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E2-85-683

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OPEN QUANTUM SYSTEMS AND FEYNMAN INTEGRALS: SOME PROBLEMS

1985

An open problem is often more important than a solved one as far as it represents a challenge. This is why lists of open problems are popular, not only the big ones which tend to cover an extensive field*), but also the smaller ones collected more or less occasionally while following some other purpose. The present paper**) offers a short list of problems related to the author's recent monograph /2/ on open quantum systems and Feynman path integrals. It does not pretend to cover the scope of this book, rather it selects five different regions within it looking for unanswered questions they can contain.

1. Continual observation of an unstable system

First of all, we mention how such a situation is described by means of the limit of a sequence of successive measurements. Consider a system whose undisturbed evolution is governed by $U_t = e^{-iHt}$ and assume that it suffers a sequence of yes-no experiments at the times $0 = \tau_0 < < \tau_1 < \ldots < \tau_n = t$, each of them being characterized by a projection E_u on the state space $\mathcal{H}$ of the system. This scheme applies particularly to non-decay measurements if the system under consideration is formed by an unstable system together with its decay products and the subspace $E_u\mathcal{H}$ refers to the unstable system alone. For the used partition $\sigma = \{\tau_j\}$ of $[0,t]$, we denote

$$\delta_k = \tau_{k+1} - \tau_k \quad , \quad \delta(\sigma) = \max_{0 \le k \le n-1} \delta_k \ . \qquad (1.1)$$

If one takes only the positive outcomes of the measurement into account, then the evolution is

*) Beside the classical problem lists like that one of Hilbert, let us mention the recent paper by Simon/1/.

**) This paper summarizes a talk given on the symposium "Theory of elementary particles and modern methods of mathematical physics" held in Alšovice, Czechoslovakia, in June 1985.

$$U(t,0;E_u,\sigma) := E_u e^{-iH\delta_{n-1}} E_u \dots E_u e^{-iH\delta_0} E_u \qquad (1.2)$$

so it seems natural to associate the limit

$$U(t,0;E_u) := \underset{\delta(\sigma)\to 0}{\text{s-lim}}\ U(t,0;E_u,\sigma) \qquad (1.3)$$

with the system on which the non-decay measurement is performed perpetually. In that case, however, one is confronted with the fact which is known as the "Zeno paradox" (cf.Ref.2, Sec.2.4, and Refs.3-5) :

<u>Theorem</u> : Suppose that $U(t,0;E_u)$ exists for all $t > 0$, and H is self-adjoint and semibounded. Moreover, let an antiunitary θ exist such that $\theta E_u \theta^{-1} = E_u$ and $\theta U_t \theta^{-1} = U_{-t}$ for all $t \in \mathbb{R}$. Then there is a projection $P \leq E_u$ and a semibounded s.a. operator $A = PAP$ such that

$$U(t,0;E_u) = e^{-iAt} P \qquad (1.4)$$

holds for all $t \neq 0$. Furthermore, $\text{Ran}\, P = \overline{E_u \mathcal{H} \cap Q(H)}$, and the operator PAP is associated with the quadratic form $q : q(\varphi) = \|(H+\gamma)^{1/2}\varphi\|^2 - \gamma\|\varphi\|^2$, where γ is some number $\geq -\inf \sigma(H)$.

Existence of the operators (1.3) is assumed here, so we have the following technical

<u>Problem 1a</u> : Find the conditions under which the operators $U(t,0;E_u)$ exist.

Notice that a sufficient condition can be found easily when the limit is taken over a special class of regular partitions (Ref.2, Proposition 2.4.2) : $\underset{n\to\infty}{\text{s-lim}}\ (E_u U_{t/n} E_u)^n$ exists if $\text{Ran}\, E_u \subset D(H)$. Does $U(t,0;E_u)$ exist then too ? Furthermore, is this condition necessary at the same time (we conjecture that it is <u>not</u>) or what is the necessary condition ?

The above theorem shows that in the cases of physical interest, the states evolving under $U(.,0;E_u)$ stay confined within $E_u\mathcal{H}$, and therefore seemingly the perpetual observation prevents decay. Fortunately, there is no paradox here, because the limit (1.3) lacks an operational meaning (for a detailed argument, see Sec.2.4 of Ref.2). In fact, we can distinguish two typical situations. In the first one, the "continual" observation means really a (dense, but finite) sequence of individual measuring acts, where we are able to register (at least, in principle) the outcome of each of them. As an example, consider monitoring the particle tracks in a bubble chamber. In that case,

evolution is described by the operators (1.2) and model calculations show that under realistic assumptions, it is difficult even to come close to the "Zeno's limit" (cf.Ref.2, Sec.2.3).

On the other hand, there are situations which can be characterized as a "true" continual observation : ascertaining of an instant when a chosen particle would decay, measuring of the arrival time/6/ or a perpetual position monitoring. In such cases, a consistent description should include analysis of the quantum system consisting of the unstable system itself together with an appropriate part of the measuring apparatus*). The task of finding an exact solution (say, for a single free neutron interacting with the many-body system of the counter medium) is, however, too difficult. Hence we have

<u>Problem 2</u> : Find a (sufficiently rigorous and realistic) <u>model of continual observation</u> in which an unstable system interacts with a <u>quantum</u> measuring apparatus.

A step in this direction was made by Kraus/7/, but his model was too simple assuming a two-level system as the measuring apparatus ; furthermore, it relied mostly on numerical examples. Nevertheless, it suggested existence of the "watchdog effect", namely that for strong enough coupling between the system and the apparatus, the decay was again suppressed. Hence solving Problem 2 one should find the conditions under which this effect appears (we conjecture : H semibounded and the coupling constant $g \to \infty$ plus possibly some technical assumptions) and explain why it does not appear under realistic physical conditions. The role played by the number of degrees of freedom of the measuring apparatus should be also clarified.

2. <u>Models of decay processes</u>

Let us start with the simplest model of a non-relativistic two-particle decay. We use the standard kinematical variables :

relative motion : $\vec{x} = \vec{x}_2 - \vec{x}_1 \quad , \quad m = \frac{m_1 m_2}{m_1 + m_2}$,

center-of-mass motion : $\vec{X} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2} \quad , \quad M = m_1 + m_2$.

The state Hilbert space is of the form

*) We avoid in this way the more fundamental question about the mechanism of reduction of the wave function.

$$\mathcal{H} = \mathcal{H}_1^X \oplus (\mathcal{H}_1^X \otimes \mathcal{H}_1^x) \quad , \tag{2.1a}$$

where $\mathcal{H}_1 = L^2(\mathbb{R}^3)$; for simplicity, we write its elements as

$$\psi : \quad \psi(\vec{X},\vec{x}) = \begin{pmatrix} \psi_1(\vec{X}) \\ \psi_2(\vec{X},\vec{x}) \end{pmatrix} \quad . \tag{2.1b}$$

The subspace of $\mathcal{H}$ referring to the unstable particle is $\mathcal{H}_u :=$ $= \{\psi : \psi_2 = 0\}$. Next one has to choose Hamiltonian of the model. We assume

$$H_g = H_0 + gV \quad , \tag{2.2a}$$

where the free part H_0 and the interaction part V are of the form

$$H_0 = \begin{pmatrix} E - \frac{1}{2M}\Delta_X & 0 \\ 0 & \overline{-\frac{1}{2M}\Delta_X - \frac{1}{2m}\Delta_x} \end{pmatrix} \quad , \tag{2.2b}$$

$$V : \quad (V\psi)(\vec{X},\vec{x}) = \begin{pmatrix} \int_{\mathbb{R}^3} d^3y \; v(\vec{y})\, \psi_2(\vec{X},\vec{y}) \\ v(\vec{x})\, \psi_1(\vec{X}) \end{pmatrix} \quad ; \tag{2.2c}$$

their matrix form refers to (2.1b). Here E is a positive number interpreted as an internal energy of the unstable particle, and $v \in L^2(\mathbb{R}^3)$. By Δ_X , Δ_x we mean the self-adjoint extensions of the corresponding Laplacians.

Notice that the mass M in the first row of H_0 cannot be replaced by some M' different from M , since the decay would be then forbidden by Bargmann superselection rule /8,9/. It becomes clear once we write down the projective representation of the Galilei group connected with our system,

$$\begin{aligned} (U(b,\vec{a},\vec{v},R)\psi)(\vec{X},\vec{x},t) &= \exp\left\{-\tfrac{i}{2}M\vec{v}^2(t-b) + iM\vec{v}.(\vec{X}-\vec{a})\right\} \times \\ &\quad \psi(R^{-1}(\vec{X}-\vec{a}-\vec{v}t+\vec{v}b), R^{-1}\vec{x}, t) \quad . \end{aligned} \tag{2.3}$$

Now we are going to present a concise discussion of the model*) :

<u>Proposition</u> : H_0 and H_g are self-adjoint on $D(H_0) = D(H_g)$.

Next we separate the center-of-mass motion. We write $\mathcal{H} = \mathcal{H}_1^X \otimes (\mathbb{C} \oplus \mathcal{H}_1^x)$,

*) The arguments are essentially the same as in the Friedrichs model (Ref.2, Sec.3.2) ; more details will be published elsewhere.

then $H = \overline{-\frac{1}{2M}\Delta_X \otimes I + I \otimes H_g^{rel}}$, where $H_g^{rel} = H_0^{rel} + gV$ with

$$H_0^{rel} = \begin{pmatrix} E & 0 \\ 0 & -\frac{1}{2m}\Delta_x \end{pmatrix} \quad , \tag{2.4a}$$

$$V : \quad V\begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{pmatrix} (v,\varphi) \\ \alpha v \end{pmatrix} \quad . \tag{2.4b}$$

Hence the evolution operator $U_t = U_t^{cm} \otimes U_t^{rel}$, where the center-of-mass part represents a free motion. For the relative part, we pass to the p-representation :

<u>Proposition</u> : H_g^{rel} is by means of $I \oplus F_3$ unitarily equivalent to the operator

$$\begin{pmatrix} E & g(\hat{v},.) \\ g\hat{v} & \frac{p^2}{2m} \end{pmatrix} \quad , \tag{2.5}$$

where $\hat{v} \equiv F_3 v$; for simplicity, we denote (2.5) again as (2.2a).

Since $E > 0$ due to the assumption, the unperturbed Hamiltonian has a simple eigenvalue embedded in the non-simple $\sigma_c(H_0) = [0,\infty)$. Recall that behaviour of the reduced propagator $V_t := E_u U_t \upharpoonright \mathcal{H}_u$ is determined by the reduced resolvent $R_u(z,H_g) :=$ $= E_u(H_g - z)^{-1} \upharpoonright \mathcal{H}_u$; it is dominated by an exponential term if $R_u(.,H_g)$ has a second-sheet pole close to the real axis (cf.Ref.2, Sec.3.1).

With this in mind, we can solve the model. We adopt the following two assumptions :

(a) $\hat{v}$ depends on $|\vec{p}|$ only,

(b) $|\hat{v}(.)|^2$ can be continued analytically across $\mathbb{R}_+$.

The first one is the consequence of Galilean invariance, the other means more explicitly that there is an analytical function which coincides with $|\hat{v}(.)|^2$ on $\mathbb{R}_+$ (for simplicity, we use the same symbol for the both). Since the interaction fulfils

$$E_u^{\perp} V E_u^{\perp} = 0 \quad , \tag{2.6}$$

where $E_u^{\perp} = I - E_u$, the reduced resolvent can be obtained algebraically; it acts as a multiplication by

$$r_u(z,H_g) = \left(-z + E + 4\pi g^2 \int_0^{\infty} \frac{p^2 |\hat{v}(p)|^2}{z - \frac{p^2}{2m}}\, dp\right)^{-1} \tag{2.7a}$$

for $\mathrm{Im}\, z > 0$. The analytical continuation of $r_u(\cdot, H_g)$ to the lower complex halfplane, $\mathrm{Im}\, z < 0$, is

$$r_u^{II}(z,H_g) = \left(-z + E + 4\pi g^2 \int_0^{\infty} \frac{p^2 |\hat{v}(p)|^2}{z - \frac{p^2}{2m}}\, dp - 8i\pi m g^2 |\hat{v}(\sqrt{2mz})|^2 \sqrt{2mz} \right)^{-1} \tag{2.7b}$$

The pole condition for (2.7b) is solved by means of the implicit-function theorem :

<u>Theorem</u> : In a neighbourhood of $g = 0$, there is an analytical function z_p such that $r_u^{II}(\cdot, H_g)$ has a simple pole at $z_p(g) = \lambda_p(g) - i\delta_p(g)$. It holds

$$\lambda_p(g) = E + 8\pi g^2 \mathcal{P}\int_0^{\infty} \frac{p^2 |\hat{v}(p)|^2}{E - \frac{p^2}{2m}}\, dp + O(g^4) \quad , \tag{2.8a}$$

$$\delta_p(g) = 8\pi^2 m g^2 |\hat{v}(\sqrt{2mE})|^2 \sqrt{2mE} + O(g^4) \quad . \tag{2.8b}$$

We remark that the model under consideration is essentially identical with the lowest sector of the "GaliLee model"[9,10]. Developing it further, one should prove the following assertions :

(i) the leading term in (2.8b) is given by Fermi rule, i.e., the decay width equals $\Gamma(g) = 2\delta_p(g) = 2\pi g^2 \frac{d}{d\lambda} (V\psi_u, E_\lambda^0 P_c(H_0) V\psi_u)\Big|_{\lambda = E}$,

(ii) the scattering amplitude in the system (H, H_g) has a pole at z_p ,

(iii) there is a (quadratic) spectral concentration as $g \to 0$,

(iv) finally, one has to justify the pole approximation, i.e., to estimate the remainder terms similarly as Demuth[11] did it for the Friedrichs model.

There are various straightforward generalizations of the model, e.g.,

- inclusion of spin
- non-vanishing final-state interaction
- the three-particle decay,

etc., but we are not going to discuss them here. It is more important for our present purpose to stress that the nice features of the model stem primarily from the Friedrichs condition (2.6) which makes it algebraically soluble.

<u>Problem 3</u> : Find alternative techniques for solving the embedded-eigenvalue problem.

We remark that two other techniques are known at present but neither of them is entirely satisfactory. The dilation-analytic method has been highly successful in solving problems as, e.g., He - autoionization, but its applicability is restricted to Schrödinger operators with well-behaved potentials[12-16]. On the other hand, there is the factorization technique whose idea goes back to Kato[17]. It was used successfully by Howland, Baumgärtel and others - see Sec.3.3 of Ref.2 for references. Its applicability is hindered, however, by the fact that one should check independence on the chosen factorization, and this task is usually difficult.

Beside the Galilean-invariant Lee model mentioned above[9,10,18], some other decay models have been worked out. Let us mention two of them : a system coupled to a fermion reservoir with persistent vacuum (cf.Ref.19), and a harmonic oscillator coupled to a massless scalar field[20]. There are also various models of quantum-mechanical tunneling decays[21-24] ; they rely on the perturbation theory of isolated eigenvalues which dissolve in the continuous spectrum once the perturbation is switched on*).

The common feature of all the models mentioned above is that they are non-relativistic. Hence we have

<u>Problem 4</u> : Construct a relativistic decay model.

Let us stress that we are looking for a dynamical model. A kinematical description of relativistic decays can be obtained on the basis of symmetry considerations - cf.Ref.2, Sec.3.5 and also Refs.28,29.

3. <u>Dissipative quantum mechanics</u>

Phenomenological non-selfadjoint Hamiltonians are widely used in some branches of nuclear physics, solid-state physics, etc. ; often it is the only way to reduce reasonably the complexity of the problem under study. A rigorous analysis, however, is usually lacking.

In this section, we shall be concerned with (contionuous contractive) semigroup evolutions $V_t = e^{-iHt}$. The operator H referring to such an evolution is called <u>pseudo-Hamiltonian</u>. This notion was introduced in Ref.30 and discussed extensively in Chap.4 of Ref.2 . There are two main groups of problems here : .

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*) Such a situation is familiar from the Stark effect. The use of this perturbative method is not, however, restricted to the decay models; in the recent series of papers by Gesztesy et al.(e.g., Refs.25-27), it is employed to derive the first-order relativistic corrections to Pauli Hamiltonian.

(i) development of the non-selfadjoint quantum mechanics,

(ii) justification of the pseudo-Hamiltonian method.

Let us start with the first of them. A densely defined H is dissipative if $\mathrm{Im}(\psi, H\psi) \leq 0$ for all $\psi \in D(H)$. Furthermore, H is maximal dissipative if it has no proper dissipative extensions ; it is essentially maximal dissipative if $\overline{H}$ is m.d. In this way, one obtains a straightforward generalization of the standard quantum-mechanical scheme :

symmetric operator	...	dissipative operator
self-adjoint operator	...	maximal dissipative
e.s.a. operator	...	e.m.d. operator
Stone theorem	...	Phillips theorem/31/ : iH generates a continuous contractive semigroup iff H is m.d.
the basic self-adjointness criterion	...	a dissipative H is e.m.d. iff $\overline{\mathrm{Ran}(H-i)} = \mathcal{H}$
von Neumann extension theory	...	the theory of dissipative extensions/32/ (important difference : <u>every</u> dissipative operator has a m.d.extension !)
Kato-Rellich theorem	...	the perturbative theorem by Nelson/33/, Gustafson/34/ and Chernoff/35/,

etc. (for more details, see Sec.4.2 of Ref.2). Many results can be also derived for Schrödinger pseudo-Hamiltonians. By this notion, we mean a Schrödinger operator $H = -\frac{1}{2}\Delta + u$ on $L^2(\mathbb{R}^d)$ with a complex Borel potential which is supposed to be regular with possible exception of a Lebesgue-zero set, fulfilling the dissipativity condition $\mathrm{Im}\, u(x) \leq 0$ a.e. in $\mathbb{R}^d$, and such that H is densely defined. As a generalization to the known self-adjointness criteria for Schrödinger operators, various conditions can be derived under which such H is e.m.d., for example

(a) H is J-selfadjoint, i.e., $H^* = \overline{-\frac{1}{2}\Delta + \bar{u}}$,

(b) $u \in L^p + L^\infty$, where $p = 2$ for $d \leq 3$ and $p > \frac{d}{2}$ for $d > 3$,

(c) $d = 3N$ and u is a sum of two-body potentials from $L^2 + L^\infty$,

(d) $u \in L^2_{loc}(\mathbb{R}^d)$ and $\inf\mathrm{ess}\,\{\mathrm{Re}\, u(x) : x \in \mathbb{R}^d\} > -\infty$

(for more details, see Sec.4.3 of Ref.2). Moreover, H is even maximal dissipative if (b) or (c) holds.

<u>Problem 5</u> : Extend the generalization described above, in particular, by adapting other known "self-adjoint" methods.

The formulation of this problem is, of course, rather vague, and a brief comment is needed. There are at least three directions in which the dissipative quantum mechanics can be developed :

(i) criteria of maximal dissipativity,

(ii) spectral properties of pseudo-Hamiltonians. The problem is substantially complicated by the possible existence of spectral singularities, and a solid information is available in some simple cases only, notably for pseudo-Hamiltonians with smooth potentials on a halfline (cf.Ref.36). The operators without the singularities (or spectral/37/) are easier to describe, but the known sufficient conditions under which a given operator is spectral are very restrictive. Various concrete suggestions can be formulated ; just to give an example, we mention the possibility of proving Lidskii theorem (Ref.2, Th.4.3.15) for $d > 1$,

(iii) non-unitary scattering theory/17,38-42/ : there are generalizations to the methods of Cook, Kato-Birman, Enss, etc., but generally our knowledge in this field is far from satisfactory.

The second group of problems is not less important and contains a lot of open questions. In fact, the pseudo-Hamiltonian description of an open system represents a generalization to the pole approximation (see Sec.4.1 of Ref.2 for a detailed discussion) ; even this approximation is justified satisfactorily in few simple decay models. It suggests that the estimation procedures for concrete systems, particularly those in which the pseudo-Hamiltonian acts on an infinite-dimensional space, are presumably difficult. As an example, let us mention the estimation connected with justification of the optical approximation in neutron scattering on nuclei - cf.Refs.40,43 and Ref.2, Sec.4.4 . Though we do not try to pick up any particular problem here, we would like to stress the demand for searches in this direction : it is one of those places in physics where a relatively successful phenomenology lacks a sufficiently rigorous footing for a long time.

4. <u>Feynman path integrals</u>

Our next problem could read "construct a theory of Feynman path integral", but certainly such a formulation is worth of nothing. Hence we try to be more specific. First of all, we restrict our attention to the quantum-mechanical systems in a flat configuration space and to the configuration-space path integrals, leaving a good many interesting problems out of the scope. For a system described by a Schrödinger (pseudo-)Hamiltonian $H = -\frac{1}{2}\Delta + u$ in $L^2(\mathbb{R}^d)$, the task is to prove Feynman-Cameron-Itô formula

$$(e^{-iHt}\psi)(x) = \int_{\Gamma_0} e^{\frac{i}{\hbar}S(\gamma+x)} \psi(\gamma(0)+x)\, D\gamma =$$

$$= \int_{\Gamma_0} e^{-\frac{i}{\hbar}\int_0^t u(\gamma(\tau)+x)\,d\tau} \psi(\gamma(0)+x)\, D\phi_s(\gamma) \qquad (4.1)$$

with the rhs defined in a suitable way. Here $s = \hbar/m$, and for convenience we set $\hbar = 1$ in the following. The symbol Γ_0 means a space of trajectories ending at $x = 0$. It can be chosen in various ways, e.g.,

Banach space $X = C_0[J^t;\mathbb{R}^d]$ with $\|\gamma\| := \max_{\tau\in J^t} |\gamma(\tau)|$,

Hilbert space $\mathcal{H} = AC_0[J^t;\mathbb{R}^d]$ with $\|\gamma\| := \left(\int_0^t |\dot\gamma(\tau)|^2 d\tau\right)^{1/2}$,

or some other path space, where $J^t = [0,t]$.

We are going to sketch now a few main ways in which the rhs of the FCI-formula (4.1) can be defined. This survey is naturally very brief ; for a more complete discussion, bibliography and notation see Chap.5 of Ref.2 .

(i) Fresnelian-type method/44/ : the algebra $\mathcal{F}(\mathcal{H})$ of "integrable functions" consists of all $f : \mathcal{H} \to \mathbb{C}$ which are Fourier images of a complex Borel measure μ_f on $\mathcal{H}$. For $f \in \mathcal{F}(\mathcal{H})$, we set

$$\int_{\mathcal{H}}^{f} f(\gamma)\, D\phi_s(\gamma) := \int_{\mathcal{H}} e^{-\frac{is}{2}\|\gamma\|^2} d\mu_f(\gamma) \ , \qquad (4.2)$$

(ii) polygonal-path approximations/45,46/: for a partition $\sigma = \{\tau_j :$ $0 = \tau_0 < \tau_1 < \ldots < \tau_n = t\}$ of J^t , we define the expression

$$I_s(f;\sigma) := \prod_{j=0}^{n-1} (2\pi i s \delta_j)^{-d/2} \int_{\mathbb{R}^{nd}} e^{\frac{i}{2s}\sum_{j=0}^{n-1}|\gamma_{j+1}-\gamma_j|^2\delta_j^{-1}} \times$$

$$f(\gamma_\sigma)\, d\gamma_0 \ldots d\gamma_{n-1} \ , \qquad (4.3a)$$

where γ_σ is a polygonal path with apices $\gamma_j \equiv \gamma(\tau_j)$. Since (4.3a) has the natural interpretation as a path integral over all such polygonal paths, one can define

$$\int_{\mathcal{H}}^{\alpha} f(\gamma)\, D\phi_s(\gamma) := \lim_{\substack{\text{refining}\\ \text{partitions}}} I_s(f;\sigma) \ . \qquad (4.3b)$$

The index α specifies the used integral over $\mathbb{R}^{nd}$ and the limit. We use, e.g.,

c as "cylindrical" for Lebesgue integral,
i for a suitable improper integral,
o for the oscillatory integral,
r for the regular limit, $\tau_j = jt/n$, $n \to \infty$,
u for the uniform limit, $\delta(\sigma) \to 0$, etc.

(iii) product formulae/33,47/: for $f(\gamma) = \exp\left\{-i\int_0^t u(\gamma(\tau)+x)\,d\tau\right\}\psi(\gamma(0)+x)$, one can use Lie-Trotter formula or its modifications to define the rhs of (4.1),

(iv) limiting F-integral/48,49/: we take a complex s which approaches the real axis from below, e.g.,

$$\int_{\mathcal{H}}^{luc} f(\gamma)\, D\phi_s(\gamma) := \lim_{\varepsilon\to 0+} \int_{\mathcal{H}}^{uc} f(\gamma)\, D\phi_{s-i\varepsilon}(\gamma) \ , \qquad (4.4)$$

(v) Itô definition (Gaussian regularized F-integral/50/) : we set

$$\int_{\mathcal{H}}^{g} f(\gamma)\, D\phi_s(\gamma) := \lim_{\varphi} \left[\det(I - \tfrac{i}{s}S)\right]^{1/2} \int_{\mathcal{H}} e^{\frac{i}{2s}\|\gamma\|^2} f(\gamma)\, d\mu_{S,\beta}(\gamma), \qquad (4.5)$$

where the limit is taken along the directed set of all correlation operators of the Gaussian measures $\mu_{S,\beta}$. This theory is worked out up to now for $d = 1$ only. Also, one would invite to have a more uniform limit,

(vi) analytical F-integral/33,49,51-53/: the F-integral $\int_X^a F(\gamma)\, D\phi_s(\gamma)$ is defined as analytical continuation of the function $\lambda \mapsto J(F;\lambda) := \int_X F(\lambda^{-1/2}\gamma)\, dw(\gamma)$, where w is the Wiener measure on X , to the point $\lambda = -i/s$.

Remark : Possible definitions of the F-integral are in no case exhausted by the above short list, and new possibilities continue to appear. In the recent series of papers, Ichinose has shown/54-56/ that the well-known no-go theorem of Cameron/48/ does not apply to Dirac operators, at least in some cases, and he has been able to construct the corresponding path measure explicitly. The non-relativistic limit of this theory offers therefore one more possibility to construct the sought path integral.

Theorem : The F-integrals $\int_{\mathcal{H}}^{\alpha} f(\gamma)\, D\phi_s(\gamma)$, $s > 0$, for $\alpha = f$, ui , luc , g , a coincide on the algebra $\mathcal{F}(\mathcal{H})$.

The statement concerning $\alpha = a$ requires a comment, since this definition employs a different path space. There is, however, a Banach algebra of functions $F : X \to \mathbb{C}$ which is isomorphic to $\mathcal{F}(\mathcal{H})$ by $F \sim f := F \restriction \mathcal{H}$ – for more details see Ref.53 .

Problem 6a : Is there a wider class of functions on which a major part of the F-integral definitions coincide ?

This is important because the integrands of (4.1) corresponding to many physically interesting potentials, such as that of harmonic oscillator, are not contained in $\mathcal{F}(\mathcal{H})$.

Problem 6b : Construct calculus for the F-integral.

This again requires a comment. The "Bateman manuscript" for Feynman integrals is very short : we can integrate, in most cases formally, the functions $\gamma \mapsto e^{q(\gamma)}$, where q is a quadratic form. In some cases, we are able to perform substitutions, in particular

translations $\gamma \to \gamma + \alpha$
regular isometric transformations $\gamma \to R\gamma$
"Cameron-Martin" transformations $\gamma \to (I+K)\gamma$ with K of the trace class

(cf.Ref.2, Secs.5.2,5.4). Moreover, though the dominated-convergence theorem is not valid for F-integral (Ref.2, Example 5.2.10), analogous assertions have been proven recently under somewhat more restrictive assumptions/57-59/. It is clear, however, that much powerful and complete calculus rules must be elaborated before Feynman path integrals could be regarded as more than the present-day very useful but mostly heuristic tool.

5. The "Feynman paths"

In the last section, we return briefly to the repeated measurements considered in Sec.2 . Now, however, we replace the fixed projection E_u by a projection-valued apparatus function $E(.)$. Then we define

$$U(t,0;E,\sigma) := E(t)\, e^{-iH\delta_{n-1}} E(\tau_{n-1})\, e^{-iH\delta_{n-2}} E(\tau_{n-2}) \dots$$

$$\dots E(\tau_1)\, e^{-iH\delta_0} E(0)$$

and

$$U(t,0;E) := \underset{\delta(\sigma)\to 0}{\text{s-lim}}\ U(t,0;E,\sigma) \quad . \tag{5.1}$$

Aharonov and Vardi/60/ suggested such a procedure as realization of a single "Feynman path". This is only partly true, however, because the limit (5.1) again has no operational meaning. Nevertheless, it represents an interesting mathematical object related to the F-integral. Let us illustrate it on the following simple example (see Ref.5, and Ref.2, Sec.6.3) :

Theorem : Let $E(t)$ be one-dimensional corresponding to $\psi_t : \psi_t(x) = \psi(x-\gamma(t))\, e^{ix.\pi(t)}$ with $\psi \in \mathcal{S}(\mathbb{R}^d)$ and γ, π continuously differentiable. Let H be maximal dissipative, $t \mapsto H\psi_t$ continuous , $D(H) \subset \mathcal{S}(\mathbb{R}^d)$, then

$$U(t,0;E)\varphi = e^{i\int_0^t L(\tau)\,d\tau} (\psi_0,\varphi)\psi_t \quad , \tag{5.2a}$$

$$L(t) = \sum_{k=1}^{d} \dot{Q}_k(t)P_k(t) - \mathcal{E}(t) - i\frac{d}{dt}\sum_{k=1}^{d} Q_k(t)P_k(t) \quad , \tag{5.2b}$$

where $Q_k(t)$, $P_k(t)$, $\mathcal{E}(t)$ are mean values of Q_k , P_k , H with respect to ψ_t , respectively.

In particular, if $H = -\frac{1}{2}\Delta + u$ and $\dot{\gamma}_k = \pi_k$, then we get

$$L(t) = \frac{1}{2}\sum_{k=1}^{d} P_k(t)^2 - U(t) - \frac{1}{2}\sum_{k=1}^{d} (\Delta P_k)^2_{\psi_t} - \frac{d}{dt}\sum_{k=1}^{d} Q_k(t)P_k(t) \ , \tag{5.3}$$

where $U(t) = (\psi_t, u\psi_t)$. Since the last term is path-independent and the third one is negligible in a quasiclassical situation, we recover the Feynman's weight factor, even for dissipative systems which may have no classical counterpart.

It makes therefore sense to expect that for a suitable E , the operators $U(t,0;E)$ can (if they exist) replace the non-existent Feynman measure. Thus we come to our last

Problem 1b : Find $U(t,0;E)$, if $\mathcal{H} = L^2(\mathbb{R}^d)$ and $E(\tau)$ projects on $L^2(M_\tau)$, $0 \leqslant \tau \leqslant t$, where M_τ is a moving space region in $\mathbb{R}^d$.

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Received by Publishing Department
on September 20, 1985.

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Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1985

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The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1985