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EGUCHI-HANSON TYPE METRICS FROM HARMONIC SUPERSPACE

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1. Introduction

It is now widely known that supersymmetric σ-models in six-dimensional ($d=6$) spacetime (or, equivalently, N=2 ones in $d=4$ or N=4 ones in $d=2$ or 3) are in one-to-one correspondence with Riemannian hyper-Kähler "target" manifolds. This was first established by a determination of the restrictions imposed on the general $d=2$ supersymmetric σ-model by additional supersymmetries /1/. It can also be deduced from the general $d=6$ superfield equation for hypermultiplets /2/, but in either case the proof that the target manifold must be hyper-Kähler yields no clues as to how its metric may be constructed. This state of affairs may be contrasted with the N=1 $d=4$ (or N=2, $d=2$ or 3) supersymmetric σ-models for which the target manifold is Kähler. Any Kähler potential furnishes us with a Kähler manifold (at least ignoring global problems) and the corresponding σ-model action has a simple manifestly supersymmetric form as the superspace integral of the Kähler potential /3/. Conversely, any manifestly supersymmetric N=1 $d=4$ σ-model immediately provides us with a Kähler metric. By an obvious extension of this reasoning one would expect a manifestly supersymmetric N=2 ($d=4$ or $d=6$) σ-model action to provide us with a hyper-Kähler metric. But until recently it was not known how to write down a manifestly supersymmetric action for interacting hypermultiplets. This problem was solved by the invention of harmonic superspace /4/. For N=2, $d=4$ supersymmetry, harmonic superspace extends the usual superspace by the inclusion of additional (bosonic) coordinates, just those of $S^2 \simeq SU(2)/U(1)$. There is an invariant subspace of this enlarged superspace that is called analytic superspace. Superfields defined over this subspace are the analogue of chiral superfields of N=1 supersymmetry, and are called analytic superfields. In particular, hypermultiplets are described by one of two types of analytic superfields (which are essentially dual forms of the same multiplet /5/). These are ω, which has zero U(1) charge and q^+ which has U(1) charge one. The general action for interacting hypermultiplets is written as

$$S = \frac{1}{\varkappa^2}\int d\zeta^{(-4)} du\, \mathcal{L}^{(+4)}(q^+, \omega, u^\pm_i, D^{++}q^+, D^{++}\omega, \dots) . \tag{1.1}$$

Here $u^{\pm}_i$ are the isospinor harmonics on SU(2)/U(1), D^{++} is the supercovariant harmonic derivative (it is of dimension zero and U(1) charge 2), and $d\zeta^{(-4)}du$ is the analytic superspace measure U(1) charge -4). The hypermultiplet superfields ω and q^+ are taken to be dimensionless, and the dimension of the coupling constant $\varkappa$ is such as to make the action dimensionless too. Arbitrary powers of ω, q^+ and their D^{++} derivatives may appear in $\mathcal{L}^{(+4)}$ provided the total U(1) charge is +4, and the dimension is zero (without the use of dimensionful parameters: their appearance in $\mathcal{L}^{(+4)}$ would require the appearance of derivatives D_α or ∂_m, and the component action would then contain interactions with higher derivatives). For further details of the formalism and conventions we refer to /4,6,7/.

Each action of the form (1.1) corresponds to some hyper-Kähler manifold; one has only to expand the equations of motion in spinor coordinates θ, omit the fermions, and solve the auxiliary field equations. Substituting the solutions into the original action and integrating over θ and u yields the sought component form of the action, from which the hyper-Kähler metric can be read off. Thus the harmonic superspace approach naturally leads to a new general procedure of obtaining and classifying the hyper-Kähler metrics. The nontrivial step is the solving of the auxiliary field equations which are differential equations on SU(2)/U(1). For a number of simple hypermultiplet actions this can be easily done and the corresponding hyper-Kähler metrics extracted.

The simplest case is the action for one q^+-superfield preserving the U(1) invariance $q^+ \to e^{i\alpha} q^+$, $\bar{q}^+ \to e^{-i\alpha}\bar{q}^+$ and without any explicit dependence on harmonics. The action is $(\lambda > 0)$

$$S_{TN} = -\frac{1}{2\varkappa^2}\int d\zeta^{(-4)}du \left[\bar{q}^+ D^{++} q^+ + \frac{\lambda}{2}(q^+)^2(\bar{q}^+)^2 \right] \qquad (1.2)$$

which describes a σ-model with Euclidean Taub-NUT space as its target manifold /6/.

The principal purpose of this article is to give the action for the N=2 supersymmetric σ-model with the Eguchi-Hanson manifold as its target space. It turns out that this has a particularly simple form in terms of one (real) ω-hypermultiplet. The action is

$$S_{EH} = -\frac{1}{4\varkappa^2}\int d\zeta^{(-4)}du \left[(D^{++}\omega)^2 - \frac{(\xi^{++})^2}{\omega^2} \right]. \qquad (1.3)$$

The dimensionless quantity ξ^{++} is given by

$$\xi^{++} = \xi^{ij} u^+_i u^+_j \qquad (1.4)$$

in terms of the real isovector coupling constant ξ^{ij}. Thus, unlike the Taub-NUT action, the Eguchi-Hanson action contains explicit harmonics. The point to be emphasized is that (1.3) provides the first manifestly supersymmetric formulation of the N=2 E-H σ model in terms of unconstrained N=2 hypermultiplet superfield.

The free action for one ω-hypermultiplet is invariant under the following SU(2) group (besides the automorphism SU(2) which rotates isospin indices of harmonics and component fields):

$$\delta\omega = -c^{+-}\omega + c^{--}D^{++}\omega \;, \quad \delta\omega = \omega'(\zeta,u) - \omega(\zeta,u). \qquad (1.5)$$

where

$$c^{+-} = c^{ij}u^+_i u^-_j \;, \quad c^{--} = c^{ij}u^-_i u^-_j \qquad (1.6)$$

and c^{ij} are group parameters. In the action (1.3), the automorphism SU(2) is explicitly broken to an U(1) subgroup (due to the presence of ξ^{++}) while SU(2) (1.5) is still respected. So the complete internal symmetry of (1.3) is U(2), in accord with the property that E-H metric has U(2) as its isometry group. Moreover, the unique potential for ω that preserves SU(2) (1.5) is ω^{-2} so the form of the action (1.3) for an interacting hypermultiplet is governed by SU(2) invariance *). The SU(2) symmetry (1.5) has a simple interpretation in terms of q^+ superfields as the Pauli-Gursey group that mixes q^+ and $\bar{q}^+$. We shall return to this point later.

The E-H N=2 σ-model was first constructed in component form by Curtright and Freedman /9/, although it was recognized as such later /10/. There is also analogous construction in terms of N=1 superfields in which the Kählerian nature of the E-H metric is manifest /11/ **). The idea of the construction is to couple an N=2 Maxwell supermultiplet to the O(2) current of the free action of two hypermultiplets, and to add a Fayet-Iliopoulos term. In the absence of a kinetic term for the Maxwell supermultiplet, its components are either auxiliary (and can be eliminated by their equations of motion) or act as Lagrange multipliers imposing constraints on the hypermultiplet fields. In the component version of this construction the triplet of auxiliary fields of the Maxwell multiplet imposes an isovector constraint, while the Maxwell gauge invariance may be used to eliminate a further scalar field. Thus by resolving the constraints and fixing the gauge

*) It is worth noting that the action of (1.3) is remarkably similar to the action of one-dimensional conformal quantum mechanics /8/.

**) This construction applies as well to more general hyper-Kähler manifolds /12/.

one arrives at an action for a single self-interacting hypermultiplet. The N=1 superspace version is similar, except that one now needs only a single (complex) chiral superfield constraint. Let us repeat this construction now for a complex N=2 analytic ω superfield *). The action is

$$S = -\frac{1}{4\varkappa^2}\int d\zeta^{(-4)} du \left\{ |\mathcal{D}^{++}\omega|^2 + \xi^{++}V^{++} \right\}, \tag{1.7}$$

where

$$\mathcal{D}^{++}\omega = D^{++}\omega + iV^{++}\omega \tag{1.8}$$

is the U(1) covariant derivative, V^{++} is the N=2 analytic Maxwell prepotential /4/, and the last term in (1.7) is the Fayet-Iliopoulos term. In this version of the construction no constraints are needed; V^{++} may be eliminated by its equation of motion. One then obtains the action (1.3) for a single real ω superfield on choosing the gauge $\omega=\bar{\omega}$. But the advantage of the form of the action (1.7) in terms of a complex ω is that one may choose a different gauge. This will be important later when we attack the problem of reducing the action to component form where the most convenient gauge turns out to be the Wess-Zumino gauge for V^{++} instead of $\omega=\bar{\omega}$.

Although we concentrate here on the simple EH action it is not difficult to generalize the construction to the multi-E-H and other interesting hyper-Kähler metrics, although in this case the action is much simpler in terms of q^+ fields. We shall comment in the conclusions on these generalizations of our results.

2. The q^+ form of S_{EH}, and its symmetries

The construction outlined in the Introduction, i.e., coupling V^{++} to two hypermultiplets and adding a F-I term, can be carried out both in terms of q^+ and ω superfields. In the q -language, e.g.,

$$S = -\frac{1}{2\varkappa^2}\int d\zeta^{(-4)} du \left[\bar{q}_1^+ D^{++} q_1^+ + \bar{q}_2^+ D^{++} q_2^+ + V^{++}(\bar{q}_1^+ q_2^+ - \bar{q}_2^+ q_1^+ + \xi^{++}) \right]. \tag{2.1}$$

This action has the invariances under the following groups:

1) O(2) gauge group

*) Let us remark at this point that the action S_{TN}(1.2) can be obtained by a similar mechanism. One couples V^{++} to the current $q^+\bar{q}^+$ of the free action for one hypermultiplet, and then adds a mass term $m^2(V^{++})^2$ for V^{++}. Then elimination of V^{++} yields the action (1.2).

$$\delta q_1^+ = \lambda(\zeta,u)\cdot q_2^+ , \quad \delta q_2^+ = -\lambda(\zeta,u)\, q_1^+$$
$$\delta V^{++} = D^{++}\lambda(\zeta,u), \tag{2.2}$$

where $\lambda = \bar{\lambda}$.

2) $U(1)_A = U(1)$ subgroup of the rigid SU(2) automorphism group of supersymmetry that leaves $\xi^{++} = \xi^{ij}u_i^+u_j^+$ invariant.

3) $SU(2)_{PG}$ = rigid Pauli-Gursey group. This includes the obvious rigid U(1) invariance of $\bar{q}^+D^{++}q^+$ but this U(1) can be extended to SU(2) by replacing the complex q^+ field by an SU(2) doublet $q_a^+ = (q^+, \bar{q}^+)$ satisfying the pseudo-reality condition /5/

$$\overline{q_a^+} = \varepsilon^{ab} q_b^+ \equiv q^{+a} = (\bar{q}^+, -q^+). \tag{2.3}$$

We shall show later that the action (2.1) reproduces the component result of ref. /9/ after performing the steps of θ integration and the auxiliary field elimination outlined in the introduction. But first we must show that this action is equivalent to the ω -action of (1.7) (and hence to (1.3)). To this end, we consider the change of variables /5/

$$q_a^+(\zeta,u) = u_a^+\,\omega(\zeta,u) + u_a^- f^{++}(\zeta,u). \tag{2.4}$$

Using completeness of the harmonics $u_i^\pm$ this change of variables can be inverted

$$\omega = u_a^- q^{+a} = u_1^-\bar{q}^+ - u_2^- q^+,$$
$$f^{++} = -u_a^+ q^{+a} = -u_1^+\bar{q}^+ + u_2^+ q^+. \tag{2.5}$$

We emphasize that $\bar{q}^+$ is not the simple complex conjugate of q^+, as the bar operation includes an additional SU(2) conjugation /4/. Only the combined conjugation preserves analyticity. Given this, the superfields ω and f^{++} are real.

Making this change of variables for both q_1^+ and q_2^+ in (2.1) one arrives at the expression

$$S = \frac{1}{4\varkappa^2}\int d\zeta^{(-4)} du \left\{ \left[(f_1^{++})^2 + 2 f_1^{++} D^{++}\omega_1 \right] + (1 \leftrightarrow 2) - V^{++}(\omega_1 f_2^{++} - \omega_2 f_1^{++} + \xi^{++}) \right\} \tag{2.6}$$

which is just (1.7) in the first order form. Eliminating V^{++}, f_1^{++} and f_2^{++} and choosing the gauge $\omega_2=0$ we reproduce the action S_{EH} of (1.3). The ω -form of the action is very simple, but the SU(2) invariance is nonmanifest. The form of the SU(2) transformation of ω can be found from the q^+ form of the action: in terms of ω and f^{++} the SU(2) transformations are ($c = (\frac{1}{2}c^{ij}c_{ij})^{1/2}$)

$$\omega_i' = \left(\cos c - \frac{\sin c}{c}\, c^{+-}\right)\omega_i - \frac{\sin c}{c}\, c^{--} f_i^{++} \tag{2.7}$$

$$f_i^{++'} = \frac{\sin c}{c}\cdot c^{++}\omega_i + \left(\cos c + \frac{\sin c}{c} c^{+-}\right) f_i^{++} \qquad (i=1,2)$$

with c^{+-} etc., as in (1.6). After elimination of f_1^{++} and f_2^{++} from (2.6) the laws (2.7) can be rewritten solely in terms of ω's. In addition, if one chooses the gauge $\omega = \bar{\omega}$ one has to combine (2.7) with a compensating SO(2) gauge transformation in order to maintain the gauge condition. The infinitesimal form of this combined transformation is just that of eq.(1.5). Remarkably, it closes without using the ω equation of motion.

It should be emphasized that the main advantage of passing to the ω-form of action is the possibility to explicitly solve the nonlinear constraint on q_i^+ which follows from (2.6) by varying with respect to V^{++}. In the ω-language, this constraint becomes the algebraic equation expressing V^{++} in terms of ω's.

3. Finding the component action

The equation of motion that follows from S_{EH} is

$$(D^{++})^2\omega = (\xi^{++})^2\omega^{-3}. \qquad (3.1)$$

The θ^0 and θ^2 parts of this equation are the equations for the auxiliary fields. In order to reduce the action to component form we would have to solve these equations. In fact, the θ^0-equation is not difficult to solve *), but the θ^2 equations are not so easy. Fortunately we can bypass these difficulties by considering the equivalent q^+-form of the action (2.1) and choosing a different gauge.

For simplicity we shall work in $d=6$. We refer to ref./13/ for details of our $d=6$ conventions and to ref./14/ for a discussion of $d=6$ harmonic superspace, but for the convenience of the reader we summarize here the essentials of $d=6$ spinor algebra and a $d=4 \Leftrightarrow d=6$ dictionary. The Lorentz group in $d=6$ is $SO(5,1) \simeq SU^*(4)$. The spinor coordinates θ_i^α $(\alpha = 1,2,3,4)$ are SU(2) doublets in the $\underline{4}$ representation of SU(4). They are complex but satisfy a pseudo-reality condition. We can construct from them the coordinates $\theta^{\alpha\pm}$ as in $d=4$ which are real with respect to the bar conjugation.

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*) The solution is

$$\omega(x,u) = \alpha\left\{\left(\cos\beta - \frac{\sin\beta}{\beta}\beta^{+-}\right)^2 + \alpha^{-4}\left[\cos\beta\cdot\xi^{+-} + \frac{\sin\beta}{\beta}\left(\beta^{--}\xi^{++} - \beta^{++}\xi^{--}\right)\right]^2\right\}^{1/2},$$

where

$$\alpha = \alpha(x), \quad \beta = \beta(x) = \left(\frac{1}{2}\beta^{ij}\beta_{ij}\right)^{1/2}, \quad \beta^{+-} = \beta^{ij}(x)u_i^+u_j^- \quad \text{etc.}$$

We have

$$\theta^{\alpha+}\theta^{\beta+}\theta^{\gamma+}\theta^{\delta+} = \varepsilon^{\alpha\beta\gamma\delta}(\theta^+)^4 \qquad (\alpha,\beta,\gamma,\delta = 1,2,3,4) \qquad (3.2)$$

which defines $(\theta^+)^4$. The multispinor-tensor correspondence is

$$V_{[\alpha\beta]} \leftrightarrow V_m, \quad V_\alpha{}^\beta\ (V_\alpha{}^\alpha = 0) \leftrightarrow V_{[mn]}$$
$$V_{(\alpha\beta)} \leftrightarrow V^+_{[mnk]}, \quad V^{(\alpha\beta)} \leftrightarrow V^-_{[mnk]} \qquad (m,n,k = 1,2,\ldots,6), \qquad (3.3)$$

where ( ) and [ ] denote symmetrization and antisymmetrization, respectively. The supercovariant harmonic derivative is

$$D^{++} = \partial^{++} - i\theta^{+\alpha}\theta^{+\beta}\partial_{\alpha\beta}. \qquad (3.4)$$

Dimensional reduction from $d=6$ to $d=4$ is achieved by setting

$$\theta^{+\alpha} = \begin{pmatrix}\theta^{+\hat{\alpha}} \\ \bar{\theta}^+_{\hat{\dot\alpha}}\end{pmatrix}, \quad \psi_\alpha = \begin{pmatrix}\psi_{\hat{\alpha}} \\ \bar{\psi}^{\hat{\dot\alpha}}\end{pmatrix} \qquad (\hat{\alpha}, \hat{\dot\alpha} = 1,2)$$

$$V^{[\alpha\beta]} = \begin{pmatrix}\varepsilon^{\hat{\alpha}\hat{\beta}}V & V^{\hat{\alpha}}{}_{\hat{\dot\beta}} \\ V_{\hat{\dot\alpha}}{}^{\hat{\beta}} & \varepsilon_{\hat{\dot\alpha}\hat{\dot\beta}}\bar{V}\end{pmatrix}, \quad V_{[\alpha\beta]} = \begin{pmatrix}-\varepsilon_{\hat{\alpha}\hat{\beta}}V & V_{\hat{\alpha}}{}^{\hat{\dot\beta}} \\ V^{\hat{\dot\alpha}}{}_{\hat{\beta}} & -\varepsilon^{\hat{\dot\alpha}\hat{\dot\beta}}\bar{V}\end{pmatrix}. \qquad (3.5)$$

We shall choose the Wess-Zumino gauge in which V^{++} takes the form

$$V^{++} = i\theta^{+\alpha}\theta^{+\beta}V_{\alpha\beta}(x) + (\theta^+)^4 P^{ij}(x)u_i^-u_j^- \qquad (3.6)$$

(omitting all fermions).

The q^+-superfield has the expansion

$$q_a^+ = F_a^+(x,u) + i\theta^{+\alpha}\theta^{+\beta}A^-_{\alpha\beta\, a}(x,u) + (\theta^+)^4 D_a^{(-3)}(x,u) \qquad (3.7)$$

again omitting all fermions and making explicit P-G SU(2) covariance. The action (2.1) yields the equations of motion

$$D^{++}q_{1a}^+ + V^{++}q_{2a}^+ = 0$$
$$D^{++}q_{2a}^+ - V^{++}q_{1a}^+ = 0 \qquad (3.8)$$

and

$$q_1^{+a}q_{2a}^+ + \xi^{++} = 0. \qquad (3.9)$$

We can now substitute the θ-expansions (3.6) and (3.7) into these equations, and collect powers of θ. At the θ^0 level we get

$$\partial^{++}F_{1a}^+ = \partial^{++}F_{2a}^+ = 0 \quad \Rightarrow \quad \begin{aligned} F_{1a}^+(x,u) &= \phi_{1a}^i(x)u_i^+ \\ F_{2a}^+(x,u) &= \phi_{2a}^i(x)u_i^+ \end{aligned} \qquad (3.10)$$

from (3.8) and then

$$\phi_1^{(ia}\phi_{2a}^{j)} + \xi^{ij} = 0 \qquad (3.11)$$

from (3.9). The latter equation is precisely the constraint of ref. [9]. At the Θ^2 level we find that

$$\left.\begin{array}{l} \partial^{++}A^{-}_{1\alpha\beta\, a} - \partial_{\alpha\beta}F^{+}_{1a} + V_{\alpha\beta}F^{+}_{2a} = 0 \\ \partial^{++}A^{-}_{2\alpha\beta\, a} - \partial_{\alpha\beta}F^{+}_{2a} - V_{\alpha\beta}F^{+}_{1a} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} A^{-\,a}_{1\alpha\beta} = \left(\partial_{\alpha\beta}\phi_1^{ia} - V_{\alpha\beta}\phi_2^{ia}\right)u_i^- \\ A^{-\,a}_{2\alpha\beta} = \left(\partial_{\alpha\beta}\phi_2^{ia} + V_{\alpha\beta}\phi_1^{ia}\right)u_i^- \end{array} \qquad (3.12)$$

from (3.8) and

$$V_{\alpha\beta} = -\frac{\phi_1^{ia}\overleftrightarrow{\partial}_{\alpha\beta}\phi_{2ia}}{\phi_1^{ia}\phi_{1ia} + \phi_2^{ia}\phi_{2ia}} \qquad (3.13)$$

on substituting (3.12) in (3.9). This is all the information we need to obtain the action. After performing the Θ -integration and using (3.10) and (3.12), the action reduces to

$$\begin{aligned} S = \frac{1}{2\varkappa^2}\int d^6x\, du \Big\{ & \left(\partial^{\alpha\beta}\phi_1^{ia} - V^{\alpha\beta}\phi_2^{ia}\right)\left(\partial_{\alpha\beta}\phi_{1a}^{j} - V_{\alpha\beta}\phi_{2a}^{j}\right) + \\ & + \left(\partial^{\alpha\beta}\phi_2^{ia} + V^{\alpha\beta}\phi_1^{ia}\right)\left(\partial_{\alpha\beta}\phi_{2a}^{j} + V_{\alpha\beta}\phi_{1a}^{j}\right)u_i^- u_j^+ + \\ & + P^{ij}\left(\phi_1^{ka}\phi_{2a}^{l} + \xi^{kl}\right)u_i^- u_j^- u_k^+ u_l^+ \Big\}. \end{aligned} \qquad (3.14)$$

Performing the u -integration we obtain (in an obvious notation),

$$S = -\frac{1}{4\varkappa^2}\int d^6x \left\{ \left(\mathcal{D}^{\alpha\beta}\phi^{ia}\right)_1\left(\mathcal{D}_{\alpha\beta}\phi_{ia}\right)_1 + (1\leftrightarrow 2) - \frac{2}{3}P_{ij}\left(\phi_1^{ia}\phi_{2a}^{j} + \xi^{ij}\right)\right\}. \qquad (3.15)$$

We have not yet used the equations (3.13) or (3.11) and we don't have to as they follow from (3.15) by variation with respect to $V_{\alpha\beta}$ and P_{ij}, respectively. Simple dimensional reduction of (3.15) to $d=4$ yields

$$S = \frac{1}{2\varkappa^2}\int d^4x \left\{ \left(D^m\phi^{ia}\right)_1\left(D_m\phi_{ia}\right)_1 + \frac{1}{3}P_{ij}\left(\phi_1^{ia}\phi_{2a}^{j} + \xi^{ij}\right)\right\} \qquad (3.16)$$

which is just the bosonic Lagrangian of ref. [9]. Then by the results of ref. [10] we are assured that our action S_{EH} indeed describes the $N=2$ supersymmetric σ -model with the Eguchi-Hanson gravitational instanton as its target manifold, as claimed.

4. Conclusions

One of the lessons of this work is that constructions of hyper-Kähler manifolds via the construction of component forms of $N=2$ $(d=4)$ supersymmetric σ -models can be carried out with a minor modification

in harmonic superspace, and that the resulting harmonic superspace action encapsulates concisely the properties of the particular hyper-Kähler manifold. Given this, it is not difficult to write down the harmonic superspace actions for multi-Eguchi-Hanson metrics [15]. These can be obtained by (i) coupling n hypermultiplets in the $\underset{\sim}{n}$ representation of SU(n) (ii) gauging the (n-1)-dimensional Abelian group generated by the Cartan subalgebra of SU(n) and (iii) adding (n-1) F-I terms (cf. [16]). In the q^+ language the harmonic superspace action is

$$S_{MEH} = -\frac{1}{2\varkappa^2}\int dZ^{(-4)}du \left\{ \bar{q}^+ D^{++} q^+ + \sum_{\kappa=1}^{n-1} V^{++}_{(\kappa)}\left[\left(\bar{q}^+\lambda_{(\kappa)}q^+\right) + \xi^{++}_{(\kappa)}\right]\right\}, \qquad (4.1)$$

where q^+ is an $\underset{\sim}{n}$ -plet of SU(n) and $\lambda_{(\kappa)}$ are the (anti-Hermitean) generators of the (n-1) Abelian subgroups of SU(n) (Cartan's subalgebra). But only for n=2 case this preserves the $SU(2)_{PG}$ group, and only for n=2 can the action be expressed simply (i.e., without explicit harmonics other than those in ξ^{++}) in terms of ω -superfields.

Having manifestly invariant off-shell N=2 superfield formulation of hypermultiplets we may combine different interactions of them to produce new examples of hyper-Kähler metrics. One can, e.g., take a sum of the interaction terms of S_{EH} and S_{TN} which gives the following action for an O(2) doublet of hypermultiplets q^+_A, $A=1,2$:

$$S = -\frac{1}{2\varkappa^2}\int dZ^{(-4)}du \left\{ \bar{q}^+_A D^{++} q^+_A + V^{++}\left(\varepsilon_{AB}\bar{q}^+_A q^+_B + \xi^{++}\right) + \frac{g}{4}\left(q^+_A\right)^2\left(\bar{q}^+_B\right)^2 \right\}. \qquad (4.2)$$

For $g=0$ this reduces to S_{EH}, so that for $g\neq 0$ it presumably yields a "perturbed" E-H metric that is also hyper-Kahler.*) But it is not simple to reduce (4.2) to the component form.

So far we have discussed only four-dimensional hyper-Kähler manifolds but, of course, one can easily extend our results to the 4n-dimensional Calabi manifolds [17] obtainable by generalizing (2.1) from 2 to n hypermultiplets. The action is

$$S = -\frac{1}{2\varkappa^2}\int dZ^{(-4)}du \left\{ \bar{q}^+_A D^{++} q^+_A + V^{++}\left(\bar{q}^+_A M_{AB} q^+_B + \xi^{++}\right)\right\}, \qquad (4.3)$$

where M is any constant anti-Hermitean $n\times n$ matrix [9]**). The further generalization to multi-Calabi metrics is straightforward.

*) It is interesting that the essentially new metric arises only if the additional q^+ -coupling breaks $SU(2)_{PG}$ -symmetry (it is the case for (4.2)). Adding an $SU(2)_{PG}$ invariant combination $\sim(\varepsilon^{AB}\bar{q}^+_A q^+_B)^2$ would produce no new situation as the resulting action reduces to S_{EH} after a redefinition of q^+ and V^{++}.

**Note that the complete agreement with ref. [9] arises if one gauges the manifest $U(1)$ subgroup of $SU(2)_{PG}$ rather than an $U(1)$ group realized on indices A,B. The resulting theories are equivalent only for $n=2$. In particular, proceeding in this way one may obtain metrics with $U(n)$ group of isometries.

Another line of extension of our results is to couple q^+ to nonabelian V^{++}. For instance, one may gauge the unitary group $U(n)$:

$$S = -\frac{1}{2\varkappa^2}\int dz^{(-4)}du\left\{\bar{q}^+_A D^{++}q^+_A + (V^{++})^i_j\left[\bar{q}^+_A (M^j_i)_{AB}\, q^+_B - \delta^j_i\, \xi^{++}\right]\right\}, \tag{4.4}$$

where q^+_A belong to some representation of $U(n)$ and M^j_i are anti-Hermitean matrices of $U(n)$-generators in this representation. The dimensionality of the latter should be greater than n^2 because the $U(n)$-gauge invariance and constraints following from (4.4) take away from the physical boson sector $4n^2$ real degrees of freedom. The action (4.4) with $A=1,\dots,n(n+m)$ presumably describes the $N=2$ σ-model having as the target space the cotangent bundle of $2nm$ dimensional Grassmann manifold ($m \geq 1$) (cf. /16/). Of course, one may choose as well other groups to be gauged. A common feature is the presence of Abelian U(1)-factors in the gauge group because only with them one can build the Fayet-Iliopoulos terms.

The ω-form of actions (4.3), (4.4) is easy to achieve but in general it involves a complicated dependence on harmonics. As has been mentioned after eq.(4.1), it does not come about only in case of unbroken $SU(2)_{PG}$-symmetry. Consider, e.g., the action (4.3). The $SU(2)_{PG}$-invariance is preserved with antisymmetric (and hence real) M_{AB}, in which case one arrives at the concise expression of (4.3) in terms of ω's:

$$S = -\frac{1}{4\varkappa^2}\int dz^{(-4)}du\left\{(D^{++}\omega)^2 + (D^{++}\omega_{A'})(D^{++}\omega_{A'}) - \frac{(D^{++}\omega_{A'}\widetilde{M}_{A'B'}\cdot\omega_{B'} - \xi^{++})^2}{\omega^2 - \omega_{A'}\widetilde{M}^2_{A'B'}\,\omega_{B'}}\right\} \qquad A',B' = 1,\ ,n-2\,. \tag{4.5}$$

In deriving (4.5), we have put M_{AB} in the block form

$$M = a\left(\begin{array}{c|c} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & 0 \\ \hline 0 & a^{-1/2}\widetilde{M} \end{array}\right)$$

(by means of an orthogonal rotation of q^+_A), absorbed a constant "a" into rescaling of ξ^{++} and chosen the gauge $\omega_2=0$. We observe that the higher dimension generalizations of the E-H action (1.3) contain, along with the potential term $\sim(\xi^{++})^2$, also some superfield metric in the kinetic terms of ω's.

In conclusion, in /6/ and the present paper we have shown that $N=2$ σ-models associated with the most of hyper-Kähler metrics appearing in the previous investigations /1,9-12,16/ admit a simple description in harmonic superspace*). The further steps should be construction of new interesting hyper-Kähler metrics starting from proper q^+ and ω-interactions and classification of all these metrics according to their $N=2$ superfield images /6/. The closely related problem is to understand how the nontrivial global properties of hyper-Kähler manifolds are coded in harmonic superfield Lagrangians. We postpone the detailed analysis of these questions to the future.

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*) $N=2$ σ-models were also constructed with making use of $N=2$ linear (tensor) multiplet /12/. In the forthcoming paper devoted to the harmonic superspace description of this multiplet we demonstrate that its any interaction is equivalent, via a dual transformation, to an interaction of ω superfields from some restricted class.

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Метрики типа Егучи-Хансона и гармоническое суперпространство

Показано, что лагранжева плотность $\mathfrak{L}^{(+4)} = \frac{1}{4}[(D^{++}\omega)^2 - (\xi^{++})^2\omega^{-2}]$ действия в гармоническом суперпространстве для одного самодействующего гипермультиплета описывает N = 2 суперсимметричную гиперкэлерову σ-модель, в которой метрикой многообразия скалярных полей служит инстантонная метрика Егучи-Хансона. Потенциал ω может быть однозначно выделен требованием инвариантности относительно группы Паули-Гюрси SU(2). Мы предлагаем еще несколько действий в гармоническом суперпространстве, которые дают другие типы гиперкэлеровых метрик, включая мульти-инстантоны Егучи-Хансона, метрики Калаби и т.п.

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Eguchi-Hanson Type Metrics from Harmonic Superspace

The harmonic superspace provides a framework for constructing general hyper-Kähler metrics. The simple example of the Taub-NUT manifold was given previously. Here we show that the harmonic superspace Lagrangian $\mathfrak{L}^{(+4)} = \frac{1}{4}[(D^{++}\omega)^2 - (\xi^{++})^2\omega^{-2}]$ for a single interacting hypermultiplet describes an N = 2 supersymmetric hyper-Kähler σ-model with the d = 4 Eguchi-Hanson instanton as its target manifold. The potential ω^{-2} is the unique one invariant with respect to a Pauli-Gursey-like SU(2) group. We present other harmonic superspace actions which we expect to yield some other interesting metrics, including the multi-Eguchi-Hanson and Calabi ones, etc.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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