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исследований
дубна

E2-85-779

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INFRARED ASYMPTOTICS OF PERTURBATIVE QCD

Renormalization Properties of the Wilson Loops in Higher Orders of Perturbation Theory

Submitted to "ЯФ"

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1985

1. Introduction

One of the most promising approaches to study the infrared behaviour of quantum chromodynamics is the attempt to formulate the non-Abelian gauge theory in the loop-space. Instead of the gauge-dependent entities and Yang-Mills equations one studies in such a formulation the properties of the gauge-invariant functionals

$$W[c] = \frac{1}{N} Tr \langle 0| T P\exp(ig\oint_c dx_\mu \hat{A}_\mu(x))|0\rangle \tag{1}$$

(the Wilson loops) and functional equations for $W[c]$ /1,2/. This approach, however, faces many problems. In particular, $W[c]$ is a nonlocal divergent functional of the gauge potential: it cannot be renormalized by the ordinary R-operation /3/ restricted to the local operators. The renormalization properties of $W[c]$ for an arbitrary contour C were studied, e.g., in refs. /4-8/, and the main conclusion thereof is the following: $W[c]$ is multiplicatively renormalizable to all orders of perturbation theory (PT). More specifically, if the loop is smooth and simple (i.e., without self-intersections) the divergent quantity $W[c]$ can be made finite by expressing it in terms of the renormalized QCD coupling constant and multiplying the result by $\exp(-KL(c))$, where K is a linear divergence and $L(C)$ the length of the contour C. It was also proved that the Wilson loop is multiplicatively renormalized in the case it has a finite number of self-intersection points and cusps corresponding to angles $\{\gamma_i\}$ (the relevant infinities are referred to as cusp singularities).

In the present paper we restrict our analysis to a simple loop (without self-intersections) and study the structure of the cusp singularities in higher orders of PT. In sect. 2 we define the regularization procedure for singularities that appear in a perturbative expansion of eq.(1), we construct there also the subtraction procedure and study some properties of both. In sect. 3 we calculate the cusp anomalous dimension to order α_s and formulate the general scheme for explicit calculations. In sect. 4 we present our results for the two-loop cusp anomalous dimension. In sect. 5 we study the general form of the cusp anomalous dimension in the limit $\gamma \gg 1$ (where γ is the Minkowskian cusp angle) for an arbitrary order of PT. In sect. 6 we analyze some properties of our results for the "timelike" cusp angles related to the Glauber singularities. In conclusion we formulate main results of the paper.

2. Regularization and Subtraction Procedure

If one expands $W[c]$ in the PT series

$$W[c] = 1 + \frac{1}{N}\sum_{n=2}^{\infty}(ig)^n \oint_c dx_1^{\mu_1} \dots \oint_c dx_n^{\mu_n}\, \theta_c(x_1 > \dots > x_n)\, \mathrm{Tr}\, G_{\mu_1 \dots \mu_n}(x_1, \dots, x_n) \qquad (2)$$

there appear the ultraviolet (UV) singularities both from the ultraviolet integration regions for the Green function $G_{\mu_1 \dots \mu_n}(x_1, \dots, x_n)$ and from "contraction into a point" of some set of contour integrations. In what follows it is always implied that all integrals are dimensionally regularized. To analyze the UV divergences of eq.(2), we incorporate the approach /4/ in which the one-dimensional fermions living on the contour C are introduced. In this approach eq.(2) can be rewritten as

$$W[c] = \langle 0 | T \bar{z}(L) z(0) | 0 \rangle$$

$$\cdot \int \mathcal{D}\bar{z}(\sigma)\, \mathcal{D}z(\sigma)\, \mathcal{D}\hat{A}_\mu\, \mathcal{D}c\, \mathcal{D}\bar{c}\, \exp\left(i S_{YM}(A, c, \bar{c}) + i\, S_{eff}(A, z, \bar{z})\right)$$

where the modified action $S_{eff.}$ is

$$S_{eff.} = i \int_0^L d\sigma \left[\bar{z}(\sigma)\partial_\sigma z(\sigma) + ig\, \bar{z}(\sigma)\, \dot{x}_\mu(\sigma)\, \hat{A}_\mu(x(\sigma))\, z(\sigma)\right] \qquad (3)$$

and furthermore the boundary conditions $x_\mu(L) = x_\mu(0)$, $z(L) = -z(0)$ are imposed.

To study the renormalization properties of the local Lagrangian (3), one can apply the ordinary R-operation since the counterterms resulting from its application have (for a smooth simple loop C) structure of the original Lagrangian /4/. In other words, after the renormalization one has

$$A \to A_R = Z_3^{-1/2} A\,,\quad c \to c_R = \tilde{Z}_3^{-1/2} c\,,\quad g \to g_R = Z_1^{-1} Z_3^{3/2} \mu^{-\frac{\epsilon}{2}} g$$
$$\alpha \to \alpha_R = Z_3^{-1}\alpha\,,\quad z(\sigma) \to z_R(\sigma) = (Z_3^F)^{-1/2} z(\sigma)\,,\quad \epsilon = 4 - n \qquad (4)$$

and incorporating in addition the Slavnov-Taylor condition

$$Z_1 / Z_3 = \tilde{Z}_1 / \tilde{Z}_3 = Z_1^F / Z_3^F$$

(where Z_1, $\tilde{Z}_1$, Z_1^F are the renormalization constants for the three-gluon, four-gluon and fermion-gluon vertices, respectively) one obtains the expression for the Wilson loop (defined on a smooth contour) which is finite in the limit $\epsilon \to 0$, $\epsilon = 4 - n$ being the dimensional regularization parameter *). Thus, for a simple smooth contour the renormalized contour average $W_R(C; g_R, \mu)$ is given by

$$W_R(C; g_R, \mu) = \lim_{\epsilon \to 0} \tilde{W}(C; g_R, \mu, \epsilon)\,,\quad \tilde{W}(C; g_R, \mu, \epsilon) = R\, W(C; g, \epsilon)\,, \qquad (5)$$

where W is a regularized, but not renormalized r.h.s. of eq.(2) and μ a subtraction point. In what follows we use the MS subtraction scheme /9/ for which the renormalization constants Z_1^F, Z_3^F are known in Feynman gauge at the two-loop level /7/.

However, if the loop C has a cusp characterized by angle γ, then W_R even after applying to it the R-operation defined by eqs. (4),(5) possesses in addition the cusp singularities resulting from integration in vicinity of the cusp. The relevant divergent subgraphs are those containing a singular point (the cusp) and being two-particle (rainbow) irreducible with respect to the lines corresponding to the one-dimensional fermions. General structure of these subgraphs is shown in fig. 1. To construct the renormalized Wilson loop, we incorporate in this case the subtraction procedure K_γ proposed in refs./2,6/. The action of K_γ on the functional $\tilde{W}(C; g_R, \mu, \epsilon)$ defined in eq.(5) produces the renormalized contour average with the cusp singularities subtracted for each divergent subgraph of fig. 1:

$$W_R(C_\gamma; g_R, \mu, \bar{C}_\gamma) = \lim_{\epsilon \to 0} K_\gamma \tilde{W}(C_\gamma; g_R, \mu, \epsilon) = \lim_{\epsilon \to 0} K_\gamma R\, W(C_\gamma; g, \epsilon)\,, \qquad (6)$$

where $\bar{C}_\gamma$ denotes a generalized subtraction point of the K_γ procedure. The cusp divergences are multiplicatively renormalizable, and the action of K_γ on a loop functional containing a single cusp singularity is defined by

$$K_\gamma \tilde{W}(C_\gamma; g_R, \mu, \epsilon) = Z_{cusp}(g_R, \gamma; \mu, \bar{C}_\gamma, \epsilon)\, \tilde{W}(C_\gamma; g_R, \mu, \epsilon)\,. \qquad (7)$$

The r.h.s. of eq.(6) would be finite if the n-th term of the PT expansion $Z_{cusp}(g_R, \gamma; \mu, \bar{C}_\gamma, \epsilon) = \sum_{n=0}^{\infty} g^{2n} Z_n$ equals (up to finite terms and taken with an opposite sign) the cusp divergence of the whole n-th-order graph contributing to $\tilde{W}$ with all subdivergences subtracted before. Fixing the finite part of Z_n one fixes a particular K_γ subtraction scheme. We shall use the two schemes described below.

The cusp singularities of an arbitrary subgraph are given in the dimensional regularization by a sum of pole terms. If one defines Z_n be given just by the sum of the poles

*) The regularization used in ref./5/ in contradistinction to the dimensional regularization violates the chiral invariance of the Z-fields Lagrangian and requires an additional renormalization of their mass in eq.(4).

$$Z_n(\gamma,\epsilon) = \sum_{k=1}^{n} \epsilon^{-k}\, a_{kn}(\gamma) \tag{8}$$

one arrives at an MS-like scheme to be referred to further as K_γ^{MS}, with the generalized subtraction point $\bar{C}_\gamma$ coinciding with the R-operation parameter μ^2 of eq.(5). Note that the coefficients of expansion (8) as well as Z_n's themselves depend in the K_γ^{MS} scheme only on the cusp angle γ, since the UV singularities of eq.(2) for an arbitrary loop C depend only on the first derivative $\dot{x}_\mu(\sigma)$ /8/, i.e. on the cusp angle in our case. Owing to this important property of the contour averages we can define $-Z_n$ corresponding to some arbitrary graph $\widetilde{W}_n$ ordered along C_γ to be equal to the contribution of the same graph (with the subdivergences subtracted beforehand) but ordered along another fixed contour $\bar{C}_\gamma$ also possessing a single cusp point with angle γ and having the length $1/\mu$. It is easy to realize that in the subtraction scheme K_γ^{MOM} defined in this way (and being an analog of the standard MOM-scheme) the following boundary condition

$$W_R(\bar{C}_\gamma; g_R, \mu, \bar{C}_\gamma) = 1 \tag{9}$$

is fulfilled. Furthermore

$$Z_{cusp}(g_R,\gamma;\mu,\bar{C}_\gamma,\epsilon) = \left(\widetilde{W}(\bar{C}_\gamma; g_R,\mu,\epsilon)\right)^{-1} = \left(\widetilde{W}(1,\gamma,\{\eta\}, g_R,\epsilon)\right)^{-1}, \tag{10}$$

where an arbitrary loop is characterized by its length, cusp angle γ, and by a set of some dimensionless parameters $\{\eta\}$.

The subtraction procedures described above posses all the necessary properties of the R-operation, and as a result, the renormalized loop average, eq.(6), satisfies the following renormalization group equation /2/:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g_R)\frac{\partial}{\partial g_R} + \Gamma_{cusp}(\gamma, g_R)\right) W_R(L\mu,\gamma,\{\eta\}, g_R) = 0, \tag{11}$$

where the anomalous dimension is given by

$$\Gamma_{cusp}(\gamma, g_R) = -\lim_{\epsilon\to 0} \frac{d}{d\ln\mu} \ln \widetilde{W}(C_\gamma; g_R,\mu,\epsilon). \tag{12}$$

As emphasized above, Γ_{cusp} depends only on a single contour parameter, the cusp angle γ.

Our ultimate goal is the calculation of the anomalous dimension in higher orders of PT. Some of its general properties can be established on the basis of the exponentiation theorem of refs./8,10/ having a straightforward generalization onto arbitrary contour averages of eq.(2). The theorem amounts to the statement that the dimensionally

regularized but nonrenormalized contour average $W[C]$ can be represented in the form

$$W(C; g,\epsilon) = \exp\Big(\sum_{n=1}^{\infty} \alpha_s^n \sum_{W\in W(n)} c_n(W)\, F_n(W)\Big), \tag{13}$$

where summation in the exponential is over all diagrams W of the set $W(n)$ of the two-particle (rainbow) irreducible (2PI) contour averages of an n-th order of PT. (It is straightforward to observe that the criterion of the two-particle irreducibility coincides with the definition of "webs" given in ref./10/). Furthermore, $F_n(W)$ denotes a contour intergral present in the expression for W and $c_n(W)$ the "maximally non-Abelian" /10/ or the "colour-connected" /11/ part of the colour factor corresponding to the contribution yielded by the diagram W to the total expression for the contour average, eq.(13). For an n-th order in α_s there exists an estimate

$$c_n(W) \sim C_F N^{n-1} \tag{14}$$

(exact definition of c_n is given in ref./11/). The diagrams whose colour factor does not possess a term of eq.(14) type do not contribute to the sum over W in eq.(13).

Of course, eq.(13) is only a formal relation unless the renormalization prescription and the renormalized analogue of eq.(13) are defined. We are interested in loops possessing the cusp singularities. To this end we apply to both sides of eq.(12) the operation $K_\gamma R$ introduced above. Note now that the exponential factor in eq.(13) is given by a sum of contour integrals. Hence, the transformation given by eq.(4) is sufficient for a consistent renormalization, i.e.,

$$R\,W(C; g,\epsilon) = \exp\Big(\sum_{n=1}^{\infty} \alpha_s^n \sum_{W\in W(n)} c_n(W)\, R F_n(W)\Big).$$

Denoting

$$\alpha_s^n F_n(W) = W_n^{2PI}(W, C_\gamma; g,\epsilon),\quad R W_n^{2PI} = \widetilde{W}_n^{2PI}(W, C_\gamma; g_R,\mu,\epsilon)$$

we find that

$$\widetilde{W}(C_\gamma; g_R,\mu,\epsilon) = \exp\Big(\sum_{n=1}^{\infty}\sum_{W\in W(n)} c_n(W)\, \widetilde{W}_n^{2PI}(W, C_\gamma; g_R,\mu,\epsilon)\Big). \tag{15}$$

Just like in the above discussion, the r.h.s. of eq.(15) possesses the noncompensated UV poles related to the cusp singularities removed by the K_γ-operation. Consider first the action of the K_γ^{MOM}-procedure on eq.(15). By virtue of eq.(10) we have

$$\begin{aligned} K_\gamma^{MOM}\widetilde{W}(C_\gamma; g_R,\mu,\epsilon) &= \left(\widetilde{W}(\bar{C}_\gamma; g_R,\mu,\epsilon)\right)^{-1} \widetilde{W}(C_\gamma; g_R,\mu,\epsilon) \\ &= \exp\Big\{\sum_{n=1}^{\infty}\sum_{W\in W(n)} c_n(W)\left[\widetilde{W}_n^{2PI}(W, C_\gamma; g_R,\mu,\epsilon) - \widetilde{W}_n^{2PI}(W,\bar{C}_\gamma; g_R,\mu,\epsilon)\right]\Big\}, \end{aligned} \tag{16}$$

where it is taken into account that all the topologically equivalent loops possessing the cusp singularity have the same colour factor $c_n(w)$. Note now that the 2PI contour averages present in the exponential factor of eq.(16) have no divergent subgraphs, and the action of the subtraction procedure K_γ^{MOM} in this case amounts to the subtraction of the contribution of the same graph containing a single pole $1/\epsilon$ but ordered along $\bar{c}_\gamma$, i.e.,

$$\widetilde{W}_n^{2PI}(w,c_\gamma;g_R,\mu,\epsilon)-\widetilde{W}_n^{2PI}(w,\bar{c}_\gamma;g_R,\mu,\epsilon)=K_\gamma^{MOM}\,\widetilde{W}_n^{2PI}(w,c_\gamma;g_R,\mu,\epsilon)$$

and, hence

$$W_R(c_\gamma;g_R,\mu,\bar{c}_\gamma)=\lim_{\epsilon\to 0}\exp\Big(\sum_{n=1}^{\infty}\sum_{w\in W(n)}c_n(w)\,K_\gamma^{MOM}R\,W_n^{2PI}(w,c_\gamma;g,\epsilon)\Big) \quad (17)$$
$$=\exp\Big(\sum_{n=1}^{\infty}\sum_{w\in W(n)}c_n(w)\,W_{R,n}^{2PI}(w,c_\gamma;g_R,\mu,\bar{c}_\gamma)\Big).$$

The validity of this important relation in the K_γ^{MS}-scheme is not obvious because of the absence of the analogue of eq.(10) for this scheme. However, it can be demonstrated (the proof is given in the Appendix) that in this case there exists a relation between the renormalization constants for the cusp singularities and the pole part of the 2PI contour averages:

$$Z_{cusp}^{MS}(g_R,\gamma,\mu,\epsilon)=\exp\Big(-\sum_{n=1}^{\infty}\sum_{w\in W(n)}c_n(w)\,\widetilde{W}_n^{2PI}(w,c_\gamma;g_R,\mu,\epsilon)\Big|_{poles}\Big). \quad (18a)$$

As a result,

$$K_\gamma^{MS}R\,W(c_\gamma;g,\epsilon)=\exp\Big(\sum_{n=1}^{\infty}\sum_{w\in W(n)}c_n(w)\,K_\gamma^{MS}R\,W_n^{2PI}(w,c_\gamma;g,\epsilon)\Big). \quad (18b)$$

Thus, the exponentiation theorem (13) is valid for the renormalized contour averages at least within the framework of the two subtraction procedures used in the present paper:

$$W_R(c_\gamma;g_R,\mu,\bar{c}_\gamma)=\exp\big(W_R^{2PI}(c_\gamma;g_R,\mu,\bar{c}_\gamma)\big), \quad (19)$$

where

$$W_R^{2PI}=\lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\sum_{w\in W(n)}c_n(w)\,K_\gamma R\,W_n^{2PI}.$$

Incorporating now the RG equation for the nonrenormalized contour averages one obtains from eq.(19) the equation for W_R^{2PI}:

$$\Big(\mu\frac{\partial}{\partial\mu}+\beta(g_R)\frac{\partial}{\partial g_R}\Big)W_R^{2PI}(c_\gamma;g_R,\mu,\bar{c}_\gamma)=-\Gamma_{cusp}(\gamma,g_R) \quad (20)$$

which has the following important consequences:

a) Using the explicit form of W_R^{2PI} we obtain the relation between the cusp anomalous dimension and the contribution of the 2PI contour integrals

$$\Gamma_{cusp}(\gamma,g_R)=-\sum_{n=1}^{\infty}\sum_{w\in W(n)}c_n(w)\frac{d}{d\ln\mu}W_{R,n}^{2PI}(w,c_\gamma;g_R,\mu,\bar{c}_\gamma) \quad (21)$$
$$=-\lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\sum_{w\in W(n)}c_n(w)\frac{d}{d\ln\mu}\widetilde{W}_n^{2PI}(w,c_\gamma;g_R,\mu,\epsilon).$$

This means that $\Gamma_{cusp}(\gamma,g_R)$, first, as expected, does not depend on the generalized subtraction point $\bar{c}_\gamma$ and, second, in an n-th order of the PT series expansion it contains only the "maximally non-Abelian" /10/ or "colour-connected" /11/ colour factors. In particular, in the QED case eq.(21) contains only the first term of the series.

b) The general form of the solution of eq.(20) is

$$W_R^{2PI}(c_\gamma;g_R,\mu,\bar{c}_\gamma)=W_R^{2PI}(c_\gamma;g_R,\bar{\mu},\bar{c}_\gamma)-\int_{g_R(\bar{\mu})}^{g_R(\mu)}dg\,\frac{\Gamma_{cusp}(\gamma,g)}{\beta(g)}. \quad (22)$$

The expression for W_R^{2PI} contains the μ-dependence in g_R and in a single logarithm $\ln\mu$, and there exists some point $\bar{\mu}$, where

$$W_R^{2PI}(c_\gamma;g_R,\bar{\mu},\bar{c}_\gamma)=0. \quad (23)$$

Hence, the general solution of eq.(11) can be written as

$$W_R(c_\gamma;g_R,\mu,\bar{c}_\gamma)=\exp\Big(-\int_{g_R(\bar{\mu})}^{g_R(\mu)}dg\,\frac{\Gamma_{cusp}(\gamma,g)}{\beta(g)}\Big), \quad (24)$$

where $\bar{\mu}$ is the solution of eq.(23) depending on the contribution to W_R only from 2PI contour averages of eq.(19).

3. One-Loop Approximation

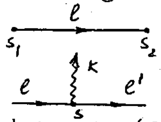
In the preceding section we established that the cusp anomalous dimension depends only on a single characteristics of the loop, the cusp angle. Hence, to calculate it, one can use the simplest loop shown in fig. 2 formed by two lines and closed at infinity. Furthermore, we restrict our analysis to the 2PI graphs yielding a nonzero contribution to the expansion (21).

To begin with, we formulate first the Feynman rules for the modified action eq.(3), in the case of the contour of fig. 2 both in momentum and configuration representations *).

*) Transformation from the configuration space into the momentum one for the one-dimensional fermions is defined by the relation

$$f(\ell)=\int_0^{\infty}d\sigma\,e^{iq\sigma-\epsilon\sigma}f(\sigma)\,,\qquad f(\sigma)=\int_{-\infty}^{\infty}\frac{d\ell}{2\pi}\,e^{-i\ell\sigma}f(\ell).$$

The latter has the dimension $n = 4-\epsilon$ for gluonic lines and is one-dimensional for the Z -fermions. In addition to ordinary QCD gluon vertices eq.(3) contains two other elements, viz., the propagator of one-dimensional fermions and the vertex describing the interaction between gluons and fermions:



$$\theta(s_2 - s_1) \qquad\qquad \frac{i}{\ell + i\epsilon} \tag{25}$$

$$i g n_\mu \hat{A}_\mu(ns) \qquad\qquad i g n_\mu \hat{A}_\mu(k)\,\delta(\ell' + (kn) - \ell)\,,$$

where $n_\mu = (p_\mu, q_\mu)$ is one of the vectors characterizing the directions of the two lines shown in fig. 2. It is worth noting here that fig. 2 may be treated as the amplitude of elastic scattering on a singlet potential of an on -mass-shell ($\ell = 0$) one-dimensional fermion. The contributions to this amplitude are due to both the self-energy corrections $\Sigma(\ell) = \Sigma(0) + \frac{\partial \Sigma(0)}{\partial \ell}\ell + \ldots$ to the fermion lines and vertex corrections $\Gamma(\ell, \ell'; \gamma)$. The latter satisfy the equality

$$\Gamma(0,0;0) = -\left.\frac{\partial \Sigma(\ell)}{\partial \ell}\right|_{\ell=0}$$

following from the gauge properties of eq.(3). Hence, the total result for the angular singularity of the diagram shown in fig. 2 is given by

$$\Gamma(0,0;\gamma) + \left.\frac{\Sigma(\ell)}{\ell}\right|_{\ell=0} = \Gamma(0,0;\gamma) - \Gamma(0,0;0) \tag{26}$$

and to calculate it one can consider only the vertex corrections which in the lowest nontrivial order in the strong coupling constant α_s are determined by the diagram shown in fig. 3a. In the following the $K_\gamma^{MS} R_{MS}$ subtraction procedure is used.

The regularized expression for the contribution of the diagram 3a in Feynman gauge is

$$\mathcal{M}_a = (ig)^2 (pq) c_F \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}} \mu^{4-n} \int_0^\infty ds \int_0^\infty dt \left[(ps+qt)^2 - i0\right]^{1-\frac{n}{2}}.$$

Using the scaling transformation $s+t=\lambda$, $s = \lambda x$ one can rewrite it as

$$\mathcal{M}_a = (ig)^2 (pq) c_F \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}} \mu^{4-n} \int_0^\infty \frac{d\lambda}{\lambda^{n-3}} \int_0^1 dx \left[(px+q\bar{x})^2 - i0\right]^{1-\frac{n}{2}}, \tag{27}$$

where the notation $\bar{x} = 1-x$ is introduced. Generally speaking, the integral over λ appearing in eq.(27) does not exists because it converges on the lower (ultraviolet) limit only if $n < 4$ whereas on the upper (infrared) limit it converges only for $n > 4$. The appearance of the IR divergences is a penalty for the (relative) simplicity of the contour chosen since the infinite length of this

contour just determines the essential scale for wavelengths of the gluons exchanged by the one-dimensional fermions. To define the λ - integral of eq.(27) correctly in the IR region, one can use another regularization scheme different from the dimensional one. More specifically, as an alternative scheme we use the fictitions gluon mass Λ , i.e., the following modification of the gluon propagator in the momentum representation

$$\frac{1}{k^2 + i0} \longrightarrow \frac{1}{k^2 - \Lambda^2 + i0} \tag{28a}$$

or, in the configuration representation

$$\frac{i}{4\pi^2}\frac{1}{-x^2+i0} \to \frac{i}{4\pi^{2-\epsilon}}\left(\frac{\Lambda^2}{-x^2+i0}\right)^{\frac{1}{2}+\frac{\epsilon}{4}} i^{\frac{\epsilon}{2}} \cos\frac{\pi\epsilon}{4}\, K_{1+\frac{\epsilon}{2}}\left(\Lambda\sqrt{-x^2+i0}\right), \tag{28b}$$

where $K_{1+\frac{\epsilon}{2}}$ is the McDonald function.

Calculation in this scheme allows one to define the λ -integral of eq. (27) as

$$\mu^{4-n}\int_0^\infty \frac{d\lambda}{\lambda^{n-3}} = \frac{1}{4-n}\left(\frac{\mu}{\Lambda}\right)^{4-n}, \tag{29}$$

where Λ has just the meaning of the IR cut-off parameter (i.e., the scale, inverse to the contour length L : $\Lambda \sim 1/L$). The x - integral remaining in eq.(27) can be easily obtained by using the following angular variables

$$\frac{x\sqrt{p^2} + \bar{x}\sqrt{q^2}e^{\gamma}}{x\sqrt{p^2} + \bar{x}\sqrt{q^2}e^{-\gamma}} = e^{2\psi}\,, \quad \frac{(pq)}{(px+q\bar{x})^2}\,dx = -\frac{\operatorname{ch}\psi}{\operatorname{sh}\psi}\,d\psi\,. \tag{30}$$

The angle γ between p and q (fig. 2) in the Minkowski space is defined by

$$\operatorname{ch}\gamma = \frac{(pq)}{\sqrt{p^2 q^2}}\,. \tag{31}$$

The corresponding Euclidean results can be obtained by a mere redefinition of the angles

$$\gamma_M = i\,\gamma_E\,. \tag{32}$$

The final result for the renormalized contribution of the diagram 3a (up to the irrelevant finite part) is

$$\mathcal{M}_{a,R}(\gamma, g_R, \mu/\Lambda) = K_\gamma^{MS} R_{MS} \mathcal{M}_a = -\frac{\alpha_s}{2\pi} c_F\, \gamma \operatorname{cth}\gamma \ln\frac{\mu^2}{\Lambda^2}\,, \tag{33}$$

where $\alpha_s = g_R^2/4\pi$. Taking into account also eq.(26) we find the one-loop contribution to the exponential factor in eq. (19)

$$W^{2PI}_{R,\text{one-loop}}(c_\gamma; g_R,\mu) = -\frac{\alpha_s}{2\pi} c_F (\gamma \operatorname{cth}\gamma - 1) \ln\frac{\mu^2}{\Lambda^2} \tag{34}$$

and the one-loop cusp anomalous dimension

$$\Gamma_{\text{cusp, one-loop}}(\gamma, g_R) = \frac{\alpha_s}{\pi} c_F (\gamma \operatorname{cth}\gamma - 1) . \tag{35}$$

4. Two-Loop Approximation

In the α_s^2 order the colour factor entering into eq.(19) is proportional to

$$c_2(W) \sim c_F N \tag{36}$$

and the set of the 2PI vertex diagrams containing the term displayed by eq.(36) in their colour factors is shown in fig. 3(b)-(e). Below we present the results of their calculation in Feynman gauge.

For the graph 3b) we have the expression

$$\mathcal{M}_b = (ig)^4 (pq)^2 c_F \left(c_F - \frac{N}{2}\right) \frac{\Gamma^2(\frac{n}{2}-1)}{16\pi^n} \mu^{2(4-n)} \times \int_0^\infty ds_1 \int_0^{s_1} ds_2 \int_0^\infty ds_4 \int_0^{s_4} ds_3 \left[((ps_1+qs_3)^2 - i0)((ps_2+qs_4)^2 - i0)\right]^{1-\frac{n}{2}}$$

which after the scaling transformation $s_2 = x s_1$, $s_3 = y s_4$, $s_1 + s_4 = \lambda$, $s_1 = \lambda z$ contains an integral over λ that can be defined in a way similar to eq.(29)

$$\mu^{2(4-n)} \int_0^\infty \frac{d\lambda}{\lambda^{2n-7}} = \frac{1}{2(4-n)} \left(\frac{\mu}{\Lambda}\right)^{2(4-n)} .$$

Calculating now the integrals over x and y gives

$$\mathcal{M}_b = \frac{g^4}{16\pi^n} c_F \left(c_F - \frac{N}{2}\right) \frac{\Gamma^2(\frac{n}{2}-1)}{2(4-n)} \left(\frac{\mu}{\Lambda}\right)^{2(4-n)} \frac{\operatorname{ch}^2\gamma}{4 \operatorname{sh}^2\gamma} \times \int_0^1 \frac{dz}{z\bar{z}} \ln\frac{z e^{\gamma} + \bar{z}}{z e^{-\gamma} + \bar{z}} \ln\frac{z + \bar{z} e^{\gamma}}{z + \bar{z} e^{-\gamma}} , \tag{37}$$

Changing further the angular variables according to eq.(30) and applying the subtraction procedure we get the regularized version of eq.(37)

$$\mathcal{M}_{b,R}(\gamma, g_R, \mu/\Lambda) = \left(\frac{\alpha_s}{\pi}\right)^2 c_F \left(c_F - \frac{N}{2}\right) \operatorname{cth}^2\gamma \int_0^\gamma d\psi\, \psi(\gamma-\psi) \operatorname{cth}\psi \ln\frac{\mu^2}{\Lambda^2} . \tag{38}$$

The calculation of the diagram 3c can be most conveniently performed in the momentum representation

$$\mathcal{M}_c = -(ig)^2 c_F\, p_\mu q_\nu \int \frac{d^n k}{(2\pi)^n} \frac{\Pi_{\mu\nu}(k)}{(kp+i0)(kq+i0)} \left(-\frac{i}{k^2}\right)^2 . \tag{39}$$

where

$$\Pi_{\mu\nu}(k) = (g_{\mu\nu} k^2 - k_\mu k_\nu) \frac{g^2 N}{2(2\pi)^n} \frac{i\pi^{\frac{n}{2}}}{(-k^2)^{2-\frac{n}{2}}} \frac{\Gamma(2-\frac{n}{2})\Gamma^2(\frac{n}{2}-1)}{\Gamma(n-2)} \frac{3n-2}{n-1}$$

is the regularized gluon polarization operator possessing a UV pole removed by the renormalization procedure eq.(4) in the MS scheme with

$$Z_3 = 1 + \frac{5}{3} N \frac{g^2}{8\pi^2} \frac{1}{4-n} .$$

After the application of the R-operation to eq.(39) it is necessary to redefine the resulting IR divergent expression according to eq.(28). This gives

$$\mathcal{M}_{c,R}(\gamma, g_R, \mu/\Lambda) = K_\gamma^{MS} R_{MS} \mathcal{M}_c = -\left(\frac{\alpha_s}{\pi}\right)^2 \frac{c_F N}{8} \left[\frac{5}{6} \ln^2\frac{\mu^2}{\Lambda^2} + \frac{31}{9} \ln\frac{\mu^2}{\Lambda^2}\right] \gamma \operatorname{cth}\gamma . \tag{40}$$

The regularized contribution of fig. 3d)

$$\mathcal{M}_d = (ig)^4 (pq) c_F \left(c_F - \frac{N}{2}\right) \int_0^\infty ds_3 \int_{s_3}^\infty ds_2 \int_{s_2}^\infty ds_1 \int_0^\infty ds_4 \frac{\Gamma^2(\frac{n}{2}-1)}{16\pi^n} \left[((ps_2+qs_4)^2 - i0)(p^2(s_1-s_3)^2 - i0)\right]^{1-\frac{n}{2}}$$

after integration over s_1, s_3 contains a UV pole corresponding to the fermion-gluon vertex correction

$$\mathcal{M}_d = \frac{g^2}{16\pi^n} (pq)(p^2)^{\frac{4-n}{2}} c_F \left(c_F - \frac{N}{2}\right) \Gamma^2\left(\frac{n}{2}-1\right) \mu^{2(4-n)} \int_0^\infty ds_2 \int_0^\infty ds_4 \frac{\left[(ps_2+qs_4)^2 - i0\right]^{1-\frac{n}{2}}}{(4-n)(n-3)} s_2^{4-n} \tag{41}$$

and removed by the renormalization of eq.(4) with

$$Z_1^F = 1 + \left(c_F - \frac{N}{2}\right) \frac{g^2}{4\pi^2} \frac{1}{4-n} .$$

Redefining the integral (41) we obtain the result of the action of the R-operation on $\mathcal{M}_d$:

$$R\mathcal{M}_d = \left(\frac{\alpha_s}{\pi}\right)^2 c_F \left(c_F - \frac{N}{2}\right) \left[-\frac{\gamma \operatorname{cth}\gamma}{(4-n)^2} \left(\frac{\mu}{\Lambda}\right)^{4-n} + \frac{1}{2(4-n)^2} \left(\frac{\mu}{\Lambda}\right)^{2(4-n)} \frac{1}{n-3} \int_0^1 \frac{dx\,(pq)(p^2x^2)^{2-\frac{n}{2}}}{\left[(px+q\bar{x})^2 - i0\right]^{3-\frac{n}{2}}}\right] ,$$

where from subtracting the cusp singularities and performing the change of angular variables (30) we find the renormalized expression for the graph 3d):

$$\mathcal{M}_{d,R}(\gamma, g_R, \mu/\Lambda) = \left(\frac{\alpha_s}{\pi}\right)^2 c_F \left(c_F - \frac{N}{2}\right) \left[\frac{1}{8} \gamma \operatorname{cth}\gamma \ln^2\frac{\mu^2}{\Lambda^2} + \frac{1}{2} \gamma \operatorname{cth}\gamma \ln\frac{\mu^2}{\Lambda^2} - \frac{\operatorname{cth}\gamma}{2} \int_0^\gamma d\psi\, \psi \operatorname{cth}\psi \ln\frac{\mu^2}{\Lambda^2}\right] . \tag{42}$$

The most complicated is the calculation of the diagram 3e) containing the three-gluon vertex. We represent the corresponding factor $V_{\mu\nu\rho}(k, \ell, -k-\ell)$ in the form

$$V_{\mu\nu\rho}(k, \ell, -k-\ell) = \bar{V}_{\mu\nu\rho}(k, \ell) + D_{\mu\nu\rho}(k, \ell)$$

(cf. refs. /12,13/), where

$$\bar{V}_{\mu\nu\rho}(k,\ell) = (2\ell+k)_\mu g_{\nu\rho} + 2k_\rho g_{\mu\nu} - 2k_\nu g_{\mu\rho} , \quad D_{\mu\nu\rho}(k,\ell) = -\ell_\nu g_{\mu\rho} - (\ell+k)_\rho g_{\mu\nu}$$

satisfies the simplest Ward identity

$$k_\mu \bar{V}_{\mu\nu\rho}(k, \ell) = \left[(k+\ell)^2 - \ell^2\right] g_{\nu\rho} . \tag{43}$$

The "D-vertex" produces the following contribution

$$\mathcal{M}_{e,D} = -\frac{c_F N}{2} g^4 \int \frac{d^n k}{(2\pi)^n} \frac{d^n \ell}{(2\pi)^n} \frac{1}{k^2 \ell^2 (k+\ell)^2} \left[\frac{2(pq)}{(qk)(pk)} + \frac{(pq)}{(qk)(p\ell)}\right] . \tag{44}$$

The first integral has the structure of that corresponding to fig. 3c) and is easily calculated. The result of applying the R-operation to eq.(44) (after necessary redefinitions) can be written in the form

$$R\, \mathcal{M}_{e,D} = \left(\frac{\alpha_s}{\pi}\right)^2 c_F N \Big[-\frac{\gamma\,\mathrm{cth}\gamma}{4(4-n)^2(n-3)}\left(\frac{\mu}{\Lambda}\right)^{2(4-n)} + \frac{\gamma\,\mathrm{cth}\gamma}{2(4-n)^2}\left(\frac{\mu}{\Lambda}\right)^{4-n}$$
$$+ \frac{1}{16(4-n)}\left(\frac{\mu}{\Lambda}\right)^{2(4-n)} \int_0^1 dy \int_0^1 dx \frac{1}{(px+q\bar{x}y)^2} \ln \frac{(px+q\bar{x}y)^2 + q^2\bar{x}^2 y\bar{y}}{q^2\bar{x}^2 y\bar{y}} \Big] .$$

After the change of angular variables, integration by parts and subtraction of cusp divergences we get

$$K_\gamma R\, \mathcal{M}_{e,D} = \mathcal{M}_{e,D,R}(\gamma, g_R, \mu/\Lambda) = \left(\frac{\alpha_s}{\pi}\right)^2 c_F N \Big[-\frac{\gamma\,\mathrm{cth}\gamma}{16} \ln^2\frac{\mu^2}{\Lambda^2} - \frac{\gamma\,\mathrm{cth}\gamma}{4} \ln\frac{\mu^2}{\Lambda^2}$$
$$+ \frac{\pi^2}{96}\gamma\,\mathrm{cth}\gamma \ln\frac{\mu^2}{\Lambda^2} + \frac{1}{4}\mathrm{cth}\gamma\,\mathrm{sh}^2\gamma \int_0^\gamma d\psi \frac{\psi\,\mathrm{cth}\psi}{\mathrm{sh}^2\gamma - \mathrm{sh}^2\psi} \ln\frac{\mathrm{sh}\gamma}{\mathrm{sh}\psi} \ln\frac{\mu^2}{\Lambda^2} \Big] . \qquad (45)$$

For the contribution of the $\bar{V}$ -term we first rewrite the vertex factor in the form

$$\bar{V}_{\mu\nu\rho}(k,\ell)\, p_\nu p_\rho = p^2 (2\ell + k)_\rho \Big[\Big(g_{\mu\rho} - \frac{p_\mu k_\rho}{(pk)}\Big) + \frac{p_\mu k_\rho}{(pk)} \Big] .$$

Now, using eq.(43), it is easy to see that the second term in this equation (longitudinal with respect to k_ρ) is cancelled by the corresponding contribution from the vertex function of diagram 3d). As a result, the sum of the $\bar{V}$ -contribution from the diagram 3e) and the total contribution of the diagram 3d) after calculating the ℓ - integral does not contain the UV-poles related to the fermion-gluon vertex

$$\mathcal{M}_d + \mathcal{M}_{e,\bar{V}} = i c_F N \frac{g^4}{16\pi^2}\mu^{4-n}\Gamma\left(3-\frac{n}{2}\right)\int\frac{d^n k}{(2\pi)^n}\frac{p^2 q_\mu}{k^2(kp+i0)(kq+i0)}$$
$$\times \int_0^1 d\lambda\, \lambda^{4-n} \int_0^1 dx (1-2x) k_\nu \Big(g_{\mu\nu} - \frac{p_\mu k_\nu}{pk}\Big)\Big[(p\bar{\lambda} - k\lambda x)^2 - k^2\lambda^2 x - i0\Big]^{\frac{n}{2}-3} , \qquad (46)$$

where $\bar{\lambda} = 1-\lambda$. Performing the IR redefinition and subtracting the cusp singularity one can rewrite eq.(46) in the form

$$\mathcal{M}_{d,R}(\gamma, g_R, \mu/\Lambda) + \mathcal{M}_{e,\bar{V},R}(\gamma, g_R, \mu/\Lambda) = -\left(\frac{\alpha_s}{\pi}\right)^2 \frac{c_F N}{4}\left(\frac{\pi^2}{24} + I\right) \ln\frac{\mu^2}{\Lambda^2} , \qquad (47)$$

where the π^2-term corresponds to the $g_{\mu\nu}$ -part of the projector present in eq.(46), while its second term yields

$$I = \mathrm{cth}\gamma \int_0^1 dx\, x(1-2x) \int_0^1 dy\, \bar{y} \int_0^\gamma d\psi \frac{\mathrm{sh}^2\psi}{\mathrm{sh}^2\gamma}\Big(x + \bar{x}y^2\frac{\mathrm{sh}^2\psi}{\mathrm{sh}^2\gamma}\Big)^{-2} = \mathrm{cth}\gamma \int_0^\gamma d\psi \frac{\mathrm{sh}^2\psi}{\mathrm{sh}^2\gamma - \mathrm{sh}^2\psi} \ln\frac{\mathrm{sh}\psi}{\mathrm{sh}\gamma} .$$

Thus, subtracting eqs.(47) and (42) we find the final result for the renormalized amplitude of $\bar{V}$ -part of diagram 3e)

$$\mathcal{M}_{e,\bar{V},R}(\gamma, g_R, \mu/\Lambda) = -\left(\frac{\alpha_s}{\pi}\right)^2 c_F N \Big(-\frac{\gamma\,\mathrm{cth}\gamma}{16}\ln^2\frac{\mu^2}{\Lambda^2} - \frac{\gamma\,\mathrm{cth}\gamma}{4}\ln\frac{\mu^2}{\Lambda^2}$$
$$+ \frac{\pi^2}{96}\ln\frac{\mu^2}{\Lambda^2} + \frac{1}{4}\mathrm{cth}\gamma\,\mathrm{sh}^2\gamma \int_0^\gamma d\psi \frac{1}{\mathrm{sh}^2\gamma - \mathrm{sh}^2\psi}\ln\frac{\mathrm{sh}\gamma}{\mathrm{sh}\psi}\ln\frac{\mu^2}{\Lambda^2}\Big) , \qquad (48)$$

It is worth noting here that the cancellation of double logarithms of μ^2 in the sum of eqs. (48), (45) is a consequence of the fact that the vertex correction of diagram 3e) gives a zero contribution to the renormalization constant Z_1^F of eq.(4).

Substituting the total contribution of the 2PI vertex diagrams 3b)-e) calculated above (with a proper account of combinational fac-

tors) we obtain the following result for the two-loop contribution to the exponential factor of eq.(19):

$$W^{2PI}_{R,\text{two-loop}}(\gamma, g_R, \mu/\Lambda) = -\left(\frac{\alpha_s}{\pi}\right)^2 c_F N \Big[\frac{11}{48}(\gamma\,\mathrm{cth}\gamma - 1)\ln^2\frac{\mu^2}{\Lambda^2}$$
$$+ \Big\{ \frac{67}{72}(\gamma\,\mathrm{cth}\gamma - 1) + \frac{1}{4} - \frac{\pi^2}{48}(\gamma\,\mathrm{cth}\gamma - 1) - \frac{1}{2}\mathrm{cth}\gamma \int_0^\gamma d\psi\, \psi\,\mathrm{cth}\psi$$
$$+ \frac{1}{2}\mathrm{cth}^2\gamma \int_0^\gamma d\psi\, \psi(\gamma-\psi)\mathrm{cth}\psi - \frac{1}{2}\mathrm{cth}\gamma\,\mathrm{sh}^2\gamma \int_0^\gamma d\psi \frac{\psi\,\mathrm{cth}\psi - 1}{\mathrm{sh}^2\gamma - \mathrm{sh}^2\psi}\ln\frac{\mathrm{sh}\gamma}{\mathrm{sh}\psi}\Big\} \ln\frac{\mu^2}{\Lambda^2} \Big] , \qquad (49)$$

For the gauge-invariant cusp anomalous dimension this gives

$$\Gamma^{\text{two-loop}}_{cusp}(\gamma, g_R) = 2\left(\frac{\alpha_s}{\pi}\right)^2 c_F N \Big[\frac{67}{72}(\gamma\,\mathrm{cth}\gamma - 1) + \frac{1}{4} - \frac{\pi^2}{48}(\gamma\,\mathrm{cth}\gamma - 1)$$
$$- \frac{1}{2}\mathrm{cth}\gamma \int_0^\gamma d\psi\, \psi\,\mathrm{cth}\psi + \frac{1}{2}\mathrm{cth}^2\gamma \int_0^\gamma d\psi\, \psi(\gamma-\psi)\mathrm{cth}\psi - \frac{1}{2}\mathrm{cth}\gamma\,\mathrm{sh}^2\gamma \int_0^\gamma d\psi \frac{\psi\,\mathrm{cth}\psi - 1}{\mathrm{sh}^2\gamma - \mathrm{sh}^2\psi}\ln\frac{\mathrm{sh}\gamma}{\mathrm{sh}\psi} \Big] . \qquad (50)$$

Continuation of eqs.(49),(50) into the Euclidean space can be performed by changing the angles as prescribed by eq.(32).

5. Asymptotic Behaviour of the Cusp Anomalous Dimension

In this section we consider the behaviour of the cusp anomalous dimension in two limiting cases for the Minkowskian angle γ :

a) $\gamma \to \infty$; in this case

$$\gamma = \ln\frac{Q^2}{m^2} , \quad Q^2 \gg m^2 , \qquad (51)$$

where

$$Q^2 = -(p-q)^2 , \quad p^2 = q^2 = m^2 .$$

and

b) $\gamma \to 0$; in this case

$$\gamma = \sqrt{Q^2/m^2} , \quad Q^2 \ll m^2 . \qquad (52)$$

In the limit (52) one can represent $\Gamma_{cusp}(\gamma, g_R)$ as

$$\Gamma_{cusp}(\gamma, g_R) \underset{\gamma\to 0}{=} \frac{\alpha_s}{\pi} c_F \frac{\gamma^2}{3} + 2\left(\frac{\alpha_s}{\pi}\right)^2 c_F N \Big[\frac{\gamma^2}{16}\Big(2 - \frac{4\pi^2}{9}\Big) + \frac{67}{72}\frac{\gamma^2}{3} \Big] \qquad (53)$$

and hence Γ_{cusp} vanishes as $\frac{Q^2}{m^2}$ when $\gamma \to 0$.

In the opposite limit (51) the two-loop term of the cusp anomalous dimension

$$\Gamma_{cusp}(\gamma, g_R) \underset{\gamma\to\infty}{=} \frac{\alpha_s}{\pi} c_F \ln\frac{Q^2}{m^2} + 2\left(\frac{\alpha_s}{\pi}\right)^2 c_F N \Big[\frac{67}{72}\ln\frac{Q^2}{m^2} - \frac{\pi^2}{24}\ln\frac{Q^2}{m^2} \Big] \qquad (54)$$

does not contain $\ln\frac{Q^2}{m^2}$ in a power higher than one, because in Feynman gauge the $\ln^3 Q^2/m^2$ -terms due to the D -parts of the diagrams of fig. 3e (see eq.(45)) are cancelled by those due to the diagram 3b (eq. (38)), and the double logarithms due to the $\bar{V}$ -parts of the diagram 3e (eq. (48)) are cancelled by those due to the diagram 3d (eq. (42)). The constant term 67/72 in eqs. (50), (53), (54) is an artifact of the scheme employed because it appears after one applies the R_{MS}-operation to remove the subdivergences from diagrams 3c,d.

It should be noted that the path-ordered exponential corresponding to the path shown in fig. 2 absorbs all the IR singularities of the amplitude of quark scattering by an external colour--singlet potential, the initial quark momentum being p and the momentum transfer Q^2 (see ref. /14/). In the limits $\gamma \to 0, \infty$ this amplitude was calculated in ref. /13/. Our results (eqs. (53),(54)) are in complete agreement with those obtained there.

Let us prove now that the cusp anomalous dimension (eq. (21)) in the limit $\gamma \gg 1$ is linear in $\ln \frac{Q^2}{m^2}$ for an arbitrary order of PT. To this end we incorporate the Feynman rules (eq. (25)) in momentum representation and note that the UV pole related to the cusp singularity of the 2PI contour averages (eq. (19)-(21)) is due to the integration over the UV region of the fermion and gluon momenta while its dependence on γ is determined by integration over small angles between the tangent vectors to the curve on which the fermions are "living" and momenta of the emitted gluons. A general structure of these angular integrals singular in the $Q^2/m^2 \gg 1$ limit can be studied by using standard methods of the factorization technique /15,16/. Note that the Lagrangian of the one-dimensional fermions (eq. (3)) has the following properties: the one-dimensional fermions interacting with gluons cannot change their "helicities", and, hence, the emission of the collinear gluons with physical polarizations by the fermions is suppressed. There exist the whole class of the so-called contour gauges defined by the gauge condition $P\exp(ig\int_c dz_\mu \hat{A}_\mu) = \hat{1}$ for which the gauge potential $\hat{A}_\mu$ is a linear functional of the field strength

$$\hat{A}_\mu(x) = \int_c dz_\nu \frac{\partial z_\rho}{\partial x_\mu} \hat{G}_{\nu\rho}(z;A) \tag{55}$$

and, hence, the gauge field has only physical degrees of freedom. Using the dimensional analysis of ref. /15/ it is now easy to find that in physical gauges the power of the logarithm $\ln \frac{Q^2}{m^2}$ is equal to the number of independent angular integrations, i.e. to unity for the 2PI subgraphs. Thus, UV cusp singularity coefficient for the 2PI diagrams of eq.(21) in the limit $Q^2 \gg m^2$ is a single logarithm $\ln \frac{Q^2}{m^2}$, and hence,

$$\Gamma_{cusp}(\gamma, g_R) = \sum_{n=1}^{\infty} \sum_{w \in W(n)} \alpha_s^n c_n(w) \left(a_n(w) \ln \frac{Q^2}{m^2} + b_n(w) \right). \tag{56}$$

The fact that in Feynman gauge the propagation of the longitudinally polarized gluons is also allowed leads to a more singular structure of angular integrals, However, higher powers appearing in separate diagrams with each other cancel in the gauge-invariant sum (eq. (38)).

It is worth emphasizing here that the cusp anomalous dimension (eqs. (35), (50)) is regular in γ everywhere except the point $\gamma = i\pi$ (or $\gamma = \pi$ in the Euclidean space-time) corresponding to the "collapse" of the contour shown in fig. 2. The contour average (renormalized as well as nonrenormalized) equals 1 in this case. This means that in the regularization scheme used in ref. /5/ the cusp singularity for $\gamma = i\pi$ possesses a linear divergence: $\exp(-K L_{coll})$ (see the Introduction), where L_{coll} is the length of the "collapsed" part of the contour, i.e. in this limiting case the very definition of the cusp anomalous dimension is meaningless.

6. Glauber Regime of the Anomalous Dimension

In the process of our calculations in sect. 4 it has been implied that all the intervals between any two points on the contour shown in fig. 2 have the same sign (this is equivalent to the statement that the three kinematic invariants $p^2, q^2, (pq)$ have the same sign), which allows one to continue analytically the results obtained into the Euclidean space using eq. (32). Note also that if the two points of the contour shown in fig. 2 are separated by a time--like interval, then the path-ordering along the contour coincides with T (or anti-T) ordering of the gauge potentials in eq. (1). Consider now the class of contours for which these properties are not valid, e.g., fig. 2 but with the change

$$q \to -q \, . \tag{57}$$

In the Euclidean space such a transformation leads only to the evident redefinition of the cusp angle

$$\gamma_E \to \pi - \gamma_E \tag{58}$$

in the final results. In the Minkowski space-time eq. (30) is not fulfilled, and moreover, the residues of gluon propagator poles produce nonzero contributions. In particular, the calculation of the diagram 3a contribution with account of eq. (57) gives

$$W^{2PI}_{R,\text{one-loop}}(C_\gamma; g_R, \mu) = -\frac{\alpha_s}{2\pi} C_F \left((\gamma - i\pi) \operatorname{cth}\gamma - 1 \right) \ln \frac{\mu^2}{\Lambda^2} \tag{59a}$$

and, correspondingly,

$$\Gamma_{cusp,\text{one-loop}}(\gamma, g_R) = \frac{\alpha_s}{\pi} C_F \left((\gamma - i\pi) \operatorname{cth}\gamma - 1 \right). \tag{59b}$$

Note that formally these relations can be obtained from eqs. (34), (36) by using eqs. (58) and (32). Let us now find the region of the momentum space of the gluons responsible for the imaginary term in eq. (59). To this end we calculate it in two ways.

Consider the frame where $p_T = q_T = 0$, $p^+ > p^-$, $q^- > q^+$ (with $\ell^\pm = (\ell_0 \pm \ell_3)/\sqrt{2}$ being the light-cone variables). Then the integral contributing to eq.(59) is

$$\mathcal{M} = -ig^2 c_F \int \frac{dk^+ dk^- d^{n-2}k_T}{(2\pi)^n} \frac{(pq)}{(2k^+k^- - k_T^2 - \Lambda^2)(k^+p^- + k^-p^+ - i0)(k^+q^- + k^-q^+ + i0)} . \qquad (60)$$

The position of the poles in the complex k^- plane is shown in fig. 4 with the numbers , representing the denominator factors of eq.(60). Taking the residue of the pole 3 for $k^+ \leqslant 0$ and of 2 for $k^+ > 0$ and using the identity $\frac{1}{x \pm i0} = P\frac{1}{x} \pm i\pi\delta(x)$ to calculate the integral over k^+ it is easy to find that

$$\mathcal{M} = \frac{i}{2} c_F \alpha_s \frac{(pq)}{p^+q^- - p^-q^+} \int_0^\infty \frac{d^{n-2}k_T}{k_T^2 + \Lambda^2} + \ldots = \frac{i}{2} c_F \alpha_s \operatorname{cth}\gamma \int_0^\infty \frac{d^{n-2}k_T}{k_T^2 + \Lambda^2} + \ldots \qquad (61)$$

where the dots stand for the real part of the integral. Thus, the imaginary part of eq.(59) is formed by the region where the gluon momentum is mostly transverse

$$k_T \gg k^+ \sim k^- \to 0 ,$$

i.e., by the Glauber regime for the gluons /17/ in which their emission does not change the virtuality of one-dimensional fermions. As $k_T \to 0$, the poles of eq.(60) in this regime move as indicated by arrows in fig. 4, and the integration contour is eventually pinched in the origin. As a result, there appears the singular imaginary part of eq.(61).

It is also instructive to calculate eq.(60) in the α -representation /3/ in which

$$\mathcal{M} = i\alpha_s c_F \int \frac{\Pi\, d\alpha_\sigma}{\alpha_1^2} \exp\left(-\frac{i}{\alpha_1}(p\alpha_2 - q\alpha_3)^2 - \epsilon(\alpha_1 + \alpha_2 + \alpha_3)\right) \qquad (62)$$

and the main contribution to the integral originates from the region in the α -parameter space where the exponential form

$$A(\alpha, p, q) = \frac{p^2}{\alpha_1}\left(\alpha_2 - \frac{Q^2}{p^2}\alpha_3\right)\left(\alpha_2 - \frac{q^2}{Q^2}\alpha_3\right)$$

vanishes /16/. The requirement $A = 0$ defines a plane

$$S_1 : \ \alpha_2 = \frac{Q^2}{p^2}\alpha_3 \qquad\qquad S_2 : \ \alpha_2 = \frac{q^2}{Q^2}\alpha_3$$

illustrated in fig. 5. According to the results of refs. /16,18/ in the region S_1 and S_2 the "pinch regime" is realized producing a regular contribution of each separate hyperplane of eq.(62). However, in our case there exists also the line $\alpha_2 = \alpha_3 = 0$ common both for S_1 and S_2 where the "intensity" of the pinch singularity is higher and the corresponding integration gives eqs. (61), (59) possessing the divergences both in UV- ($\alpha_1, \alpha_2, \alpha_3 \to 0$ on S_1, S_2) and in IR- regions ($\alpha_1 \to \infty$ on S_1, S_2). Thus, the Glauber regime of the gluon momenta corresponds to a pinch regime in the α -parameter space. One can prove that for the nonzero Glauber regime contribution

in eq.(59) to exist it is necessary (though not sufficient) that the k integration contour be pinched at the origin, while the presence of the pinch regime in the α -space is both necessary and sufficient.

Note now that in the allowed region $\alpha_i \geqslant 0$ the pinch hyperplanes S_1 and S_2 of fig. 5 (as well as the imaginary contributions to eq. (59)) disappear if $p^2, q^2 < 0$, $(pq) > 0$, i.e., if any two points on the contour shown in fig. 2 are in the space-like region and the T-ordering in eq.(1) can be omitted. This means, in particular, that eq. (59) as a function of the vectors p, q has a singularity on the light cone.

In view of the connection between the IR asymptotics of the contour averages and those of the hard QCD processes /14/ we may conclude from the above discussion that contributions due to the Glauber regime (which for some time was thought to be a possible source of contributions violating /17/ the factorization for hard QCD processes) are completely taken into account by the 2-loop cusp anomalous dimension.

7. Conclusions

In the present paper we studied the renormalization properties of the cusp singularities of the contour averages. Incorporating some properties of the $K_\gamma R$ -procedure responsible for subtracting the divergences due to cusp singularities we established the general form of the PT series for the cusp anomalous dimension in the limit of large Minkowskian cusp angles. The two-loop contribution to the cusp anomalous dimension was explicitly calculated and its connection to the nonleading IR behaviour of the quark form factor was demonstrated. We observed also that there exist two sources of non-analyticity of the results obtained with respect to the cusp angle γ : first, for $\gamma_M = i\pi$ ($\gamma_E = \pi$) there appears a linear divergence and second, the Glauber gluons in the Minkowski space give a nonzero contribution to the cusp singularity.

Acknowledgements

We are grateful to A.V.Efremov, S.V.Mikhailov, D.V.Shirkov for stimulating discussions and for the interest to this investigation.

APPENDIX

Let us prove that in the K_γ^{MS} -subtraction scheme there exists a connection between the renormalization constants Z_{cusp} and the pole part of the 2PI contour averages (eq. (18)). To this end we note that in this scheme Z_{cusp} (given by a sum of poles of eq. (8)) can always be represented in the form

$$Z_{cusp}(\gamma,\epsilon)=\exp\Big(-\sum_{\substack{n=1\\ \ell\le n}}^{\infty}\frac{g^{2n}}{\epsilon^{\ell}}\,f_{n\ell}(\gamma)\Big),$$

where $f_{n\ell}(\gamma)$ are some yet unknown functions of the cusp angle γ. Substituting this expression and the one for the regularized contour average determined by the contribution of the 2PI graphs

$$W(C_\gamma;g,\epsilon)=\exp\big(W^{2PI}(C_\gamma;g,\epsilon)\big)=\exp\Big(\sum_{n=1}^{\infty}\frac{(g^2(\mu L)^{\epsilon})^n}{\epsilon}\,\Phi_n(\gamma,\{\eta\},\epsilon)\Big)$$

(where $\Phi_n(\gamma,\{\eta\},\epsilon)$ is some regular function of ϵ) into eq.(7) we find that

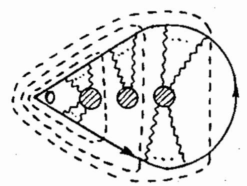
$$W_R(C_\gamma;g_R,\mu)=\lim_{\epsilon\to 0}\exp\Big(\sum_{n=1}^{\infty}\frac{g^{2n}}{\epsilon}(\mu L)^{\epsilon n}\Phi_n(\gamma,\{\eta\},\epsilon)-\sum_{\substack{n=1\\ \ell\le n}}^{\infty}\frac{g^{2n}}{\epsilon^{\ell}}f_{n\ell}(\gamma)\Big).$$

The requirement that the ϵ-poles be absent in both sides of this equation unambiguously fixes the $f_{n\ell}$ coefficients:

$$f_{n\ell}(\gamma)=0\,,\quad \ell>1 \qquad\qquad f_{n1}(\gamma)=\Phi_n(\gamma,\{\eta\},0)\,.$$

Hence, the final expression for the cusp singularity renormalization constant is given by an exponential of the pole part of the 2PI contour averages depending only on the cusp angle γ /8/.

Fig. 1. General structure of the rainbow-irreducible subgraphs. The dashed line denotes the contour integration in a vicinity of the cusp O . The blob denotes an arbitrary gluon subprocess.



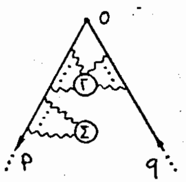


Fig. 2. Self-energy and vertex corrections to the contour average.

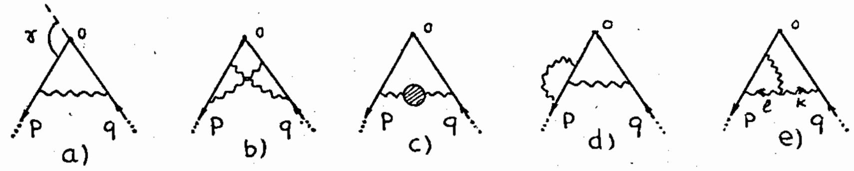
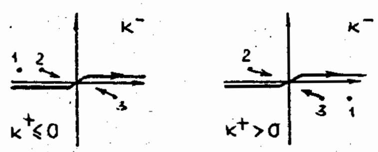


Fig. 3. Total set of diagrams contributing to the two-loop cusp anomalous dimension.

Fig. 4. Position of poles related to the gluon propagators of eq. (60) in the complex k^--plane in the Glauber regime.



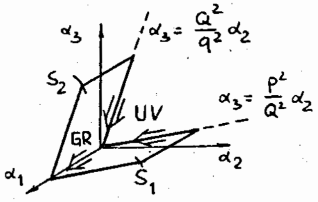


Fig. 5. Pinch hyperplanes S_1 , S_2 in the α-parameter space containing the ultraviolet (UV) and the Glauber (GR) regimes of the gluon momenta.

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Received by Publishing Department
on October 31,1985.

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Инфракрасная асимптотика пертурбативной КХД. Ренормализационные свойства вильсоновских петель в высших порядках теории возмущений

Подход, основанный на использовании формализма контурных средних, применен к исследованию ренормализационных свойств простых вильсоновских петель в высших порядках теории возмущений /ТВ/. Рассмотрены свойства процедуры вычитания угловых особенностей P-упорядоченных экспонент. В порядке α_s^2 вычислена угловая аномальная размерность. Общий вид ее разложения в ряд ТВ найден в пределе больших углов излома контура в пространстве Минковского. Исследована аналитичность полученного выражения по углу излома. Продемонстрирована их связь с нелидирующей инфракрасной асимптотикой кваркового формфактора. Полученные результаты могут применяться для нахождения аномальных размерностей ренормгрупповых уравнений, описывающих ИК поведение пертурбативной КХД.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1985

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Infrared Asymptotics of Perturbative QCD. Renormalization Properties of the Wilson Loops in Higher Orders of Perturbation Theory

The approach based on the use of the loop space formalism is applied for the investigation of renormalization properties of the Wilson loops in higher orders of perturbation theory. The properties are considered of the subtraction procedure for the cusp singularities of path-ordered exponentials . The cusp anomalous dimension is culculated in the order α_s^2. The general form of its perturbative expansion is obtained in the limit of large Minkowskian cusp angles γ. The analyticity properties with respect to γ of the results obtained are investigated. The relation of these results to the nonleading infrared behaviour of the quark form factor is demonstrated. The final expressions can be applied for construction of the anomalous dimensions of the renormalization group equations describing the infrared asymptotics of perturbative QCD.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1985