $c \in R_+$ such that

$$\forall_{f \in H_{-1}(R^2)} \int \phi^2(f)\, \mu_{s-G}(d\phi) \le c \|f\|^2_{-1}$$

and μ_{s-G} is completely regular iff there exists a constant C such that

$$\forall_{f \in H_{-1}(R^2)} : \int e^{\phi(f)}\, \mu_{s-G}(d\phi) \le e^{C\|f\|^2_{-1}}.$$

The set of tempered, regular (respectively completely regular) sine-Gordon measures we denote by $\mathcal{G}^t_r(z)$ (resp. $\mathcal{G}^t_{cr}(z)$).

It can be shown [3,4] that for $\mu_{s-G} \in \mathcal{G}^t(z)$ the following formulas for the conditional expectation values hold:

$$E_{\mu_\Lambda}\{F(\phi)|\Sigma(\Lambda^c)\}(\eta)$$

$$\overset{\mu}{\underset{\eta}{\forall}} : \; = (Z^\eta_\Lambda(z))^{-1} \int \mu_0^{\partial\Lambda}(d\phi) F(\phi + \Psi_\eta^{\partial\Lambda})\, e^{z\int_\Lambda :\cos\epsilon(\phi+\Psi_\eta^{\partial\Lambda}):dx} .$$

$$Z^\eta_\Lambda(z) = \int \mu_0^{\partial\Lambda}(d\phi)\, e^{z\int_\Lambda :\cos\epsilon(\phi+\Psi^{\partial\Lambda}):(x)\,dx}$$

Here $\mu_0^{\partial\Lambda}$ means the free field measure with the Dirichlet boundary condition on $\partial\Lambda$ and $\Psi_\eta^{\partial\Lambda}$ is the (unique) solution of the following Dirichlet problem

$$\begin{cases} (-\Delta+1)\,\Psi_\eta^{\partial\Lambda}(x) = 0\,; & x \in \operatorname{Int}\Lambda \\ \Psi_\eta^{\partial\Lambda}(x) = \eta(x)\,; & x \in \partial\Lambda \end{cases}$$

The quantity (assumed to exist)

$$p^\eta_\infty(z) \equiv -\lim_{\Lambda\uparrow R^2} \frac{1}{|\Lambda|} \ln Z^\eta_\Lambda(z) \equiv \lim_{\Lambda\uparrow R^2} \lim p^\eta_\Lambda(z)$$

(where the meaning of the symbol $\lim\limits_{\Lambda\uparrow R^2}$ will be specified below) is called the infinite volume vacuum energy density conditioned by η.

Our result is:

Theorem.

Let $\mu_{s-G} \in \mathcal{G}^t_r(z)$. Then for μ a.e. $\eta \in S'(R^2)$ and $|\epsilon| < \dfrac{2}{1-1/2\pi}$,

$p^\eta_\infty(z)$ exists and is equal to $p^0_\infty(z) = p^{\eta=0}_\infty(z) = p^{HD}_\infty(z)$.

2. INFINITE VOLUME PRESSURE-INDEPENDENCE OF THE BOUNDURY CONDITION

In this note we will concentrate on the proof that the infinite volume pressure does not depend on the typical boundary condition "$\eta \in \operatorname{supp}\mu$" whenever $\mu \in \mathcal{G}^t_r(z)$.

Several results on the independence of the infinite volume preasure for the so-called classical boundary conditions have been obtained /6/. However, it seems to us very likely that the class of classical boundary conditions is not of the measure one

from the point of the Gibbsian approach to this problem. Therefore, all the results /6/ are incomplete for the present applications.

2.1. Shape Independence

Let $\mu \in \mathcal{G}^t_r(z)$. Then, we define the finite volume pressures:

$$p^\circ_\Lambda(z) = -\frac{1}{|\Lambda|} \ln Z^\circ_\Lambda(z)\,, \tag{2.1}$$

$$p^\eta_\Lambda(z) = -\frac{1}{|\Lambda|} \ln Z^\eta_\Lambda(z) \tag{2.2}$$

for "$\eta \in \operatorname{supp}\mu$". The corresponding infinite volume limits will be denoted by $p^\circ_\infty(z)$ and $p^\eta_\infty(z)$ respectively.

Lemma 2.1. (Shape independence).

Let $\mu \in \mathcal{G}^t_r(z)$. Whenever $\Lambda\uparrow R^2$ in the sense of Van Hove and such that $\partial\Lambda$ are piecewise $-C^1$, then:

$$\overset{\mu}{\underset{\eta}{\forall}} \quad \lim_{\Lambda\uparrow R^2} p^\eta_\Lambda(z) = p^\eta_\infty(z) \tag{2.3}$$

exists and does not depend on the chosen sequence $\Lambda\uparrow R^2$ as above.

Proof:

Let us rewrite Z^η_Λ in the following way:

$$Z^\eta_\Lambda(z) = \int \mu_0^{\partial\Lambda}(d\phi) \exp\Big(z\int_\Lambda :c((\phi):_1(x)\,d^2x\Big) \times$$

$$\times \exp\Big(z\int_\Lambda :c(\phi):_1(x)\,(c(\Psi_\eta^{\partial\Lambda})(x) - 1)\,d^2x\Big) \times \tag{2.4}$$

$$\times \exp\Big(z\int_\Lambda :s(\phi):_1(x)\,s)\Psi_\eta^{\partial\Lambda})(x)\,d^2x\,.$$

where $:\;:_1$ means the normal ordering with respect to the covariance $(-\Delta+1)^{-1}$, i.e.

$$:c(\phi)_1:(x) \;\text{"="}\; \exp\Big(\frac{\epsilon^2}{2}S(0)\Big)\cos\epsilon\phi(x)\,,$$
$$\tag{2.5}$$
$$:s(\phi)_1:(x) \;\text{"="}\; \exp\Big(\frac{\epsilon^2}{2}S(0)\Big)\sin\epsilon\phi(x)\,.$$

Using the L_2-estimate following from the (proof of) Theorem 3.4. in /1/ for half-Dirichlet state (at this point one can use conditioning inequalities), we have that there exist constants

c_1, c_2 independent of η such that

$$Z^{\eta}_{\Lambda}(z) \leq c_1 \exp c_2 z^2 |\Lambda| . \qquad (2.6)$$

Above we also used the following trivial bounds: $|c(\Psi^{\partial\Lambda}_{\eta})| \leq 1$ and $|c(\Psi^{\partial\Lambda}_{\eta})(x)| \leq 1$ poinwise on S'. Recall that here the corresponding quantities are not Wick-ordered.

From the estimate (2.6) it follows that whenever $\Lambda_n \uparrow R^2$ in some well prescribed sence there exists a subsequence $(n') \subset (n)$ such that $p^{\eta}_{\Lambda_{n'}}(z)$ is then convergent.

In the case when $\{\Lambda_n\}$ is such that $\partial\Lambda_n$ are C^1-piecewise and there exists $\epsilon > 0$ such that

$$\lim_{n\to\infty} \frac{|\Lambda_n|}{|\partial\Lambda_n|^{1+\epsilon}} = 0 \qquad (2.7)$$

it follows (see § 2.4) that every accumulation point of the sequence $\{p^{\eta}_{\Lambda_n}(z)\}$ is equal to $p^{\circ}_{\infty}(z)$ and this proves the claimed convergence and shape independence.

q.e.d.

2.2. Estimates on $\Psi^{\partial\Lambda}_{\eta}$

Several local decay properties of the solutions of the stochastic Dirichlet problem (1.3) have been proved in the basic paper[3]. However, the results obtained in [3] are not sufficient for our purposes. As we will show below, some a'priori bounds are needed for the estimation of the quantities like

$$\int_{\Delta} |\Psi^{\partial\Lambda}_{\eta}|^p(x)\, dx , \qquad (2.8)$$

where Δ is a typically unit cube in R^2 and $p \geq 1$.

Such estimates follow easily from the application of the Tchebyshev inequality.

Let us denote

$$K^{\partial\Lambda}(x,y) \equiv (-\Delta+1)^{-1}(x,y) - (-\Delta^{\partial\Lambda}+1)^{-1}(x,y) . \qquad (2.9)$$

It is well known that $K^{\partial\Lambda}(x,x)$ is a smooth function for $x \notin \partial\Lambda$ and has exponential decay as $\mathrm{dist}(x,\partial\Lambda) \uparrow \infty$. Moreover, as $x \to \partial\Lambda$, than $K^{\partial\Lambda}(x,x)$ behaves like $+\frac{1}{2\pi}\ln|\mathrm{dist}(x,\partial\Lambda)|$ (see, i.e. [3]).

Lemma 2.2.

Let $\mu \in \mathcal{G}^t_r(z)$. Then for any unit cube $\Delta \subset R^2$ and any bounded $\Lambda \subset R^2$ with C^1-piecewise boundary, there exists a constant $C_3(\eta,\Lambda)$ finite for μ a.e. η such that for all $\beta<1$ the following estimate holds:

$$\int_{\Delta} (\Psi^{\partial\Lambda}_{\eta})^2(x)\, dx \leq c_3(\eta,\Lambda) [\int_{\Delta} K^{\partial\Lambda}(x,x)]^{\beta} dx . \qquad (2.10)$$

Proof:

Let $c(R^2) = \cup \Delta_j$ be the partitioning of R^2 onto unit cubes such that $\Delta \in c(R^2)^j$. Take $\delta > 0$ arbitrary and fixed. For $\mu \in \mathcal{G}^t_r(z)$ we have:

$$\mu\{\eta \in S'(R^2) | \underset{j}{J} \int_{\Delta_j} (\Psi^{\partial\Lambda}_{\eta})^2(x)\,dx \geq \frac{1}{\delta}(\int_{\Delta_j} K^{\partial\Lambda}(x,x)\,dx)^{\beta}\} \leq$$

$$\leq \sum_j \mu\{\eta \in S'(R^2) | \int_{\Delta_j} (\Psi^{\partial\Lambda}_{\eta})^2(x)\,dx \geq \frac{1}{\delta}[\int_{\Delta_j} K^{\partial\Lambda}(x,x)\,dx]^{\beta}\} \leq$$

(by the application of the Tchebyshev inequality)

$$\leq \delta \sum_j [\int_{\Delta_j} K^{\partial\Lambda}(x,x)]^{-\beta} \int d\mu(\eta) (\int_{\Delta_j} (\Psi^{\partial\Lambda}_{\eta})^2(x)\,dx \leq$$

$$\leq \delta \cdot \mathrm{const}(\sum_j (\int_{\Delta_j} K^{\partial\Lambda}(x,x)\,dx)^{1-\beta}).$$

Whenever $\beta<1$ the sum $\sum_j$ is finite due to the exponential decay of $K^{\partial\Lambda}$. Since δ is arbitrary, the proof follows.

q.e.d.

For the case of completely regular Gibbs measure one can generalize this Lemma:

Lemma 2.3.

Let $\mu \in \mathcal{G}^t_{cr}(z)$. Then for any unit cube $\Delta \subset R^2$ and any bounded $\Lambda \subset R^2$ with a C^1-piecewise boundary there exists a constant $c_4(\eta,\Lambda)$ finite for μ-almost every η and such that for all $\beta < n/2$ the following estimate holds:

$$\int_{\Delta} |\Psi^{\partial\Lambda}_{\eta}|^n(x)\,dx \leq c_4(\eta,\Lambda)\{\int_{\Delta} K^{\partial\Lambda}(x,x)\,dx]^{\beta} . \qquad (2.11)$$

Proof:

The main argument is again the Tchebyshev inequality applied as in the proof of Lemma 2.2. The additional argument comes from the assumed complete regularity of μ. From $\mu \in \mathcal{G}^t_{cr}(z)$ there follows from the Cauchy integral formula the following estimate:

$$|\int \mu(d\eta) \prod_{i=1}^{n} \eta(f_i)| \leq (n!)^{1/2} \cdot \mathrm{const} \cdot \prod_{i=1}^{n} ||f_i||_{-1} . \qquad (2.12)$$

q.e.d.

The following estimates of the quantities like $\int_\Delta (\nabla \Psi_\eta^{\partial\Lambda})^2 (x)\,dx$ for $\mathrm{dist}(\Delta, \partial\Lambda) > \epsilon > 0$ should be useful in the future application

Lemma 2.4.

Let $\mu \in \mathcal{G}_r^t(z)$. Let $\Lambda \subset R^2$ be a bounded with C^1-piecewise boundery subset, and let Δ be a unit cube in R^2 such that $\mathrm{dist}(\Delta, \partial\Delta) = \delta > 0$. There exists a constant $c_5(\eta, \Lambda, \delta)$ finite for μ a.e. η and such that:

$$\int_\Delta |\nabla \Psi_\eta^{\partial\Lambda}(x)|^2 dx \le \tag{2.13}$$

$$\le c_5(\eta, \Delta)\,\delta)\,\Big(\int_\Delta \Delta K^{\partial\Lambda}(x,x)\,dx + \big(\int K^{\partial\Lambda}(x,x)\,dx\big)^{\beta}\Big)$$

for any $\beta < 1$.

Proof:

By elementary calculations we have:

$$\Delta K^{\partial\Lambda}(x,x) + 2K^{\partial\Lambda}(x,x) = \tag{2.14}$$

$$= 2 \int_{\partial\Lambda}\int_{\partial\Lambda} (\nabla_x P^{\partial\Lambda}(x, z_1))\,(-\Delta+1)^{-1}(z_1,z_2)(\nabla_x P^{\partial\Lambda}(x, z_2))\,dz_1 dz_2$$

for $x \notin \partial\Lambda$.

Moreover, $\Delta K^{\partial\Lambda}(x,x)$ has still exponential decay in $\mathrm{dist}(x, \partial\Lambda)$ argument. Therefore, we may again apply Tchebyshev inequality in the spirit of the proof of Lemma 2.1.

q.e.d.

The unpleasant feature of the obtained estimates is the a'priori dependence of the constants c_3, c_4 and c_5 of Λ. However, this dependence is not very essential as the following estimate shows.

Estimate

Take $\mu \in \mathcal{G}_r^t(z)$ and let $\Delta \in c(R^2)$ be given. Let $\{\Lambda_n\}$ be any sequence of bounded subsets of R^2 with C^1-piecewise boundaries and such that $\Lambda_n \uparrow R^2$ monotonously and by inclusion. There exists a subsequence $(n') \subset (n)$ and a constant $D(\eta, \beta)$ finite for μ a.e. η such that for all

$$\int_\Delta (\Psi_\eta^{\partial\Lambda_{n'}}(x))^2 dx \le D(\eta,\beta)\Big[\int_\Delta K^{\partial\Lambda_{n'}}(x,x)\,dx\Big]^{\beta}. \tag{2.15}$$

Proof

We use again the Tchebyshev inequality. Let us take $\rho > 0$ to be arbitrary. Then:

$$\mu\{\eta \in S'(R^2) \mid \int_\Delta (\Psi_\eta^{\partial\Lambda_{n'}}(x))^2 dx \ge \frac{1}{\delta}\Big[\int_\Delta K^{\partial\Lambda_{n'}}(x,x)\,dx\Big]^{\beta}\} \le$$

$$\le \sum_n \mu\{\eta \in S'(R^2) \mid \int_\Delta (\Psi_\eta^{\partial\Lambda_{n'}}(x))^2 dx \ge \frac{1}{\delta}\Big[\int_\Delta K^{\partial\Lambda_{n'}}(x,x)\,dx\Big]^{\beta}\} \le$$

$$\le \delta \sum_{n'} \frac{\int d\mu(\eta)\big(\int_\Delta (\Psi_\eta^{\partial\Lambda_{n'}}(x))^2 dx\big)}{\Big[\int_\Delta K^{\partial\Lambda_{n'}}(x,x)\,dx\Big]^{\beta}} \le \delta \sum_{n'} \Big[\int_\Delta K^{\partial\Lambda_{n'}}(x,x)\,dx\Big]^{1-\beta}.$$

A typical contribution of the integrals to the sum $\sum_{n'}$ is bounded by $O(1)\exp(-\mathrm{dist}(\Delta, \partial\Lambda_{n'}))$. Let us denote $a_{n'} = \mathrm{dist}(\Delta, \partial\Lambda_{n'})$. Applying the root criterion we easily conclude that the series $\sum_{n'}$ is convergent whenever $\liminf_{n'} (a_{n'}/n') > 0$. From the assumptions made on the sequence $\{\Lambda_n\}$ it follows that a subsequence of that type $(n') \subset (n)$ may always be chosen. Because δ is arbitrary the proof follows.

q.e.d.

The value of the proved estimate is the following one. In some situation we know from the very beginning that the thermodynamic limits of some quantities of interest do exist. Therefore, it is enough to control these thermodynamic limits by passing to an arbitrary subsequence. It follows from the proof that the most natural case of the applications is the case when $\Lambda_n \uparrow R^2$ in the sense of Fisher.

Remark

There exists the corresponding version of this astimate for the case of completely regular measures. In particular, they have been applied to prove convergence of the high temperature cluster expansion in the $P(\phi)_2$ models (however, nonuniform in the boundary data). In this paper, we will not use it, therefore we will not write them explicitly.

The following Lemma also shows mild dependence on the volume of the constants c_3, c_4 and c_5 in the above proved Lemmas.

Lemma 2.5.

Let $\{\Lambda_n\}$ be any sequence of bounded subsets of R^2 with a piecewise- C^1 boundaries $\{\partial\Lambda_n\}$ and such that $\Lambda_n \uparrow R^2$ monotonously and by inclusion.

1. Let $\mu \in \mathcal{G}_r(z)$ and let a number $\rho > 0$ be given. Then, there exists a subsequence $(n') \subset (n)$ and a function $c_6(\eta, \rho)$ finite μ a.e. and such that:

$$\int_{(\partial_1 \Lambda_{n'})} (\Psi_\eta^{\partial\Lambda_{n'}}(x))^2 dx \leq c_6(\eta, \rho) \, |\partial\Lambda_{n'}|^{1+\rho}, \tag{2.16}$$

where

$$\partial_1 \Lambda_{n'} = \{ x \in \Lambda \mid \mathrm{dist}(x, \partial\Lambda) \leq 1 \},$$

2. Let $\mu \in \mathcal{G}_r(z)$ and let a number $\rho > 0$ be given. Then, there exists a constant $c_7(\eta, \rho)$ finite μ a.e. and a subsequence $(n') \subset (n)$ such that

$$\int_{\partial_\epsilon^1 \Lambda_{n'}} |\nabla \Psi_\eta^{\partial\Lambda_{n'}}(x)|^2 dx \leq c_7(\eta, \rho) \, |\partial\Lambda_{n'}|^{1+\rho}. \tag{2.17}$$

Proof:

Estimates (2.16) and (2.17) are obtained rigorously again by the application of the Tchebyshev inequality and the assumed regularity of μ. Instead of writing the formal proofs in detail, we explain why these estimates are true. Taking $\epsilon > 0$ we have

$$\int \mu(d\eta) \frac{\int_{\partial_1 \Lambda_n} (\Psi^{\partial\Lambda_n}(x))^2 dx}{|\partial_1 \Lambda_n|^{1+\epsilon}} \leq$$

$$\leq |\partial_1 \Lambda_n|^{-1-\epsilon} \int_{\partial_1 \Lambda_n} dx \int \mu(d\eta) (\Psi_\eta^{\partial\Lambda_n}(x))^2 \leq \mathrm{const} \cdot |\partial\Lambda_n|^{-\epsilon}.$$

The last estimate follows from the well known fact that there exists a constant c such that for every $\Delta_j \in c(R^2)$ we have $\| K^{\partial\Lambda} \|_{L^1(\Delta)} < c$ (see Prop. 7.8.7. in /3/). Using additionally formula (3.14) the evidence of the validity of (2.17) can be seen by the similar arguments.

q.e.d.

For the completely regular measures we note the following estimates:

Lemma 2.6.

Let $\{\Lambda_n\}$ be as in Lemma 2.5. Assume that $\mu \in \mathcal{G}_{cr}^t(z)$ and let $\rho > 0$ be given. For every integar $k \geq 1$, there exists a constant $c_8(\eta, \rho, k)$ finite μ a.e. and a sequence $(n') \subset (n)$ such

$$\int_{\partial_1 \Lambda_{n'}} (\Psi_\eta^{\partial\Lambda_{n'}}(x))^k dx \leq c_8(\eta, \rho, k) \, |\partial\Lambda_{n'}|^{\rho+1}. \tag{2.18}$$

2.3. Shift Transformation

Now we are ready to demonstrate that the effect of the conditioning is a typically boundary effect and in the case of pressure it vanishes in the thermodynamic limit.

For a given bounded $\Lambda \subset R^2$ with C^1-piecewise boundary $\partial\Lambda$ let us denote (here $0 < \epsilon < 1$):

$$Y = \{ x \in \Lambda \mid \mathrm{dist}(x, \partial\Lambda) \geq 1 \},$$

$$Y_\epsilon = \{ x \in \Lambda \mid \mathrm{dist}(x, Y) < \epsilon \}, \quad Y^\epsilon = \Lambda - Y_\epsilon :$$

Let χ_ϵ be a function (indexed by Λ) such that: $\chi_\epsilon \in C_0^\infty(R^2)$,

$$0 < \chi_\epsilon(x) \begin{cases} = 1, & x \in Y, \\ \leq 1, & x \in Y_\epsilon, \\ 0, & x \in Y^\epsilon. \end{cases} \tag{2.19}$$

and such that:

$$\sup_{x \in \Lambda} \max \{ |\partial_1 \chi_\epsilon|(x), |\partial_2 \chi_\epsilon|(x) \} = c_{10}(\Lambda),$$
$$\sup_{x \in \Lambda} |\Delta \chi_\epsilon| = c_{11}(\Lambda). \tag{2.20}$$

In the formula defining Z_Λ^η let us perform the following shift transformation:

$$\phi \to \phi - \chi_\epsilon \cdot \Psi_\eta^{\partial\Lambda}.$$

Using

$$\frac{d\mu_0^{\partial\Lambda}(\phi - \chi_\epsilon \cdot \Psi_\eta^{\partial\Lambda})}{d\mu_0^{\partial\Lambda}(\phi)} = \tag{2.21}$$
$$= \exp(-\phi(J_\eta^\epsilon)) \exp\left(\frac{1}{2} \int \chi_\epsilon(x) \Psi_\eta^{\partial\Lambda}(x) J_\eta^\epsilon(x) dx\right),$$

where

$$J^{\epsilon}_{\eta}(x) \equiv (-\Delta + 1)(\chi_{\epsilon} \cdot \Psi^{\partial\Lambda}_{\eta})(x)$$

is given by:

$$J^{\epsilon}_{\eta}(x) = \begin{cases} 0, & \text{for } x \in Y \\ (-\Delta\chi_{\epsilon})\Psi^{\partial\Lambda}_{\eta}(x) + 2(\nabla\chi_{\epsilon}(x))(\nabla\Psi^{\partial\Lambda}_{\eta})(x) & \text{for } x \in Y_{\epsilon} - Y, \\ 0, & \text{for } x \in \Lambda - Y_{\epsilon} = Y^{\epsilon}. \end{cases} \tag{2.22}$$

Note that because $\overset{\mu}{\underset{\eta}{\forall}} : \Psi^{\partial\Lambda}_{\eta}$ is a C^{∞} function inside Λ as it is a solution (in S') of the elliptic homogeneous equation $(-\Delta + 1)\Psi^{\partial\Lambda}_{\eta}(x) = 0$. This is a reason why the transformation made above has a perfectly right mathematical sense.

Using these formulas we have:

$$\frac{Z^{\eta}_{\Lambda}(z)}{Z^{\circ}_{\Lambda}(z)} = \exp\left(\frac{1}{2} \int \chi_{\epsilon}(x)\, \Psi^{\partial\Lambda}_{\eta}(x)\, J^{\epsilon}_{\eta}(x)\, dx\right) \times$$

$$\times \left\langle \exp\left(z \int_{\Lambda} [c(\phi + (1-\chi_{\epsilon}) \cdot \Psi^{\partial\Lambda}_{\eta})(x) - c(\phi)(x)] dx\right) \exp(-\phi(J^{\epsilon}_{\eta}))\right\rangle^{\eta=0}_{\Lambda}(z). \tag{2.23}$$

By the application of the Cauchy-Schwartz inequality we have:

$$\frac{Z^{\eta}_{\Lambda}(z)}{Z^{\circ}_{\Lambda}(z)} \leq \Pi^{1}_{\Lambda}(\eta)\,(\Pi^{2}_{\Lambda}(\eta))^{1/2}\,(\Pi^{3}_{\Lambda}(\eta))^{1/2}, \tag{2.24}$$

where we have defined:

$$\Pi^{1}_{\Lambda}(z) = \exp\left(\frac{1}{2} \int \chi_{\epsilon}(x) \cdot \Psi^{\partial\Lambda}_{\eta}(x)\, J^{\epsilon}_{\eta}(x)\, dx\right), \tag{2.25}$$

$$\Pi^{2}_{\Lambda}(z) = \left\langle \exp\left[2z \int_{\Lambda - Y} [c(\phi + (1-\chi_{\epsilon})\Psi^{\partial\Lambda}_{\eta})(x) - c(\phi)(x)\, dx\right]\right\rangle^{\circ}_{\Lambda}(z), \tag{2.26}$$

$$\Pi^{3}_{\Lambda}(z) = \langle \exp - 2\phi(J^{\epsilon}_{\eta}) \rangle^{\circ}_{\Lambda}(z). \tag{2.27}$$

Now we prove that all these factors have a typical behaviour like $\exp O(\eta)|\partial\Lambda|$.

In the next three lemmas we assume that $\{\Lambda_n\}$ is a sequence as described in the Lemma 2.5. above. Additionally for a given sequence $\{\Lambda_n\}$ we choose a sequence χ^{n}_{ϵ} such that

$$c_{10} \equiv \sup_n c_{10}(\Lambda_n) < \infty, \quad c_{11} = \sup_n c_{11}(\Lambda_n) < \infty. \tag{2.28}$$

Lemma 2.7.

Let $\{\Lambda_n\}$ be as above. Let $\mu \in \mathcal{G}^{t}_{r}(z)$ and $\rho > 0$ be given. There exists a constant $c_{12}(\eta)$ finite "on supp μ" and a subsequence $(n') \subset (n)$ such that:

$$|\Pi^{1}_{\Lambda_{n'}}(\eta)| \leq \exp c_{12}(\eta, \rho)\, |\partial\Lambda_{n'}|^{1+\rho}. \tag{2.29}$$

Proof:

It is due to the factor J^{ϵ}_{η} in the integral over dx that this integration is made over the set $\partial^{+}_{\epsilon}\Lambda \equiv Y_{\epsilon} - Y$. Using the definition of J^{ϵ}_{η} given by (2.22), the properties (2.20) of χ^{n}_{ϵ} and the Cauchy-Schwartz inequality we have:

$$\Pi^{1}_{\Lambda_n}(\eta) \leq \exp\left(\frac{1}{2} c_9 \cdot c_{10} \int_{\partial^{+}_{\epsilon}\Lambda_n} (\Psi^{\partial\Lambda_n}_{\eta})^2(x)\, dx\right) \times$$

$$\times \exp\left(\frac{1}{4} c_9 \cdot c_{10} \left(\int_{\partial^{+}_{\epsilon}\Lambda_n} (\nabla\Psi^{\partial\Lambda_n}_{\eta})^2(x)\right)^{1/2}\right) \times \tag{2.30}$$

$$\times \exp\left(\frac{1}{2} c_9 \cdot c_{10} \cdot \left(\int_{\partial^{+}_{\epsilon}\Lambda_n} (\Psi^{\partial\Lambda_n}_{\eta})^2(x)\, dx\right)^{1/2}\right).$$

Given the sequence $\{\Lambda_n\}$ as in the assumptions and using then the Lemma 2.5, we conclude that there exists a subsequence $(n') \subset (n)$ such that:

$$|\Pi^{1}_{\Lambda_{n'}}(\eta)| \leq \exp c_{12}(\eta, \rho)\, |\partial\Lambda_{n'}|^{1+\rho}. \tag{2.31}$$

q.e.d.

Lemma 2.8.

Let $\{\Lambda_n\}$ be as above. For any $\mu \in \mathcal{G}^{t}_{r}(z), \rho > 0$ there exists a constant $c_{13}(\eta, \rho)$ "finite on supp μ" and a subsequence $(n') \subset (n)$ such that:

$$|\Pi^{3}_{\Lambda_n'}(\eta)| \le \exp c_{13}(\eta,\rho)\,|\partial\Lambda_{n'}|^{1+\rho}. \tag{2.32}$$

Proof:

From the correlation inequality (see /7/) it follows that for every $f \in H_{-1}(R^2)$ we have:

$$\langle e^{\phi(f)}\rangle^{\eta=0}_{\Lambda}(z) \le \exp\frac{1}{2}\,||f||^2_{-1,\partial\Lambda} \tag{2.33}$$

uniformly in the volume Λ. Applying this observation we have:

$$|\Pi^{3}_{\Lambda_n}(\eta)| = \langle\exp(-2\phi(J^{\epsilon}_{\eta}))\rangle^{\circ}_{\Lambda}(z) \le \exp(2\,||(-\Delta^{\partial\Lambda}+1)(\chi_\epsilon\cdot\Psi^{\partial\Lambda}_{\eta})||^2_{-1}) =$$

$$= \exp(2\int dx\, J^{\epsilon}_{\eta}(x)(\chi_\epsilon\cdot\Psi^{\partial\Lambda}_{\eta})(x)) .$$

We see now that the integral to be estimated is an almost identical to that met in Lemma 2.7.

q.e.d.

We proceed now to estimate the factor $\Pi^2_{\Lambda}(\eta)$. Here, we use the $\cos\epsilon\phi$ -bound of Fröhlich /1/. From the use of this bound there follows our technical restriction on the size of ϵ. The $\cos\epsilon\phi$ -bound says that for every regular f we have:

$$\langle e^{c(\phi+\theta)(f)}\rangle^{\circ}_{\Lambda}(z) \le e^{(||f||_1+||f||_{p(\epsilon)})} \qquad 0\le\theta<2\pi \tag{2.34}$$

uniformly in the volume Λ. Here we have to assume that

$$p(\epsilon) > \frac{1}{1-\epsilon^2/4\pi}. \tag{2.35}$$

Lemma 2.9.

Let $\{\Lambda_n\}$ be a sequence as in the Lemma 2.5.
Let $\mu\in\mathcal{G}^t(z)$. There exists a constant $c_{14}(\eta)$ finite μ a.e. such that:

$$|\Pi^{2}_{\Lambda_n'}(\eta)| \le \exp c_{14}(\eta)\,|\partial\Lambda_{n'}|. \tag{2.36}$$

Proof:

By a little algebra we have:

$$\Pi^2_{\Lambda_n}(\eta) \le \langle\exp 2z\int_{\Lambda_n} dx :c(\phi):_{\partial\Lambda_n}(x)\,(:c(1-\chi_\epsilon\cdot\Psi^{\partial\Lambda}_{\eta}):(x)-1)\rangle^{\circ}(z)\rangle^{1/2}_{\Lambda}\ \times$$

$$\times\langle\exp -2z\int_{\Lambda_n} :s(\phi):_{\partial\Lambda_n}(:s(1-\chi_\epsilon\ \Psi^{\partial\Lambda}_{\eta}):(x)\rangle^{\circ}_{\Lambda}(z))^{1/2} . \tag{2.37}$$

The functions

$$:c(1-\chi_\epsilon\cdot\Psi^{\partial\Lambda}_{\eta}):(x) - 1 =$$

$$= \exp(\frac{\alpha^2}{2}(1-\chi_\epsilon)^2 K^{\partial\Lambda}(x,x))\ \cos(\alpha(1-\chi_\epsilon)\Psi^{\partial\Lambda}_{\eta})(x) - 1 \tag{2.38}$$

and

$$:c(1-\chi_\epsilon\cdot\Psi^{\partial\Lambda}_{\eta}):(x) =$$

$$= \exp(\frac{\alpha^2}{2}(1-\chi_\epsilon)^2 K^{\partial\Lambda}(x,x))\ \sin(\alpha(1-\chi_\epsilon)\Psi^{\partial\Lambda}_{\eta})(x) \tag{2.39}$$

are both supported on the set Λ-Y and are bounded there by:

$$|:c((1-\chi_\epsilon)\Psi^{\partial\Lambda}_{\eta}):(x) - 1| \le 2(1-\chi^{\epsilon})\exp(\frac{\alpha^2}{2}K^{\partial\Lambda}(x,x)) \tag{2.40}$$

and

$$|:s(1-\chi_\epsilon)\cdot\Psi^{\partial\Lambda}_{\eta})(x):| \le (1-\chi_\epsilon)\exp\frac{\alpha^2}{2}K^{\partial\Lambda}(x,x) . \tag{2.41}$$

Note that we have changed ϵ in formula (2.37) defining interaction by α in order to exclude the possible missing of the symbols used.

The functions $K^{\partial\Lambda}$ have locally integrable singularities on the set $\partial\Lambda$. They have the behaviour like

$$K^{\partial\Lambda}(x,x) \sim -\frac{1}{2\pi}\ln|\text{dist}(x,\partial\Lambda)|\ \exp -\text{dist}(x,\partial\Lambda) \tag{2.42}$$

as $x \to \partial\Lambda$. In applying the $\cos\epsilon\phi$ -bound we need to have $\epsilon^2 < \frac{2}{1-(1/2\pi)}$.

Assuming this holds, we can apply Lemma 3.6 to both the factors in the estimate (2.37).

q.e.d.

2.4. $p_\infty^\eta = p_\infty^\circ$

Summarizing our discussion we have the following theorem.

Theorem 2.10

Let $\epsilon^2 < \frac{2}{1-(1/2\pi)}$ and $\mu \in \mathcal{G}_r^t(z)$.

Then

$$\overset{\mu}{\underset{\eta}{\forall}} : \lim_{\Lambda \uparrow R^2} p_\Lambda^\eta(z) = p_\infty^\circ(z) . \tag{2.43}$$

Here $\Lambda \uparrow R^2$ means any sequence $\{\Lambda_n\}$ of bounded, with C^1-piecewise boundaries subsets of R^2 such that $\Lambda_n \uparrow R^2$ monotonously and by inclusion and such that for some $\rho > 0$:

$$\lim_{n\to\infty} \frac{|\partial\Lambda_n|^{1+\rho}}{|\Lambda_n|} = 0 .$$

Proof:

From the Lemma 2.1 we know that there exists for μ a.e. η a subsequence $(n') \subset (n)$ such that the limit

$$\Theta_{n'}^\eta = \lim_{\Lambda_{n'} \uparrow R^2} \ln\left(\frac{Z_{\Lambda_{n'}}^\eta(z)}{Z_{\Lambda_{n'}}^\circ(z)}\right)^{-1/|\Lambda_{n'}|}$$

exists. From formula (2.23) and the Lemmas 2.7 - 3.9 it follows that for μ a.e. η we have $\Theta_{n'}^\eta = 0$. From this we conclude that $p_\infty^\circ(z)$ is the only accumulation point of the sequence $\{p_{\Lambda_n}^\eta(z)\}$.

q.e.d.

Concluding remark

In the paper [5] we have applied the result proven here to extend the uniqueness part of the work [3].

REFERENCES

1. Fröhlich.J. Comm.Math.Phys. 1976, 47, p.233.
2. Fröhlich J., Park Y.M. Helv.Phys.Acta, 1978, 50, p.315.
3. Albeverio S., Hoegh-Krohn, Comm.Math.Phys., 1979, 68, p.95.
4. Dobrushin R.L., Minlos R.A.In: Problems of Mechanics and Mathematical Physics, "Mir", Moscow, 1976 (in Russian).
5. Gielerak R. JINR, E2-85-898, Dubna, 1985.
6. Guerra F., Rosen L., Simon B. Ann.L'Inst.Henn-Poincare, 1975, 15, p.223.
7. Fröhlich J., Park Y.M. Comm.Math.Phys. 1978, 59, p.235.

Received by Publishing Department
on December 13, 1985.

Гелерак Р. E2-85-899

Утверждение Ван-Хова для двумерной модели сина-Гордона

Исследуются равновесные уравнения типа Добрушина – Ланфорда – Рюэля в случае двумерной модели сина-Гордона. Задача о единственности решения сводится к задаче о независимости давления от регулярных граничных условий. При использовании некоторых новых оценок на решения стохастической задачи Дирихлета, которые доказываются в этой работе, удалось получить общую версию теоремы Ван-Хова. Теорема Ван-Хова утверждает, что свободная энергия в термодинамическом пределе не зависит от класса граничных условий, которые имеют полную меру относительно любой регулярной меры.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1985

Gielerak R. E2-85-899

The Van Hove Theorem for the Two-Dimensional Sine-Gordon Model

In the space of regular measures on $\{S'(R^2), \mathcal{B}\}$ stochastic Dirichlet problem is investigated. The new, local estimates on the solutions of the stochastic Dirichlet problem are proved. These estimates are then applied to prove the independence of the infinite volume free energy in the two-dimensional sine-Gordon model on the tempered and regular boundary conditions.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1985