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A SIMPLE APPROACH TO THE ABJ AXIAL ANOMALY

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The most familiar interpretation of the celebrated ABJ axial anomaly [1] refers to the ultraviolet properties of the VVA triangle graph (see, e.g., [1-3]). On the other hand, dispersive analysis of the triangle diagram, initiated by Dolgov and Zakharov [4], shows that the anomaly is related to a threshold singularity of the corresponding absorptive part. (For a recent comparison of various approaches to axial anomaly, see ref [5]). Dolgov and Zakharov [4], followed by Huang [6] also proposed a new explanation of the physical origin of the ABJ anomaly. These authors suggest that the anomaly manifested in the absorptive part of the triangle graph is an analogy of the "Lee-Nauenberg effect" occurring in the massless limit of spinor electrodynamics: Some total transition rates, corresponding to processes forbidden for massless fermions by chiral invariance, remain finite in the limit $m_f \to 0$ of the massive theory owing to mass singularities of the perturbation expansion [7].

The aim of this note is twofold: First, we examine critically the above-mentioned physical interpretation of the ABJ anomaly. Second, we reformulate in a more satisfactory way the crude analysis of the absorptive part of the VVA triangle graph presented in ref. [6] and arrive thus at a very simple semi-quantitative derivation of the anomaly.

To begin with, we shall briefly review some earlier well-known results which will be needed in our discussion. The contribution of the familiar VVA triangle diagram is formally given by

$$T_{\alpha\mu\nu}(k,p) = \Gamma_{\alpha\mu\nu}(k,p) + \Gamma_{\alpha\nu\mu}(p,k);$$
$$\Gamma_{\alpha\mu\nu}(k,p) = \int \frac{d^4r}{(2\pi)^4} \frac{\mathrm{Tr}((\not r - \not k + m)\gamma_\mu(\not r + m)\gamma_\nu(\not r + \not p + m)\gamma_\alpha\gamma_5}{((r-k)^2 - m^2)(r^2 - m^2)((r+p)^2 - m^2)}. \tag{1}$$

The k, p are external four-momenta outgoing from vector vertices and m is the fermion mass. In what follows, we shall restrict ourselves to k, p such that

$$k^2 = p^2 = 0; \tag{2}$$

the amplitude (1) then corresponds to the creation of a pair of real photons from an external axial-vector source. The requirement of the gauge invariance (vector Ward identities) reads

$$k^{\mu} T_{\alpha\mu\nu}(k, p) = p^{\nu} T_{\alpha\mu\nu}(k, p) = 0 . \tag{3}$$

The integral in (1) is an ill-defined quantity. If one defines it so as to preserve (3), the corresponding axial Ward identity contains an anomalous term [1]:

$$q^{\alpha} T_{\alpha\mu\nu}(k, p) = 2m\, T_{\mu\nu}(k, p) + \frac{1}{2\pi^2} \epsilon_{\mu\nu\rho\sigma} k^{\rho} p^{\sigma}, \tag{4}$$

where $q = k + p$ and the $T_{\mu\nu}(k, p)$, corresponding to the "normal" term may be obtained from (1) by the replacement $\gamma_{\alpha} \to 1$. For k, p satisfying (2), the $T_{\alpha\mu\nu}(k, p)$ restricted by (3) may be written in terms of two invariant amplitudes (form factors) as follows (for a detailed discussion of this point, see, e.g., ref. [8])

$$T_{\alpha\mu\nu}(k, p) = F_1(q^2; m)\, \epsilon_{\mu\nu\rho\sigma} k^{\rho} p^{\sigma} q_{\alpha} + F_2(q^2; m)(p_{\nu} \epsilon_{\alpha\mu\rho\sigma} - k_{\mu} \epsilon_{\alpha\nu\rho\sigma}) k^{\rho} p^{\sigma}. \tag{5}$$

For $T_{\mu\nu}(k, p)$ one has

$$T_{\mu\nu}(k, p) = G(q^2; m)\, \epsilon_{\mu\nu\rho\sigma} k^{\rho} p^{\sigma}. \tag{6}$$

Equation (4) may be then recast as

$$q^2 F_1(q^2 : m) = 2m\, G(q^2; m) + \frac{1}{2\pi^2}. \tag{7}$$

Thus, the second formfactor F_2 is irrelevant for the discussion of axial anomaly and will not be considered further; correspondingly, we shall write simply $F(q^2; m)$ instead of $F_1(q^2; m)$ in the sequel. The absorptive (imaginary) parts of $T_{\alpha\mu\nu}$, F and G, resp., will be denoted consecutively by $A_{\alpha\mu\nu}$, A and B. These are nonvanishing for $q^2 > 4m^2$ and the form factors F, G may be represented by means of unsubtracted dispersion relations

$$F(q^2; m) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{A(t; m)}{t - q^2 - i0} dt,$$

$$G(q^2; m) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{B(t; m)}{t - q^2 - i0} dt. \tag{8}$$

Obviously, A and B fulfill normal axial Ward identity

$$q^2 A(q^2; m) = 2m\, B(q^2; m). \tag{9}$$

From (8) and (9) it is easy to see that the presence of the anomalous term in eq. (7) is equivalent to a "sum rule"

$$\int_{4m^2}^{\infty} A(q^2; m)\, dq^2 = -\frac{1}{2\pi}. \tag{10}$$

Since the normal term on the right-hand side of eq. (7) (or eq. (9) resp.) vanishes for $m \to 0$, one must have

$$A(q^2; m) \xrightarrow[m \to 0]{} -\frac{1}{2\pi} \delta(q^2). \tag{11}$$

The results (10) and (11) have been first obtained by means of an explicit calculation in ref. [4] and later generalized in [9–11].

Let us now discuss an interpretation of the result (11), suggested by Dolgov and Zakharov [4] and later made more explicit by Huang [6], which we have already mentioned in the beginning of our paper. For this purpose, we shall represent the absorptive part $A_{\alpha\mu\nu}(k, p)$ in the following form:

$$2i A_{\alpha\mu\nu}(k, p) = -\frac{1}{32\pi^2} \frac{|\vec{P}|}{\omega} \sum_{s, s'} \int d\Omega\, [\bar{u}(P, s) \gamma_{\alpha} \gamma_5 v(P', s')] \left[\bar{v}(P', s') \gamma_{\nu} \frac{\not{P} - \not{k} + m}{(P - k)^2 - m^2} \gamma_{\mu} u(P, s)\right]. \tag{12}$$

The last result may be obtained either by using directly a unitarity relation for the amplitude $T_{\alpha\mu\nu}(k, p)$, or with the help of the well-known Cutkosky rules [3,12]. (Note that the absorptive parts of $\Gamma_{\alpha\mu\nu}(k, p)$ and $\Gamma_{\alpha\nu\mu}(p, k)$ in (1) are the same). For simplicity, we have written (12) in the c.m. system of the final-state photon pair; the k and p may be conveniently chosen as, e.g., $k = (\omega, 0, 0, \omega)$, $p = (\omega, 0, 0, -\omega)$. Then $q^2 = 4m^2$ and hence we have a restriction $\omega > m$ for $A_{\alpha\mu\nu}$ to be nonvanishing. The four-momenta P, P' satisfy $P^2 = P'^2 = m^2$, $P_0 = P_0' = \omega$, $P + P' = q$. The integration is performed over the directions of $\vec{P}$ (note that $|\vec{P}| = (\omega^2 - m^2)^{1/2}$). The bispinor amplitudes u, v are normalized by $\bar{u}u = 2m$, $\bar{v}v = -2m$. We assume tacitly that (12) is multiplied by polarization vectors ϵ_1, ϵ_2 of physical photons such that $k \cdot \epsilon_1 = p \cdot \epsilon_2 = 0$; it is clear from (5) that only the invariant amplitude $A(q^2; m)$ then survives in (12). In this way, the $A(q^2; m)$ is manifestly expressed in terms of the amplitudes of two successive processes: First, the axial-vector source creates a fermion-antifermion pair, which subsequently is converted into two photons. Following Huang [6], helicity arguments indicate that both processes involved in (12) are forbidden by chiral invariance in the massless limit, if one uses the familiar connection between helicity and chirality for $m = 0$. (It turns out that fermion and antifermion in the intermediate state must have opposite chiralities for $m \to 0$ and therefore the matrix elements in the numerator of eq.(12) vanish in the chiral limit). For a finite m, the considered amplitude should be correspondingly suppressed by powers of m. Thus, naively, one would expect that $A(q^2; m) \to 0$ for $m \to 0$. However, the denominator of the propagator in (12) may also vanish (at least for a particular kinematical configuration) for $m \to 0$ and thereby could in principle compensate the vanishing matrix elements in the numerator. An example of such a situation has been discussed earlier, in a somewhat different context, by Lee and Nauenberg in their classic paper [7]. Now let us see, whether such a compensation may indeed occur in (12). For the invariant amplitude $A(q^2; m) \equiv A(\omega; m)$ we have, according to (5) and (12)

$$A(\omega; m) = -\frac{1}{16\pi}\,\frac{(\omega^2 - m^2)^{1/2}}{\omega^2}\,\frac{m^2}{\omega^2}\int d(\cos\theta)\,\frac{1}{\omega - (\omega^2 - m^2)^{1/2}\cos\theta} = \tag{13a}$$

$$= \frac{1}{16\pi}\,\frac{m^2}{\omega^4}\,\ln\frac{\omega - (\omega^2 - m^2)^{1/2}}{\omega + (\omega^2 - m^2)^{1/2}}\,. \tag{13b}$$

(it is in order to remark here that the corresponding formulae used in ref. [6] slightly differ from ours; we shall clarify this later). In ref. [6] it is correctly argued that no compensation occurs for $\omega > m$ (even without passing to (13b) it is clear that the integral in (13a) may produce only a logarithmic singularity like $\ln(\omega/m)$ which is not sufficient to compensate the suppression factor m^2/ω^2). However, on the threshold, i.e., for $\omega = m$, a "compensation" of the powers of m is claimed to occur since for $\omega = m$ the integrand in (13a) is independent of θ and the denominator then yields the desired negative power of m. Furthermore, the dimension of the $A(q^2; m)$ is $(\text{mass})^{-2}$ (this is obvious from (1) and (5)) and the author of ref. [6] therefore concludes that for $\omega = m$, A should diverge like m^{-2}. Such an intuitive picture is in qualitative agreement with the formal result (11) (though, however, no simple argument is offered in [6] to demonstrate that the integral in (10) is indeed nonzero).

In our opinion, the just described crude argument [6] has two flaws. The first is a technical one: Writing the relevant formulae for $A(\omega; m)$, Huang [6] incorrectly ignored the phase-space factor stemming from the sum over the intermediate states (cf. the factor $|\vec{P}|$ in (12)) so that his formulae (11.46) and (11.50), resp., differ from the correct ones (see (13a) and (13b) resp.)

by an extra factor $(1 - m^2/\omega^2)^{-1/2}$ (cf. also refs. [4,5,8]). Thus, instead of diverging like m^{-2}, the $A(\omega; m)$ is exactly zero for $\omega = m$ (see (13b)). Nevertheless, it is clear that this technicality does not change the qualitative picture, underlying the result (11): Including the correct phase-space factor, the peak of $A(\omega; m)$ (still proportional to m^{-2} for dimensional reasons) is only shifted away from threshold, approaching the origin for $m \to 0$. The second flaw which we want to mention is conceptual: In fact, there is no question of a "conpensation of the powers of m" on the threshold (even if we discard for a moment the vanishing phase-space factor) since there is no physically substantiated (chirality-induced) suppression of the relevant matrix elements in (12): For $\omega = m$ the fermion-antifermion pair in the intermediate state is at rest and thus the helicity - chirality arguments used for $\omega > m$ cannot be applied. Another way to see this is to realize that the relevant "chirality-suppression factor" is not simply m but rather m/ω; however, $m/\omega = 1$ on the threshold. Of course, the matrix elements in (12) may be proportional to m for $\omega = m$, but this is entirely due to our normalization conventions. It is evident that the threshold behaviour of amplitude in question is determined completely by trivial dimensional arguments, as the only available energy scale is then m.

In this sense, the earlier interpretation [4,6] of the result (11) as an analogy of the afore-mentioned Lee-Nauenberg (LN) effect [7] seems to be untenable.

The LN phenomenon [7] is a true effect of mass singularities of perturbation theory, compensating the suppression of an amplitude due to approximate chiral invariance. For completeness, we shall illustrate this point on a specific example. Let us consider the two-photon annihilation of an electron-positron pair. Again, we shall work in the c.m. system and keep the notation employed in equations (12), (13). Suppose that both electron and positron have positive helicity and both final-state photons have right-handed circular polarization. Then in the massless limit the initial-state particles have opposite chiralities; hence, for $m \ll \omega$ we may expect a suppression of the relevant matrix element by a power of the ratio m/ω. Indeed, using the general formulae of McMaster [13] (see also [14]), we obtain for the corresponding differential cross section:

$$\frac{d\sigma}{4\Omega} = \frac{\alpha^2}{4\beta\omega^2(1 - \beta^2\cos^2\theta)^2}(1 - \beta^2)(1 + \beta)^2, \tag{14}$$

where α is the fine structure constant and $\beta = |\vec{P}|/\omega = (1 - m^2/\omega^2)^{1/2}$ is the electron velocity. It is seen that the suppression factor $1 - \beta^2 = m^2/\omega^2$ is compensated by the denominator for $\theta = 0$ or π.

Integration of the factor $(1-\beta^2\cos^2\theta)^{-2}$ in the vicinity of these points produces a singularity of the type m^{-2} for $m \to 0$. The total cross section, corresponding to (14) should therefore be nonzero in the massless limit. To verify this, we integrate (14) over the angles and obtain

$$\sigma_{tot}(\omega; m) = \frac{\pi\alpha^2}{2\beta\omega^2}(1+\beta)^2\left(1+\frac{1-\beta^2}{2\beta}\ln\frac{1+\beta}{1-\beta}\right); \quad (15)$$

for $m \to 0$ $(\beta \to 1)$ one then gets from (15)

$$\lim_{m \to 0} \sigma_{tot}(\omega; m) = \frac{2\pi\alpha^2}{\omega^2}. \quad (16)$$

Thus, we see that the compensation of the suppression factor m^2/ω^2 manifested in (14) through (16) is due to "collinear mass singularities" familiar also from other applications of perturbative quantum field theory.

In the rest of this paper we shall present a very simple semi-quantitative derivation of equation (10) or (11), resp., which we believe is more satisfactory than the crude argument given in /6/. Let us start with the following simple observation: Since the dimension of $A(q^2; m)$ is $(\text{mass})^{-2}$ the integral on the left-hand side of eq.(10) is dimensionless. However, the result of the integration may only depend on m and hence it must be a pure number constant C. It remains to show that $C \neq 0$. To this end, we shall employ the Ward identity (9). A little algebra reveals that $B(q^2; m)$ is proportional to (imaginary part of) the contribution of a scalar triangle diagram (which corresponds to replacement of the numerator in (1) by unity):

$$B(q^2; m) = \text{const. } m\, B^{(scal.)}(q^2; m). \quad (17)$$

Equations (9) and (17) thus imply

$$A(q^2; m) = \text{const. } \frac{m^2}{q^2} B^{(scal.)}(q^2; m). \quad (18)$$

The quantity $B^{(scal.)}(q^2; m)$ may be estimated with the help of a unitarity relation analogous to eq.(12). In the scalar case one may write straightforwardly

$$B^{(scal.)}(q^2; m) = f(q^2; m) \int d\Omega \frac{1}{(k-P)^2 - m^2}, \quad (19)$$

where the function $f(q^2; m)$ includes all numerical constants, normalization factors, and the phase-space factor descending

from the sum over the intermediate states. Since $P^2 = m^2$ and $k^2 = 0$, the energy-momentum conservation obviously implies that $P-k$ cannot be on mass shell. That is, the denominator $(k-P)^2 - m^2$ is different from zero for any considered values of q^2 and Ω involved in (19) and therefore (as a continuous function of these variables) does not change its sign. This in turn means that the integrand in eq.(19) has a constant sign over the whole range of relevant variables. Beside that, the $f(q^2; m)$ also does not change its sign. Hence, substituting (18) and (19) into the left-hand side of eq.(10), one may immediately conclude that

$$\int_{4m^2}^{\infty} A(q^2; m)\, dq^2 = C \neq 0$$

and this is the desired result. Of course, the precise numerical value of C can only be determined by means of an explicit calculation.

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Простой подход к ABJ аксиальной аномалии

Приводится простой полуколичественный вывод аксиальной аномалии Адлера, Белла и Джекива, основанный на изучении свойств абсорбтивной части треугольной диаграммы типа VVA и дисперсионных соотношениях. Существенными для вывода являются нормальное тождество Уорда для мнимой части треугольной диаграммы, размерный анализ и унитарность. Подвергается критике объяснение физической природы аксиальной аномалии, предложенное раньше другими авторами в рамках такого дисперсионного подхода. В частности, показано, что интерпретация треугольной аномалии как аналога эффекта Ли - Науенберга, имеющего место в безмассовом пределе спинорной электродинамики, является необоснованной.

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A Simple Approach to the ABJ Axial Anomaly

A very simple semi-quantitative derivation of the Adler-Bell-Jackiw (ABJ) axial anomaly is given, based on an investigation of the absorptive part of the VVA triangle graph and dispersion relations. Essential ingredients of our discussion are: normal Ward identities for the absorptive part of the relevant diagram, dimensional analysis, unitarity, and energy-momentum conservation. An explanation of the physical origin of axial anomaly, proposed in some earlier treatments within such a dispersive framework, is critically examined. In particular, the interpretation of the ABJ anomaly as an analogy of the Lee-Nauenberg effect occurring in the massless limit of spinor electrodynamics is shown to be fallacious.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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