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SOLUTIONS OF THE BETHE-ANSATZ EQUATIONS FOR THE XXX ANTIFERROMAGNET OF ARBITRARY SPIN WITH A FINITE NUMBER OF SITES

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Introduction

In the last years, a remarkably-growing interest is called to exactly integrable models like the generalized Heisenberg magnet. This model describes a pairwise interaction of neighbouring spins s in a periodic chain with N sites. The model has been proposed /1/ in the context of the quantum inverse scattering method and for the first time investigated/2/ on the basis of the "string" hypothesis about the structure of the Bethe-ansatz (BA) solutions. However, a further analysis/3-5/ showed that this hypothesis is only a convenient approximation which has not yet found a complete mathematical comprehension. In refs./3-7/, deviations from the string picture have been discovered for excited states. However, it has been assumed in refs. /6,7/ that, although for $N\to\infty$ excitations may have a nonstring form, they exist on the background of a sea of perfect 2s-strings. In the present paper, we consider corrections to this idealized picture for finite N. In the $s=\frac{1}{2}$ case /3,4/, the question about a change in the shape of 2s-strings did not arise because they degenerated to real roots. This allowed us to estimate/8/ an influence of the N finiteness on their density by the method /9/ of evaluating finite-size corrections. For $s>\frac{1}{2}$, it is not less important to take into account the 2s-string deformation that may prove to be by no means exponentially small.

1. The Equations and the String Hypothesis

The Bethe-ansatz equations (BAE) for the model/1/ have the form /2/

$$\left(\frac{\lambda_j+is}{\lambda_j-is}\right)^N=-\prod_{k=1}^{M}\frac{\lambda_j-\lambda_k+i}{\lambda_j-\lambda_k-i}\ ,\qquad (j=1\ldots M). \tag{1.1}$$

The energy, momentum, and spin of a state of the magnet are expressed through a solution of system (1.1), a set of M complex numbers $\{\lambda_j\}$;

$$E=-J\sum_{j=1}^{M}\frac{s}{\lambda_j^2+s^2}\ ; \tag{1.2}$$

$$P \stackrel{2\pi}{=} \frac{1}{i} \sum_{j=1}^{M} \ln \frac{\lambda_j + is}{\lambda_j - is}, \qquad (\mathrm{mod}\ 2\pi); \tag{1.3}$$

$$L = sN - M \geqslant 0. \tag{1.4}$$

For the low-lying antiferromagnetic (AF) states ($J>0$; in the following, we set $J=1$) the spin (1.4) is not high, and hence, $M \approx sN$. It is clear that a system of a large but finite number of nonlinear equations can be solved only numerically. However, before applying a numerical method, one has to specify some preliminary information about the solution. This is because the total number of solutions to eqs.(1.1) is extremely large, and not all of them are physical. There are a lot of solutions with coinciding roots ($\lambda_j = \lambda_k$ for some $j \neq k$), and just several of them obey additional restrictions /10/ that ensure the existence of (nonzero) Bethe vectors.

The only available information about solutions is based on the "string" hypothesis /11,2/. According to it, as $N \to \infty$, any solution to eqs.(1.1) consists of "n-strings" of the form

$$\lambda = x + i\left[\tfrac{1}{2}(n+1) - m\right], \qquad (m = 1 \dots n), \tag{1.5}$$

where x is a real number (the position of the center), and n is a positive integer (the length of the string). Deviations from the limit picture (1.5) are supposed to be exponentially small. If one accepts the hypothesis and specifies a set of n_j-strings ($j=1\dots K$), then, for their centers x_j, the following equations /2/ can be derived from eqs.(1.1):

$$Q_{n_j}(x_j) \stackrel{1}{=} Q_{n_j}(+\infty), \qquad (\mathrm{mod}\ 1), \tag{1.6}$$

where the function $Q_n(x)$ for the specified configuration is

$$Q_n(x) = \frac{N}{\pi} \sum_{k=1}^{\min(n,2s)} \mathrm{atan} \frac{x}{s + \frac{1}{2}(n+1) - k} - \frac{1}{\pi} \sum_{j=1}^{K} \Big\{ \Big[\underline{\text{if}}\ n = n_j\ \underline{\text{then}}\ 0$$

$$\underline{\text{else}}\ \mathrm{atan} \frac{2(x-x_j)}{|n-n_j|} \Big] + 2 \sum_{k=\frac{1}{2}|n-n_j|+1}^{\frac{1}{2}(n+n_j)-1} \mathrm{atan} \frac{x-x_j}{k} + \mathrm{atan} \frac{2(x-x_j)}{n+n_j} \Big\}, \tag{1.7}$$

$$Q_n(+\infty) = \tfrac{1}{2} N \min(n, 2s) - \sum_{j=1}^{K} \min(n, n_j) + \tfrac{1}{2}\Big(\sum_{j=1}^{K} \delta_{n n_j} - 1\Big).$$

The choice of branches in eqs.(1.6) is restricted /2/ by

$$|Q_{n_j}(x_j)| \leq Q_{n_j}(+\infty). \tag{1.8}$$

The number of configurations that satisfy eq.(1.8) turns out /12/ to be exactly the one implied by the completeness requirement. The problem, however, consists in the fact that the string sets thus presented may differ strongly /3-7/ from the true solutions to eqs.(1.1), even as $N \to \infty$. Besides, the restriction (1.8) has no substantiation in the framework of the string hypothesis itself. On the one hand, eq.(1.6) is known to have solutions that do not obey eq.(1.8). On the other hand, as shown in ref./5/, not to every choice of branches in accordance with eqs.(1.8), does a solution correspond uniquely: counter-examples appear for extreme branches at sufficiently large N.

2. The Algorithm of Numerical Computations

Despite all these difficulties, for a preliminary classification of the BAE solutions and for finding out initial approximations, we have to use the string picture. It should be noted that eqs.(1.6) for the (real) string centers are written in terms of arctangents (1.7) and do not involve poles in the unknowns, as the initial exact BAE (1.1). This fact simplifies essentially their numerical solution (for example, by Newton's method, see below). Eqs. (1.6) are analogous to the equations for the sea of real roots at $s=\frac{1}{2}$ (solved, for instance, in refs./13,14/). Thus, a determination of the string centers for an initial approximation entails no fundamental difficulties. Evaluating deviations of strings from eq.(1.5) is a more serious task.

It can be performed in zeroth approximation. We assume the deviations to be small and neglect them in eqs.(1.1) on the background of the unit-order quantities and distances between the string centers. This trick has been used in ref./15/ for checking the consistency of the string picture. In this connection we must stress that approximating products (sums) over roots with a density function/13/ may be incorrect if the initial expression involves a singularity (then, a rule of passing over it determines the answer). But this is just the case when trying/15/ to evaluate the deviations (see below, sect. 5). In addition, if the number of strings grows proportionally to N, just the $O(1/N)$ deviations (see sect.4) may result in a finite correction that is left as $N \to \infty$. So, even without the density-function approximation, to zeroth order we obtain only a rough estimate of the actual deviations.

For achieving a better accuracy, we use Newton's iterations:

$$\sum_k \left[\lambda_k^{(n+1)} - \lambda_k^{(n)}\right] \nabla_{jk}^{(n)} = -\phi_j^{(n)}, \tag{2.1}$$

$$\phi_j^{(n)} = N \ln \frac{\lambda_j^{(n)} + is}{\lambda_j^{(n)} - is} - \sum_{k \neq j} \ln \frac{\lambda_j^{(n)} - \lambda_k^{(n)} + i}{\lambda_j^{(n)} - \lambda_k^{(n)} - i}, \tag{2.2}$$

$$\nabla_{jk}^{(n)} = \partial \phi_j^{(n)} / \partial \lambda_k^{(n)}; \qquad (j,k = 1 \dots M; \quad n = 0, 1, 2, \dots). \tag{2.3}$$

At step n of the iterations, it is necessary to find the vector (2.2) of logarithmic errors in the BAE (1.1), to compute the $M \times M$ matrix of first derivatives (2.3), and to solve the system of M linear equations (2.1).

Eqs.(2.1)-(2.3) involve complex numbers. However, we can make use of the fact[16] that physical solutions of eqs.(1.1) are self-conjugate: $\{\lambda_j^*\} = \{\lambda_j\}$. Then, the problem is reduced to a system of M real equations for independent real parameters.

Newton's method (2.1) is a most general method for solving nonlinear equations with a rapid convergence in a vicinity of a solution. However, in this method it is necessary to keep a large matrix of derivatives in the computer memory (in our computations, we were limited to $M \leqslant 128$), and besides, initial approximations should be accurate enough because of the following. Formation of strings and string-like configurations[3,4,6,7] entails that there are substantially nonlinear singularities near the solutions of eqs.(1.1). As a result, a "zone of attraction" for the linear algorithm (2.1) turns out to be very small. On the other hand, a lot of "parasitic" solutions with equal roots have a rather wide zone of attraction. For this reason, we have not succeeded in computing a sufficiently full picture of excitations for high spins s when the above-described "string" zeroth approximation for the deviations proves to be too rough.

3. The Full Set of States for $s=1$, $N=6$

A search for the BAE solutions at relatively small N and s allows us, on the one hand, to verify the hypothesis of the BA completeness, and on the other hand, to make definite conclusions about the structure of the solutions, that can be extended to higher N and s.

In the $s=1$, $N=6$ case, the sector of singlet states ($L=0$, $M=6$) is most instructive. In table 1, complete data of these solutions are presented: the string prototype $\{n_Q\}$, a configuration[2] of n-strings with the corresponding $Q_n(x)$ (1.7); the number H of holes, physical excitations[2,7] over the AF vacuum; the energy (1.2) E $(J=1)$ and momentum (1.3) $P\frac{N}{2\pi}$ (mod N); the values of the $\{\lambda_j\}$ parameters, their arrangement on the complex plane is sketched in fig. 1. Asterisks point out "doublet" solutions that have a symmetric "partner" $\{-\lambda_j\} \neq \{\lambda_j\}$ with the same energy and opposite momentum.

One can make the following conclusions from considering the picture of the solutions. First, the string approximation gives a correct qualitative classification of states. The AF vacuum - the $2_{-1}2_0 2_1$ state with the minimum energy - is in fact a sea of 2s-strings[2]. However, deviations from perfect strings (1.5) are not always small: only strings of length $2s+1=3$ (or fragments of longer strings for high-excited states) have really small deviations. The least string-like is the $2_0 4_0$ state, in which a 2 -string and a 4-string with centers at zero have united into a symmetric complex, the 4-string

remarkably "bent". Among the solutions, there are examples ($1_0 2_0 3_0$ and $1_0 5_0$) with a double root at zero. These symmetric solutions that include the 1-string and the perfect 3-string at zero satisfy the restrictions of ref.[10], and therefore, are physical.

Table 1. Configurations, number of holes, energies, momenta, and BAE solutions for singlet states: $L=0$, $M=N=6$, $s=1$

$\{n_Q\}$	H	E	$P\frac{N}{2\pi}$	$\{\lambda_j\}$
$2_{-1} 2_0 2_1$	0	-6.219 352 5	0	±.438 492 14 ±.548 302 99i, ±.520 705 59i
$1_0\ 2_0\ 3_0$	2	-5.081 138 8	0	0, 0, ±.474 497 87i, ±i
$1_0\ 2_1\ 3_0^*$	2	-3.941 227 2	2	-.236 263 09, -.266 972 74, -.252 556 77 ±1.000 476 733i, .504 174 68 ±.535 930 69i
$2_0\ 4_0$	4	-3.876 298 6	0	±.065 745 96 ±.502 532 08i, ±1.557 215 51i
$2_1\ 4_0^*$	4	-3.250 000 0	5	.279 770 34 ±.491 697 27i, -.142 231 69 ±.500 177 165i, -.137 538 66 ±1.513 981 98i
$2_2\ 4_0^*$	4	-2.275 770 9	4	.686 976 56 ±.534 944 56i, -.338 110 18 ±.500 133 728i, -.348 866 38 ±1.503 962 44i
$1_{-1/2}\ 1_{1/2}\ 4_0$	4	-2.657 823 6	0	±.536 888 34, ±.500 096 663i, ±1.510 188 98i
$3_{-1/2}\ 3_{1/2}$	4	-2.477 655 6	0	±.539 986 89, ±.532 038 44 ±1.000 634 356i
$1_0\ 5_0$	6	-1.918 861 2	0	0, 0, ±i, ±2.107 491 03i
$1_1\ 5_0^*$	6	-1.533 001 9	2	.757 071 86, -.146 381 35=x, x+1.764 633 × 10^{-7}±(1-2.304 466 × 10^{-8})i, -.158 963 72 ± 2.084 969 85i
6_0	8	-.768 869 75	0	±(.5+2.255 552 × 10^{-7})i, ±1.500 150 077i, ±2.775 428 00i

However, the picture obtained cannot, obviously, be considered as an evidence that the string hypothesis is always true. Essentially non-string states are formed[3-7] at sufficiently large N on the background of a dense 2s-string sea, and it is clear that one-three strings are simply insufficient for this.

Computations have been also performed in higher-spin sectors, $L=1...5$ $(M=5...1)$. The results are presented in table 2 in the same form as in table 1 except the $\{\lambda_j\}$ parameters. The number of solutions in each sector agrees exactly with the prediction based on the completeness in the spin space; for $s=1$, this is

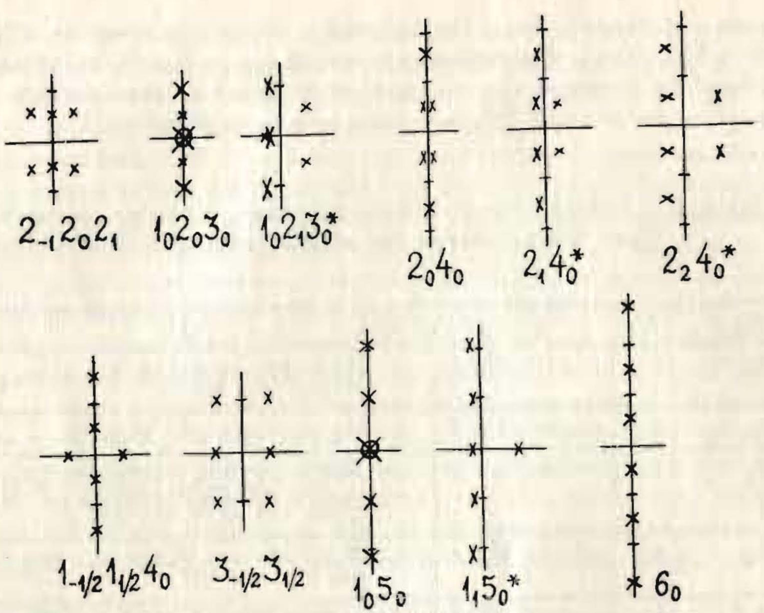

Fig.1

$$\sum_m \frac{N!}{m!(M-2m)!(N-M+m)!} - \sum_m \frac{N!}{m!(M-1-m)!(N-M+1+m)!} \,. \tag{3.1}$$

Every Bethe vector is the highest member of a spin multiplet of $2L+1$ states with the same energy and momentum. The total number of states, together with the ferromagnetic vacuum ($M=0$, $L=6$), is equal to $3^6 = (2s+1)^N$. This verifies the BA completeness.

4. The AF-Vacuum Computations

It is clear that the accuracy of strings cannot be studied on the basis of calculations only at one N. It is necessary to consider a sequence of analogous states with different N values, and then to make conclusions about the N dependence. In this section we examine the BAE solutions related to the AF-vacuum state. Its spin (1.4) equals zero, the number of the $\{\lambda_j\}$ parameters $M=sN$. The corresponding configuration is the sea of $\frac{1}{2}N$ 2s-strings. If we neglect deviations from the idealized picture (1.5), then, for the density of the string centers and for the energy in the $N\to\infty$ limit, we obtain the expressions[7]:

$$\sigma_\infty(x) = \frac{N}{2\cosh(\pi x)} \,, \tag{4.1}$$

Table 2. Configurations, number of holes, energies, and momenta for states of spin $L=1\ldots5$; $N=6$, $s=1$.

L	$\{n_Q\}$	H	E	$P\frac{N}{2\pi}$	L	$\{n_Q\}$	H	E	$P\frac{N}{2\pi}$
1	$1_0 2_{1/2} 2_{-1/2}$	2	-5.662 917 6	3	2	$1_1 3_2^*$	6	-1.750 000 0	0
1	$1_0 2_{3/2} 2_{1/2}^*$	2	-4.667 421 8	5	2	$1_1 3_1^*$	6	-2.070 914 8	1
1	$1_0 2_{3/2} 2_{-1/2}^*$	2	-4.447 428 2	4	2	$1_1 3_0^*$	6	-2.021 547 3	2
1	$1_0 2_{3/2} 2_{-3/2}$	2	-3.255 542 0	3	2	$1_1 3_{-1}^*$	6	-1.681 270 7	3
1	$2_0 3_0$	4	-4.189 694 9	3	2	$1_1 3_{-2}^*$	6	-1.198 283 9	4
1	$2_1 3_0^*$	4	-3.460 827 2	5	2	4_0	8	-1.069 643 2	0
1	$2_2 3_0^*$	4	-2.436 793 6	4	2	4_1^*	8	-.978 653 47	5
1	$2_2 3_1^*$	4	-2.483 070 3	3	2	4_2^*	8	-.739 282 65	4
1	$2_1 3_1^*$	4	-3.773 000 4	4	3	$1_0 2_0$	6	-3.651 387 8	3
1	$2_0 3_1^*$	4	-3.633 754 5	5	3	$1_0 2_1^*$	6	-3.149 552 7	2
1	$2_{-1} 3_1^*$	4	-2.750 000 0	0	3	$1_0 2_2^*$	6	-2.230 647 5	1
1	$2_{-2} 3_1^*$	4	-1.851 987 9	1	3	$1_0 2_3^*$	6	-1.500 000 0	0
1	$1_{1/2} 1_{-1/2} 3_0$	4	-2.950 916 4	3	3	$1_1 2_3^*$	6	-1.634 401 8	5
1	$1_{1/2} 1_{-1/2} 3_1^*$	4	-2.653 524 5	5	3	$1_1 2_2^*$	6	-2.500 000 0	0
1	$1_0 4_0$	6	-2.081 540 4	3	3	$1_1 2_1^*$	6	-3.092 240 0	1
1	$1_0 4_1^*$	6	-1.919 598 2	4	3	$1_1 2_0^*$	6	-3.000 000 0	2
1	$1_1 4_1^*$	6	-1.698 823 7	1	3	$1_1 2_{-1}^*$	6	-2.366 025 4	3
1	$1_1 4_0^*$	6	-1.673 179 6	2	3	$1_1 2_{-2}^*$	6	-1.561 618 7	4
1	$1_1 4_{-1}^*$	6	-1.406 929 7	3	3	$1_1 2_{-3}^*$	6	-.922 488 04	5
1	5_0	8	-.859 388 69	3	3	$1_1 1_0 1_{-1}$	6	-1.848 612 2	3
1	5_1^*	8	-.783 660 38	5	3	3_0	8	-3/2	0
2	$2_{1/2} 2_{-1/2}$	4	-5.047 731 9	0	3	3_1^*	8	-1.370 222 7	5
2	$2_{3/2} 2_{1/2}^*$	4	-4.447 831 5	4	3	3_2^*	8	-1.038 828 6	4
2	$2_{3/2} 2_{-1/2}^*$	4	-4.090 576 2	5	3	3_3^*	8	-.633 974 60	3
2	$2_{3/2} 2_{-3/2}$	4	-3.101 521 3	0	4	2_0	8	-2.671 461 5	0
2	$2_{5/2} 2_{3/2}^*$	4	-3.386 695 6	2	4	2_1^*	8	-2.362 372 4	5
2	$2_{5/2} 2_{1/2}^*$	4	-3.568 729 3	3	4	2_2^*	8	-1.690 389 3	4
2	$2_{5/2} 2_{-1/2}^*$	4	-3.070 243 2	4	4	2_3^*	8	-1.000 000 0	3
2	$2_{5/2} 2_{-3/2}^*$	4	-2.174 674 5	5	4	2_4^*	8	-.468 778 65	2
2	$2_{5/2} 2_{-5/2}$	4	-1.403 255 5	0	4	$1_{1/2} 1_{-1/2}$	8	-1.735 341 7	0
2	$1_{1/2} 1_{-1/2} 2_0$	4	-3.877 848 1	0	4	$1_{3/2} 1_{1/2}^*$	8	-1.340 832 1	4
2	$1_{1/2} 1_{-1/2} 2_1^*$	4	-3.413 710 2	5	4	$1_{3/2} 1_{-1/2}^*$	8	-1.137 627 6	5
2	$1_{1/2} 1_{-1/2} 2_2^*$	4	-2.367 213 9	4	4	$1_{3/2} 1_{-3/2}$	8	-.593 196 75	0
2	$1_0 3_0$	6	-5/2	0	5	1_0	10	-1	3
2	$1_0 3_1^*$	6	-2.268 901 9	2	5	1_1^*	10	-3/4	2
2	$1_0 3_2^*$	6	-1.771 470 7	1	5	1_2^*	10	-1/4	1

$$E_\infty = -\left[\underline{\text{if}}\ s=\textit{integer}\ \underline{\text{then}} \sum_{k=1}^{s}\frac{1}{2k-1}\ \underline{\text{else}}\ \ln 2+\sum_{k=1}^{s-\frac{1}{2}}\frac{1}{2k}\right]N. \tag{4.2}$$

For finite N, corrections to these formulas arise, and the shape of 2s-strings is changed: they "stretch" or "shrink" (for $s\geqslant 1$, the intervals between the imaginary parts of the string members deviate from unit by a Δ ; for the vacuum, $\Delta>0$) and "curve" (for $s\geqslant\frac{3}{2}$, the real parts become different).

Our numerical computations performed for $2s=2\ldots 9$, $sN\leqslant 128$, give the following qualitative picture. Maximum deviations from eq. (1.5) are observed for extreme strings, the remotest from the origin. The string-curving effect is relatively small as compared to their stretching. The following formula approximates roughly the stretching as a function of the string-center coordinate:

$$\Delta(x)\approx C\,\frac{2\cosh(\pi x)}{N}, \qquad |x|\leqslant\frac{1}{\pi}\ln\frac{2N}{\pi}. \tag{4.3}$$

The coefficient C is of an order of 0.1 and decreases with increasing s. The majority of strings have the $O(1/N)$ deviations, and for the extreme strings the deviation probably approaches a constant, $O(1)$. The mean value of the deviations is of an order of $O(\ln N/N)$. Hence, the effects that break the string picture are by no means exponentially small as it was supposed formerly. It is interesting to note, however, that the string deformations affect weakly the integral quantities like the energy (1.2): the finite-size correction to the AF-vacuum energy,

$$\Delta E_N = E_N - E_\infty , \tag{4.4}$$

behaves like $O(1/N)$, just as in the $s=\frac{1}{2}$ case/8/, i.e. the relative correction to eq.(4.2) is only $O(1/N^2)$.

In tables 3 and 4, the N dependence is illustrated for the $s=1$ and $s=\frac{9}{2}$ cases. The numerical data are presented for the minimum, mean, and maximum deviations from exact strings and also for the energy correction (4.4) with proper N factors. One can see that for higher spin, when the "number of degrees of freedom" for string deformations grows large, a quantitative agreement with the empirical formula (4.3) becomes worse although the character of the dependence is retained.

The spin dependence of the string accuracy and of the energy correction is shown in table 5. Through the computer data up to $sN\leqslant 128$, an extrapolation has been made of the leading-asymptotics coefficients for Δ_{min}, Δ_{max}, and ΔE_N as $N\to\infty$. With the growth in s, an improvement of the string accuracy is observed.

Table 3. The string deformations Δ and energy corrections ΔE_N for the AF vacuum at $s=1$

N	$\Delta_{min} N$	$\Delta_{mean} N/\ln N$	Δ_{max}	$\Delta E_N\, N$
6	.248 467 0	.261 891 3	.0966 059 8	-1.316 114
8	.263 256 6	.245 636 3	.0947 895 0	-1.285 479
10	.236 365 7	.235 766 3	.0940 232 9	-1.270 731
26	.226 156 0	.208 477 8	.0931 021 3	-1.245 094
64	.223 030 6	.194 011 3	.0931 293 8	-1.239 031
126	.221 686 6	.186 619 4	.0932 071 1	-1.237 217
128	.221 736 8	.186 471 6	.0932 088 7	-1.237 185

Table 4. The string deformations Δ and energy corrections ΔE_N for the AF vacuum at $s=\frac{9}{2}$

N	$\Delta_{min} N$	$\Delta_{mean} N/\ln N$	Δ_{max}	$\Delta E_N\, N$
6	.0460 430 5	.0819 270 3	.0433 584 9	-2.305 609
8	.0467 326 8	.0755 645 3	.0424 694 1	-2.213 779
16	.0373 252 0	.0654 011 4	.0418 261 7	-2.109 443
28	.0341 454 0	.0601 558 6	.0418 106 4	-2.074 654

Table 5. The spin dependence of the $N\to\infty$ asymptotics for the deviations from perfect strings Δ and the finite-size energy correction ΔE_N in the AF vacuum

s	$\lim_{N\to\infty}(\Delta_{min} N)$	$\lim_{N\to\infty}\Delta_{max}$	$\lim_{N\to\infty}(\Delta E_N\, N)$
1	.220	.0933	-1.235
3/2	.153	.070	-1.484
2	.093	.060	-1.65
5/2	.072	.053	-1.77
3	.053	.049	-1.86
7/2	.043	.046	-1.93
4	.034	.044	-1.98
9/2	.030	.042	-2.03

5. A String-Deformation Estimate for the s=1 Vacuum

In explaining the computer results for finite N, one has to regard a correction to the string-center density (4.1) and the string deformation. While the first problem can be solved by a perfect analogy to the $\zeta=\frac{1}{2}$ case [8], there are a lot of difficulties involved in evaluating the deformations. As an instructive example, we consider the simplest case of the $\zeta=1$ vacuum. To demonstrate our approximation, we present computations in detail.

Instead of perfect 2-strings (1.5), we start with complex-conjugate pairs

$$\lambda_j^{\pm}=x_j\pm i\,\tfrac{1}{2}(\Delta_j+1)\,,\qquad\qquad (j=1\ldots\tfrac{1}{2}N). \tag{5.1}$$

Our aim consists in finding a function $\Delta(x)$ that describes the deviations of eq.(5.1) from eq.(1.5), $\Delta(x_j)=\Delta_j$, in a first nontrivial approximation as $N\to\infty$. We are not going to calculate the density correction which is $O(1/N)$ [8]. Therefore, in the first approximation, we disregard the shifts of roots with respect to their discrete positions,

$$x_j=\frac{1}{\pi}\ln\tan\left[\frac{\pi}{N}\left(j-\tfrac{1}{2}\right)\right], \tag{5.2}$$

that correspond to the σ_∞ density (4.1). One can get an information about these shifts from considering the phase balance in BAE (1.1). We study the modulus squared of eq.(1.1) for $\lambda_j^{\pm}$ (5.1):

$$\left[\frac{x_j\pm i(\frac{1}{2}\Delta_j+\frac{3}{2})}{x_j\pm i(\frac{1}{2}\Delta_j-\frac{1}{2})}\right]^N=\prod_{k=1}^{\frac{1}{2}N}\left[\frac{x_j-x_k\pm i(\frac{1}{2}\Delta_j-\frac{1}{2}\Delta_k+1)}{x_j-x_k\pm i(\frac{1}{2}\Delta_j-\frac{1}{2}\Delta_k-1)}\;\frac{x_j-x_k\pm i(\frac{1}{2}\Delta_j+\frac{1}{2}\Delta_k+2)}{x_j-x_k\pm i(\frac{1}{2}\Delta_j+\frac{1}{2}\Delta_k)}\right]. \tag{5.3}$$

Here and below, the abbreviation $\pm$ implies that factors with both signs should be included. The logarithm of eq.(5.3) is rewritten identically:

$$N\ln\frac{x_j^2+(\frac{1}{2}\Delta_j+\frac{3}{2})^2}{x_j^2+(\frac{1}{2}\Delta_j-\frac{1}{2})^2}=\int dx\left\{\left[\sum_k\delta(x-x_k)-\sigma_\infty(x)\right]+\sigma_\infty(x)\right\} \tag{5.4}$$
$$\times\ln\left\{\frac{x_j-x\pm i[\frac{1}{2}\Delta_j-\frac{1}{2}\Delta(x)+1]}{x_j-x\pm i[\frac{1}{2}\Delta_j-\frac{1}{2}\Delta(x)-1]}\;\frac{x_j-x\pm i[\frac{1}{2}\Delta_j+\frac{1}{2}\Delta(x)+2]}{x_j-x\pm i[\frac{1}{2}\Delta_j+\frac{1}{2}\Delta(x)]}\right\}.$$

In eq.(5.4), we have picked out a correction to the "discreteness" of roots. This correction is small, of the same order as Δ (see below). Therefore, we can omit Δ in calculating the corresponding contribution which is then reduced to

$$\int dx\left[\sum_k\delta(x-x_k)-\sigma_\infty(x)\right]\ln\frac{x_j-x\pm 2i}{x_j-x\pm i\Delta_j}$$
$$=\sum_k\ln\frac{(x_j-x_k)^2+4}{(x_j-x_k)^2+\Delta_j^2}-N\int_0^\infty\frac{dp}{p}\,\frac{2\cos(x_jp)}{2\cosh(\frac{1}{2}p)}\left(e^{-\Delta_jp}-e^{-2p}\right) \tag{5.5}$$
$$\approx 2\ln\frac{2}{\Delta_j}+\sum_{k\neq j}\ln\frac{(x_j-x_k)^2+4}{(x_j-x_k)^2}-N\ln\frac{x_j^2+\frac{9}{4}}{x_j^2+\frac{1}{4}}\;.$$

Here, as well as below, we use formulas from the appendix of ref. [7]. The constant Δ_j in eq.(5.5) ensures a correct passing over the singularity at $x=x_j$.

The main difficulty consists in evaluating the integral, where the imaginary part under logarithm involves a nontrivial dependence $\Delta(x)$. One succeeds in obtaining an analytic expression only by making an expansion in $\Delta(x)$:

$$\ln\left\{\frac{x_j-x\pm i[\frac{1}{2}\Delta_j-\frac{1}{2}\Delta(x)+1]}{x_j-x\pm i[\frac{1}{2}\Delta_j-\frac{1}{2}\Delta(x)-1]}\;\frac{x_j-x\pm i[\frac{1}{2}\Delta_j+\frac{1}{2}\Delta(x)+2]}{x_j-x\pm i[\frac{1}{2}\Delta_j+\frac{1}{2}\Delta(x)]}\right\}$$
$$=\ln\left[\frac{x_j-x\pm i(\frac{1}{2}\Delta_j+1)}{x_j-x\pm i(\frac{1}{2}\Delta_j-1)}\;\frac{x_j-x\pm i(\frac{1}{2}\Delta_j+2)}{x_j-x\pm i\frac{1}{2}\Delta_j}\right] \tag{5.6}$$
$$+\left[-2\frac{1}{(x_j-x)^2+1}+\frac{\frac{1}{2}\Delta_j+2}{(x_j-x)^2+(\frac{1}{2}\Delta_j+2)^2}-\frac{\frac{1}{2}\Delta_j}{(x_j-x)^2+(\frac{1}{2}\Delta_j)^2}\right]\Delta(x)+O\left[\Delta^2(x)\right].$$

The integral of the first term of eq.(5.6) is evaluated to

$$\int dx\,\sigma_\infty(x)\ln\left[\frac{x_j-x\pm i(\frac{1}{2}\Delta_j+1)}{x_j-x\pm i(\frac{1}{2}\Delta_j-1)}\;\frac{x_j-x\pm i(\frac{1}{2}\Delta_j+2)}{x_j-x\pm i\,\frac{1}{2}\Delta_j}\right]$$
$$=N\ln\left[\frac{x_j^2+(\frac{1}{2}\Delta_j+\frac{3}{2})^2}{x_j^2+(\frac{1}{2}\Delta_j-\frac{1}{2})^2}\;\frac{\cosh(\pi x_j)-\sin(\frac{1}{2}\pi\Delta_j)}{\cosh(\pi x_j)+\sin(\frac{1}{2}\pi\Delta_j)}\right] \tag{5.7}$$
$$\approx N\ln\frac{x_j^2+(\frac{1}{2}\Delta_j+\frac{3}{2})^2}{x_j^2+(\frac{1}{2}\Delta_j-\frac{1}{2})^2}-\frac{N}{2\cosh(\pi x_j)}\,2\pi\Delta_j\;.$$

Then, in the first approximation (5.5)-(5.7), eq.(5.4) takes the form

$$0\approx 2\ln\frac{2}{\Delta_j}+\sum_{k\neq j}\ln\frac{(x_j-x_k)^2+4}{(x_j-x_k)^2}-N\ln\frac{x_j^2+\frac{9}{4}}{x_j^2+\frac{1}{4}} \tag{5.8}$$
$$-2\pi\Delta_j\sigma_\infty(x_j)+\int dx\,\Delta(x)\sigma_\infty(x)\left[-2\frac{1}{(x_j-x)^2+1}+\frac{\frac{1}{2}\Delta_j+2}{(x_j-x)^2+(\frac{1}{2}\Delta_j+2)^2}-\frac{\frac{1}{2}\Delta_j}{(x_j-x)^2+(\frac{1}{2}\Delta_j)^2}\right].$$

The computer results of sect. 4 suggest that it is reasonable to look for a self-consistent solution to eq.(5.8) of the form (4.3): $\Delta(x)\sigma_\infty(x)\approx C$. Then, the integral in eq.(5.8) is taken easily, and we obtain a simple equation for C:

$$C+\frac{1}{2\pi}\ln\frac{C}{N}=F_j\,, \tag{5.9}$$

$$F_j = -\frac{1}{2\pi}\ln\cosh(\pi x_j) + \frac{1}{4\pi}\Big[\sum_{k\neq j}\ln\frac{(x_j-x_k)^2+4}{(x_j-x_k)^2} - N\ln\frac{x_j^2+\frac{9}{4}}{x_j^2+\frac{1}{4}}\Big]. \quad (5.10)$$

The consistency condition requires that all the F_j's must be the same. This is not difficult to verify by substituting eq.(5.2) into eq.(5.10).

The results are included into table 6. The F_j values increase monotonically with $|x_j|$, therefore, only the extremes are presented. For computing C we used F_{max} in eq.(5.9). Also the Δ data for the C value obtained are calculated through eq.(4.3). A comparison of tables 6 and 3 shows that our estimate has the least accuracy for extreme strings. This should be expected after the approximations made, (5.5)-(5.7). The errors are connected with the C-order quantities and grow with an increase in deviations.

Table 6. The first approximation of the string deviations for the AF vacuum at $s=1$

N	F_{min}	F_{max}	C	$\Delta_{min}N$	$\Delta_{mean}N/\ln N$	Δ_{max}
6	-0.545	-0.538	.1054	.211	.196	.0703
8	-0.594	-0.585	.1047	.227	.186	.0684
10	-0.633	-0.622	.1043	.209	.180	.0675
26	-0.793	-0.777	.1032	.206	.164	.0659
64	-0.941	-0.921	.1027	.206	.156	.0654
126	-1.052	-1.030	.1025	.205	.153	.0652
128	-1.054	-1.032	.1025	.205	.152	.0652
512	-1.278	-1.253	.1021	.204	.147	.0651

It is worth mentioning that the main effect originates from the difference (5.5) between the sum in eq.(5.10) and its approximation through the integral. The relative error of such an approximation is rather large - in our case, $O(\ln N/N)$ - just on account of the singularity at $x=x_j$. This difficulty should be taken into consideration when analyzing critically the verification of the string hypothesis in ref.[15]. The self-consistent deviations can diminish as $N\to\infty$ for the majority of strings, however, not exponentially.

6. The Lowest Excitations for $s=1$ and $s=\frac{3}{2}$

With the purpose of verifying the string approximation, it is worth comparing numerical data with the theoretical computations based on the perfect-string sea picture[7]. The lowest excitations over the AF vacuum for even N are the triplet ($L=1$) and singlet ($L=0$)

states with $H=2$ holes located symmetrically at a maximum distance from the origin. Using formulas from ref.[7], we obtain the following expansions for the corresponding excitation energies:

$$E_\infty^{(t)} - E_\infty = \frac{\pi^2}{4N}\Big[\frac{1}{s} - \frac{1}{\ln N} + \frac{\ln(8s/\pi)}{\ln^2 N} + O\Big(\frac{1}{\ln^3 N}\Big)\Big], \quad (6.1)$$

$$E_\infty^{(s)} - E_\infty = \frac{\pi^2}{4N}\Big[\frac{1}{s} + \frac{3}{\ln N} - 3\frac{\ln(8s/\pi)}{\ln^2 N} + O\Big(\frac{1}{\ln^3 N}\Big)\Big]. \quad (6.2)$$

Here, E_∞ is the energy of the vacuum (4.2), $E_\infty^{(t)}$ is that of the triplet, and $E_\infty^{(s)}$ of the singlet, in the approximation of exact 2s-strings for $N\to\infty$. Our estimates (6.1) and (6.2) are asymptotically correct for $\ln N \gg s$.

In the $s=1$ case, both states include a real root (the singlet includes a 3-string in addition) at the point $x_0=0$ due to the symmetry of the hole positions $(x_1=-x_2)$. The 3-string at zero remains perfect, so the singlet includes the double zero root. Like for $N=6$ (sect. 3), solutions of this type are physical for any even N [10]. Thus, the structure of the lowest states for $s=1$, just as their momentum (π for the triplet and for the singlet), agree with the string picture.

It is interesting to observe the accuracy of eqs. (6.1) and (6.2) for finite N (table 7). A good agreement proves to take place only if we compare eqs.(6.1) and (6.2) with the triplet $E_N^{(t)}$ and singlet $E_N^{(s)}$ energies at finite N over the "theoretical" vacuum E_∞ rather than the actual one E_N. The reason is the following. Finite-N energies involve a considerable correction $\Delta E_N = O(1/N)$. This correction is caused, on the one hand, by a change in the string-center positions[8], and on the other hand, by the string deformation. For the vacuum state, these two contributions have identical signs and add. In contrast, for the lowest triplet and singlet, a shrinking of strings is observed instead of their stretching: $\Delta^{(t,s)}<0$. As a result, a definite compensation of the contributions to ΔE_N occurs. So, the best agreement is attained when subtracting the E_∞, where no $O(1/N)$ term is present. It should be mentioned also that in formulas (6.1) and (6.2), one has to take equally into account nonleading terms which arise from an expansion in the reciprocal of the hole coordinates $-x_1 = x_2 \approx \frac{1}{\pi}\ln\frac{8sN}{\pi}$.

The comparison of the sea-string deviations for the lowest excitations with the vacuum ones (table 3) gives no reasons to tell of a "string stabilization", an improvement of the string accuracy, after adding holes. At $H=2$, no indications appear that the exponential accuracy can be restored at a large number of excitations, as it would follow from assertions about a thermodynamic limit $H\to\infty$ (see e.g., ref.[15]).

Table 7. The string deformations and comparison of the excitation energy with the perfect-string approximation for the lowest triplets and singlets at $s=1$.

N	$\Delta^{(t)}_{\vert min\vert}N$	$\Delta^{(t)}_{mean}N/\ln N$	$\Delta^{(t)}_{\vert max\vert}$	$[E_N^{(t)}-E_\infty]N$	$[E_\infty^{(t)}-E_\infty]N$
6	-0.565 066 3	-.315 369 5	-.0941 777 2	2.022 494	1.809
8	-0.617 807 3	-.329 113 7	-.0897 069 2	2.038 441	1.814
10	-0.701 019 8	-.342 091 0	-.0874 367 4	2.050 427	1.831
26	-0.969 213 6	-.412 014 9	-.0828 073 7	2.098 167	1.927
64	-1.236 151 9	-.490 365 1	-.0810 064 9	2.138 276	2.007
126	-1.444 001 3	-.553 546 4	-.0801 341 6	2.165 009	2.056
N	$\Delta^{(s)}_{\vert min\vert}N$	$\Delta^{(s)}_{mean}N/\ln N$	$\Delta^{(s)}_{\vert max\vert}$	$[E_N^{(s)}-E_\infty]N$	$[E_\infty^{(s)}-E_\infty]N$
6	-0.306 025 6	-.170 796 1	-.0510 042 6	5.513 167	4.443
8	-0.406 223 0	-.195 352 0	-.0507 778 8	5.181 174	4.427
10	-0.473 175 4	-.214 622 7	-.0504 692 8	4.953 954	4.377
26	-0.779 617 0	-.301 611 0	-.0517 273 6	4.264 123	4.088
64	-1.068 860 7	-.390 250 2	-.0539 063 7	3.888 304	3.847
126	-1.286 528 2	-.459 957 9	-.0554 236 3	3.698 988	3.702

Completing the section, we consider the lowest singlets for $s=\frac{3}{2}$, $N=16$ and $N=18$. These states demonstrate convincingly an inadequacy of the string approach. The "perfect" string hypothesis predicts, on the background of the 3-string sea, a configuration of two additional strings, of length $2s+1=4$ and $2s-1=2$, both at $x_0=0$. Only the 4-string at zero may remain exact, eq.(1.5); the other are deformed.

Neglecting the sea deformation and taking into account the finite hole coordinates $(-x_1=x_2\approx.9)$, we would obtain[7], instead of the 2-string $(\lambda=\pm\frac{1}{2}i)$, a narrow pair (NP) $\lambda\approx\pm.7i$. We can find the analytic solution to the NP equations[7] for arbitrary s, too, if the holes are supposed to go to $\pm\infty$. Then, the $(2s-1)$-string stretches homogeneously by a factor of $2s/(2s-1)$. For $s=\frac{3}{2}$, this leads to $\lambda=\pm\frac{3}{4}i$.

However, at $N=16$, the computer gives $\lambda=\pm.928\ 869\ 091$. Thus, the NP adjoins the sea 3-strings which shrink to .950 234 28 as an average. Also, a considerable curving of strings occurs: the difference between the string-member real parts is not less than half of the imaginary-parts deviation from unit. Still more significant is the string-picture violation for the $s=\frac{3}{2}$, $N=18$ lowest singlet (fig.2). A "collectivization" of the zero root happens between the bent 3-string and the NP which form together a symmetric complex $\pm$ .047 676 33 $\pm$ .947 753 251.

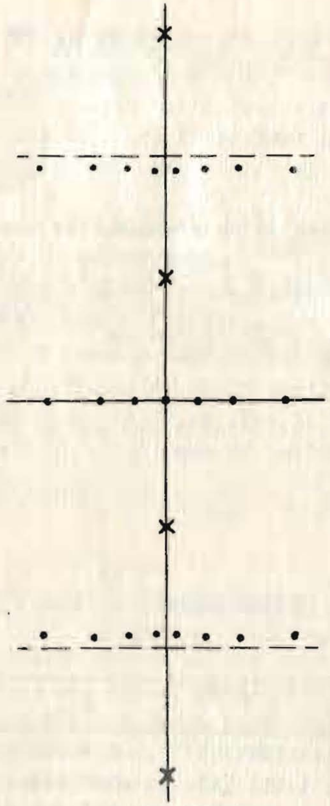

Fig.2

Nevertheless, such an essential deviation from the string picture affects the energy relatively little. The computer values of $[E_N^{(s)}-E_\infty]N$ for $N=16$ and $N=18$ are 3.520 646 and 3.434 215. They are close enough to $[E_\infty^{(s)}-E_\infty]N$ from eq.(6.2), 3.024 and 3.018, although the N values are far from ensuring $\ln N \gg s$. The weak sensitivity of the energy to string deformations seems to be a rather general phenomenon.

7. A Spin Wave for $s=1$, $N=128$

With an assumption that the perfect $2s$-string sea is present, in the $N\to\infty$ limit, physical excitations are realized as holes. A hole at a point x gives additive contributions[7] to the energy and momentum:

$$E^{(h)}(x)=\frac{\pi}{2\cosh(\pi x)}, \qquad (7.1)$$

$$P^{(h)}(x)=-2\,\mathrm{atan}(e^{-\pi x}). \qquad (7.2)$$

From eqs.(7.1) and (7.2), the dispersion law follows that coincides with the $s=\frac{1}{2}$ case[13]:

$$E^{(h)}=-\tfrac{1}{2}\pi\sin P^{(h)}. \qquad (7.3)$$

With the purpose of verifying eq.(7.3), the energies and momenta have been computed for some triplet two-hole states at $s=1$, $N=128$. For convenience of the analysis, the states have been selected in which the second hole is situated at $x_2\approx 0$, i.e. the corresponding to it $Q_2(x_2)=0$ (1.7) is absent in the set of 2-strings. The energy and momentum contributions of this hole are fixed about $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$. In turn, the position of the first hole x_1 can alter and thereby influence the energy and momentum of the state. Using the dispersion relation (7.3) for the first hole and setting $x_2=0$, we can calculate a theoretical excitation energy as a function of the momentum,

$$E^{(*)}=\tfrac{1}{2}\pi\,[1-\cos P_N^{(t)}], \qquad (7.4)$$

and compare eq.(7.4) with the actual value $E_N^{(t)}-E_\infty$

We can also take into account a shift of the second hole. The BAE solution contains a real root between the holes,

$$x_0 = \tfrac{1}{2}(x_1 + x_2). \tag{7.5}$$

From eqs.(7.2) for both holes, it follows that

$$\tan\left[\tfrac{1}{2} P_N^{(t)}\right] = \frac{e^{-\pi x_1} + e^{-\pi x_2}}{e^{-\pi(x_1+x_2)} - 1} \,. \tag{7.6}$$

Solving eqs.(7.5) and (7.6) together allows us to evaluate the sum of the theoretical hole energies (7.1), $E^{(2h)}$, corrected to $x_2 \neq 0$.

The results of the verification are displayed in table 8. The real-root position x_0, the momentum of the state $P_N^{(t)}$, and the energy over the vacuum $E_N^{(t)} - E_\infty$ are the numerical-solution data. On the other hand, there are theoretical predictions based on eqs.(7.1)--(7.3): the simplest estimate $E^{(*)}$ with $x_2 = 0$, eq.(7.4), and the corrected value $E^{(2h)}$ with the x_2 determined by eqs.(7.5), (7.6).

Table 8. The numerical verification of the dispersion law for the triplet spin wave at $s=1$, $N=128$

x_0	$P_N^{(t)} \frac{N}{2\pi}$	$E_N^{(t)} - E_\infty$	$E^{(*)}$	$E^{(2h)}$	x_2
-.959 474 0	32	1.574 455	1.570 8	1.578 383	-.001 541 0
-.332 183 2	37	1.952 364	1.952 5	1.955 245	-.000 580 7
-.220 362 8	42	2.308 973	2.311 3	2.310 978	.000 065 8
-.151 793 6	47	2.621 673	2.625 7	2.622 916	.000 753 4
-.099 868 0	52	2.871 558	2.876 9	2.872 196	.001 688 4
-.055 908 1	57	3.043 589	3.049 8	3.043 808	.003 477 0
-.015 683 2	62	3.127 427	3.134 0	3.127 437	.010 264 3

Even a rough check through $E^{(*)}$ shows a satisfactory agreement. Regarding the second-hole shift improves the agreement twice. The square-mean difference between $E^{(2h)}$ and $E_N^{(t)} - E_\infty$ is about .002. A comparison with smaller-N cases allows us to conclude that this difference does not exceed $O(1/N)$. The string deformations for the states considered are of the same order as for the vacuum (sect. 4) and for the lowest excitations (sect. 6). A new phenomenon is a change of sign of $\Delta(x)$ as x passes through the hole positions. If the number of holes grows large, and they fill the real axis densely enough, this may lead to oscillations and a decrease in $\Delta(x)$ in the thermodynamic limit $H \to \infty$ /15/.

8. The S-Matrix for Some States

Reasoning along the line of ref. /3/, by means of formulas from ref. /7/, we can get an expression for the S-matrix of scattering physical excitations (holes) on each other and on complex-roots configurations. The result can be represented as the following relation:

$$\exp[-iNP^{(h)}(x)] = (-)^{N-1} \prod_j^H S^{-1}(x - x_j) \tag{8.1}$$

$$\times \prod_j^{M_f} \left(e^{-i\pi/s} \frac{\sinh\{\frac{\pi}{2s}[x - x_j \pm i(y_j - s) - \frac{1}{2}i]\}}{\sinh\{\frac{\pi}{2s}[x - x_j \pm i(y_j - s) + \frac{1}{2}i]\}} \right) \prod_j^{M_m + M_w} \frac{x - x_j \pm i(y_j - s) + \frac{1}{2}i}{x - x_j \pm i(y_j - s) - \frac{1}{2}i} \,.$$

Here the notation corresponds to ref. /7/; $P^{(h)}(x)$ is the hole contribution to the momentum, (7.2);

$$S^{-1}(x) = \exp\left\{ i\left[\pi\left(1 - \frac{1}{4s}\right) + \varphi(x)\right]\right\} \frac{\Gamma\left(\frac{1}{2s} - i\frac{x}{2s}\right)\Gamma\left(1 + i\frac{x}{2s}\right)}{\Gamma\left(\frac{1}{2s} + i\frac{x}{2s}\right)\Gamma\left(1 - i\frac{x}{2s}\right)} \,, \tag{8.2}$$

$$\varphi(x) = \int_0^\infty \frac{dp}{p} \sin(xp)\, e^{(s-1)p} \frac{\tanh(\frac{1}{2}p)}{\sinh(sp)} \,. \tag{8.3}$$

The interpretation of the expressions on the r.h.s. of eq.(8.1) as S-matrices can be motivated with the use of the explicit coordinate-BA formula for the Bethe wave vector, generalized to the arbitrary-spin XXX model in ref. /10/. The factors for scattering on free NPs, on multiplets, and on wide pairs agree with the BAE form /7/ for these configurations. It is interesting that holes scatter on themselves, too: $S^{-1}(0) = \exp[i\pi(1 - \frac{1}{4s})]$. As $s \to \infty$ at fixed x, the hole S-matrix (8.2) tends to

$$S^{-1}(x) \xrightarrow{s\to\infty} -\left[\frac{\Gamma(1 + i\frac{1}{2}x)\,\Gamma(\frac{1}{2} - i\frac{1}{2}x)}{\Gamma(1 - i\frac{1}{2}x)\,\Gamma(\frac{1}{2} + i\frac{1}{2}x)}\right]^2 , \tag{8.4}$$

just the $s = \frac{1}{2}$ S-matrix squared. For two-hole states at even N, formulas (8.1)-(8.4) agree with ref. /6/.

For $s=1$, we can express $\varphi(x)$ (8.3) through the known mathematical functions:

$$s=1, \quad x \geqslant 0 \implies$$

$$\varphi(x) = \tfrac{1}{4}\pi + x \ln \tanh\left(\tfrac{1}{2}\pi x\right) + \tfrac{1}{\pi}\left[\mathrm{Li}_2(-e^{-\pi x}) - \mathrm{Li}_2(e^{-\pi x})\right], \tag{8.5}$$

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dx}{x} \ln(1-x) \overset{|x|<1}{=} \sum_{k=1}^\infty \frac{x^k}{k^2} \,. \tag{8.6}$$

Eq.(8.1) is obtained for perfect 2s-strings as $N\to\infty$. It is interesting to check it for finite N. The lowest triplets for $s=1$ (table 7, sect. 6) have been considered. Because the precise hole positions are unknown, we compute them through the excitation energy with eq.(7.1), remembering the symmetry of the states. The energy corrections are sufficiently small, so one may hope that the error involved does not change the answer qualitatively. If the state includes two holes x_1, x_2 and a real root $\frac{1}{2}(x_1+x_2)$, then formulas (8.1) and (8.2) lead to the equation

$$\exp[2i\,\mathrm{atan}(e^{-\pi x_1})] = \exp\{2i\,\mathrm{atan}\exp[-\tfrac{1}{2}\pi(x_1-x_2)]-\tfrac{1}{2}i\pi+i\varphi(x_1-x_2)\}$$
$$\times\frac{\Gamma[1+i\frac{1}{2}(x_1-x_2)]\,\Gamma[\frac{1}{2}-i\frac{1}{2}(x_1-x_2)]}{\Gamma[1-i\frac{1}{2}(x_1-x_2)]\,\Gamma[\frac{1}{2}+i\frac{1}{2}(x_1-x_2)]}\,. \tag{8.7}$$

Using eqs.(8.5),(8.6) and taking the values of the complex Γ function from the tables [17], we can compute the phase difference φ between the l.h.s. and r.h.s. of eq.(8.7).

Table 9. The accuracy of eq.(8.7) for the lowest triplets at $s=1$

N	φ	$\varphi \ln N$
8	.049	.101
10	.048	.110
26	.043	.140
64	.037	.154
126	.033	.160

One may see from table 9 that the phase difference diminishes like $1/\ln N$ with a small coefficient, and the φ value is comparable with Δ_{max} (table 7). An analogous examination can be performed for the other two-hole states of sects. 6,7. For nonsymmetric states the hole positions can be determined through the momentum and the real-root location by solving eqs.(7.5), (7.6). For singlets, besides the terms of eq.(8.7), one should introduce an additional factor on the r.h.s., $[\frac{1}{2}(x_1-x_2)+\frac{1}{2}i]/[\frac{1}{2}(x_1-x_2)-\frac{1}{2}i]$, due to the 3-string. The results are qualitatively the same as in table 9; e.g., for the first state from table 8, $\varphi=-.023$. Thus the scattering phase that has logarithmic deviations is more sensitive to the string deformation than the energy.

Summary

The analysis of the BAE numerical solutions for the model [1] leads to the inference that the string picture gives only a qualitative description of the situation.

At $s=1$, $N=6$, the BA completeness is confirmed, and the hypothesis [2] is checked about the AF ground state: a sea of 2s-strings. For different s, the sea strings deviate from their nominal shape by $O(1/N)$ or more, and one cannot generally neglect these deviations in a further analysis. For the AF vacuum, the string deformations decrease with the growth in s.

The deformations influence the energy rather weakly: the relative correction is $O(1/N^2)$, the absolute one is $O(1/N)$. This phenomenon is confirmed for low-lying excitations too, particularly, in verifying the dispersion law for holes. A similarity is noticeable in the behaviour of the energy corrections to the $s=\frac{1}{2}$ case [8].

The expected diminishing [15] of the sea-string deformation with enlarging the number of holes is not observed for the two-hole states. However, we have found a change of sign of the deformations when passing holes. This may be a reason for a cancellation of the string deviations.

As concerns the true situation of the BAE roots on the complex plane, it can differ rather strongly from the string prescriptions for excited states at high s. This leads to considerable difficulties in searching for new solutions.

In the perfect-sea approximation [7], the phase-balance equation is derived for scattering physical excitations on allowed configurations. At finite N, the equation is satisfied with the $O(1/\ln N)$ accuracy.

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Решение уравнений анзатца Бете для XXX-антиферромагнетика произвольного спина при конечном числе узлов

Изучается точно интегрируемое обобщение изотропной XXX-цепочки Гейзенберга из N узлов на случай произвольного спина S. Получены численные решения уравнений анзатца Бете, отвечающие состоянию антиферромагнитного вакуума /для $SN \le 128$/ и простейшим возбуждениям над ним. В случае s=1 проверена полнота базиса бетевских векторов для N=6 и сделана полуаналитическая оценка структуры вакуумного решения при конечных N. Обнаружено, что для s=1...9/2 имеются отклонения от "струнной" картины решений по крайней мере на $O(1/N)$. Тем не менее, относительная поправка к энергии вакуума и низших возбуждений составляет $O(1/N^2)$. Однако уравнение баланса фаз рассеяния физических возбуждений выполняется лишь с точностью $O(1/\ln N)$.

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Solutions of the Bethe-Ansatz Equations for the XXX Antiferromagnet of Arbitrary Spin with a Finite Number of Sites

We study the exactly integrable generalization of the Heisenberg isotropic XXX chain with N sites for the case of an arbitrary spin S. The numerical solutions of the Bethe-ansatz equations are obtained that correspond to the antiferromagnetic-vacuum state (for $SN \le 128$) and to the simplest excitations over it. In the s=1 case, the completeness of the Bethe-vector set for N=6 is verified, and a semianalytic estimate of the vacuum-solution structure at finite N is performed. For s=1...9/2 it is found that there are at least $O(1/N)$ deviations from the "string" picture of solutions. Nevertheless, the relative correction to the energy of the vacuum and low-lying excitations is $O(1/N^2)$. However, the phase-balance equation for the physical-excitation scattering is satisfied only with the $O(1/\ln N)$ accuracy.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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