объединенный
институт
ядерных
исследований
дубна

E2-85-597

V.V.Nesterenko

ON THE DERIVATION OF THE FORMULA FOR THE HAMILTONIAN FUNCTIONAL INTEGRAL IN THEORIES WITH THE FIRST- AND SECOND-CLASS CONSTRAINTS

Submitted to "ТМФ"

1985

1. Introduction

The Faddeev well-known paper [1] laid the foundation of the quantization of the theories with the singular Lagrangians by the functional integration in the phase space. The important application of this approach is the construction of quantum theory of the gauge fields [2,3] and gravitation [4].

In paper [1] the most interesting from the application point of view first-class constraints [5] were considered. For simplicity the gauge conditions were supposed to be in involution between themselves. In addition it was assumed that the constraints and the gauge conditions do not contain the time explicitly. However, there are field models with the second-class constraints, for example, the massive Yang-Mills field. And for the Lagrangians homogeneous of the first-degree in the velocities the gauge conditions must be explicitly time-dependent [6,7].

In paper [8] at the same assumptions as in [1] the second-class constraints were included into consideration. In paper [7] the Faddeev proof of the formula for the Hamiltonian functional integral was extended to the gauge conditions explicitly time-dependent. In [9] an attempt which is not completely consistent, to our opinion, was made to consider the gauge conditions explicitly time dependent and noninvolutary between themselves simultaneously.

The basic peculiarity of the theories with degenerate Lagrangians is the following. The physical dynamics develops not in the whole phase space Γ, but only on its submanifold Γ^* defined by the constraints and the gauge conditions. The physical submanifold Γ^* of the symplectic manifold and the canonical coordinates can be introduced on it. The same statement is right also for $\bar{\Gamma} = \Gamma \setminus \Gamma^*$, i.e., for the difference of Γ and Γ^*. In the case of the first-class constraints and gauge conditions involutory between themselves, the corresponding canonical coordinates can be specified immediately [1,7]. But if there are second-class constraints or the gauge conditions are noninvolutory between themselves then the consideration of the paper/1/ is not applicable here. In this case one has to use the mathematical theorem (ref. /10/ theorem VII.24) about the canonical form of equations which specify the submanifolds. The Poisson brackets of the left-hand sides of these equations equal one or zero[1].

Then one has to prove that the dynamics on the physical submanifold Γ^* of the phase space is the Hamiltonian dynamics, i.e., the equations of motion on Γ^* are the Hamiltonian equations and the corresponding Hamiltonian must be determined. In papers devoted to the quantization of the systems with the singular Lagrangians via the path integration in the phase space, these facts do not proved but implicitly are supposed beforehand/1,2/. And one usually assumes that the effective Hamiltonian generating the dynamics on Γ^* is the contraction of the canonical Hamiltonian H on Γ^*. The example of the degenerate Lagrangians homogeneous in the velocities shows that in the general case this is not so. The canonical Hamiltonian in such theories equals zero identically.

In the present paper we propose a simple and consistent derivation of the formula for the Hamiltonian functional integral for the theories with the constraints of the most general kind: they may be the first- and second-class constraints and they can contain time explicitly. The gauge conditions can be noninvolutory between themselves and can be explicitly time-dependent as well. In contrast with other papers much attention will be paid to proving the Hamiltonian form of the theory on the physical submanifold of the phase space. It will be shown that Δ^{-1}, where Δ is the Faddeev-Popov determinant, is just the volume element of the submanifold Γ expressed in terms of the noncanonical coordinates defined by constraints and gauge conditions. This simplifies the interpretation of the final formula for the path integral. As far as we know, this property of Δ was not noted in the previous papers devoted to this problem. The final formula for the path integral in the phase space does not depend on the choice of the gauge conditions. Usually, this property of the Hamiltonian path integral is demonstrated by infinitesimal changes of the gauge conditions/1,8/. Here, this statement will be proved by transition to an absolutely new set of gauge equations.

The material is arranged as follows. In the second section the constraint equations in theories with degenerate Lagrangians are analysed. The third sectinn is devoted to the derivation of the equations of motion in the phase space. In the fourth section the

1) In paper/11/ this theorem has been proved anew without references to /10/.

path integral representation for the matrix element of the evolution operator is constructed. In the appendix the derivation of the formula for the volume element of the symplectic manifold in terms of the arbitrary noncanonical coordinates is given.

2. Different forms of constraints

Let us consider the system with a finite number n of degrees of freedom described by a singular Lagrangian $L(q,\dot{q},t)$

$$\operatorname{rank} \left\| \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right\| < n, \quad 1 \leq i,j \leq n. \tag{2.1}$$

Here q and $\dot{q}$ are the generalized coordinates and velocities $q = (q_1,\dots,q_n)$, $\dot{q} = (\dot{q}_1,\dots,\dot{q}_n)$, $\dot{q} = dq/dt$. For the generality we assume the explicit time dependence in L, therefore the constraints explicitly time-dependent will be taken into consideration.

We suppose that the complete set of the functionally independent constraints in the theory under consideration is known, i.e. all the primary and secondary constraints are known

$$\omega_s(q,p,t) = 0, \quad s = 1,\dots,m < n, \tag{2.2}$$

$$\operatorname{rank} \left\| \frac{\partial(\omega_s)}{\partial(q,p)} \right\|_{\omega_s = 0} = m, \tag{2.3}$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1,\dots,n. \tag{2.4}$$

According to (2.3) the equations of constaints (2.2) determine in the $2n$-dimensional phase space Γ with the coordinates q,p the $2n-m$-dimensional submanifold M.

The complete set of constraints can be obtained by the known iterative procedure proposed by Dirac. It is based on the requirement of fulfilling all the constraint equations during the evolution. The primary constraints which are the consequences of the condition (2.1) are given by the specific form of the degenerate Lagrangian.

The complete set of constraints in the phase space can be obtained in the framework of the Lagrangian formalism as well/12/. From the view point of this formalism the secondary constraints are the Lagrangian constraints in the theory (i.e. the Euler equations that do not contain the secondary derivatives with respect to time) and all the derivatives of the Lagrangian constraints with respect to time up to some fixed degree. In order to pick out the first-class and second-class constraints in the complete set (2.2), one must make some assumptions on the properties of the skew-symmetric matrix $\|(\omega_s, \omega_{s'})\|$, $s, s' = 1,\dots,m$, the elements of which are the Poisson brackets of the constraints $\omega_s(q,p,t)$, $s=1,\dots,m$

$$(\omega_s, \omega_{s'}) = \sum_{i=1}^{n} \left(\frac{\partial \omega_s}{\partial q_i} \cdot \frac{\partial \omega_{s'}}{\partial p_i} - \frac{\partial \omega_s}{\partial p_i} \cdot \frac{\partial \omega_{s'}}{\partial q_i} \right). \quad (2.5)$$

Let on the submanifold M we have

$$\operatorname{rank} \|(\omega_s, \omega_{s'})\|_{|M} = 2m_2 < m, \quad (2.6)$$

$$s, s' = 1,\dots,m.$$

This means that the matrix $\|(\omega_s, \omega_{s'})\|$ has on M exactly $m_1 = m - 2m_2$ linearly independent eigenvectors with zero eigenvalues[2)]

$$\xi^a_s(q,p,t)\,(\omega_s, \omega_{s'}) = 0, \quad (2.7)$$

$$s, s' = 1,\dots,m, \qquad a = 1,\dots,m_1.$$

We suppose as usual the summation with respect to the repeated endices in the corresponding limits. The requirement of the completeness of the set of constraints (2.2) is written in the following form:/12/

$$\xi^a_s \left[\frac{\partial \omega_s}{\partial t} + (\omega_s, H) \right]_{|M} = 0, \quad (2.8)$$

$$a = 1,\dots,m_1.$$

The constraints (2.2) can be numbered in such a way that the matrix $\|(\omega_s, \omega_{s'})\|$ will have $2m_2$ linearly independent last rows and accordingly $2m_2$ linearly independent last columns. Now we go from the initial constraints set (2.2) to the equivalent set of the constraints according to the formulae

$$\varphi_a(q,p,t) = \xi^a_s(q,p,t)\,\omega_s(q,p,t), \quad a = 1,\dots,m_1, \quad (2.9)$$

---

2) The coefficients in the condition of the linear independence of the vectors ξ^a_s have to be considered as the functions of q, p, t.

$$\theta_\alpha(q,p,t) = \omega_{m_1+\alpha}(q,p,t), \quad \alpha = 1,\dots,2m_2. \quad (2.10)$$

As the vectors $\xi^a_s(q,p,t)$ are linearly independent then the equations of the new constraints

$$\varphi_a(q,p,t) = 0, \quad a = 1,\dots,m_1, \quad (2.11)$$

$$\theta_\alpha(q,p,t) = 0, \quad \alpha = 1,\dots,2m_2 \quad (2.12)$$

determine the same submanifold M as the equations of the initial constraints (2.2).

Taking into account (2.7) and (2.8), one verifies easily the following equalities on the submanifold M:

$$(\varphi_a, \varphi_b) \overset{\varphi,\theta}{\approx} 0, \quad \frac{\partial \varphi_a}{\partial t} + (\varphi_a, H) \overset{\varphi,\theta}{\approx} 0, \quad (\varphi_a, \theta_\alpha) \overset{\varphi,\theta}{\approx} 0, \quad (2.13)$$

$$\operatorname{rank} \|(\theta_\alpha, \theta_\beta)\| \overset{\varphi,\theta}{\approx} 2m_2, \quad (2.14)$$

$$a, b = 1,\dots,m_1, \qquad \alpha, \beta = 1,\dots,2m_2.$$

The sign $\overset{\varphi,\theta}{\approx}$ means the weak equality /5/, i.e. at the outset one has to evaluate the Poisson brackets or to make the differentiation in the left-hand side from this sign and then to put $\varphi = 0$, $\theta = 0$. Thus, regrouping of the constraints (2.9) and (2.10) chooses the first-class (2.11) and second-class (2.12) constraints which obey on M the conditions (2.13), (2.14).

As is well-known, the first-class constraints result in the functional freedom in the equations of motion in the phase-space Γ. In order to remove this freedom one has to impose on the canonical variables q, p gauge conditions /1/ in addition to the constraints (2.11) and (2.12)

$$\chi_a(q,p,t) = 0, \quad a = 1,\dots,m_1. \quad (2.15)$$

These conditions must have the following property:

$$\det \|(\chi_a, \varphi_b)\|_{|\varphi=0,\,\theta=0,\,\chi=0} \neq 0. \quad (2.16)$$

The gauge conditions (2.15) cut out from M the physical submanifold Γ^* of the phase space. The dimension of Γ^* is $2(n-m_1-m_2)$.

In contrast with papers /1,2,8/ we shall not demand the involution of the gauge conditions (2.15) between themselves. As we consider the most general case of the constraints (2.2) with the explicit time-dependence, the gauge conditions (2.15) have to be time-dependent too. Moreover, if the canonical Hamiltonian H is identically equal to zero (in this case the Lagrangian $L(q,\dot{q},t)$ is the homogeneous function of the first-degree in the velocities $\dot{q}$) , then the gauge conditions (2.15) must be explicitly time-dependent /6,7/ beyond the dependence of the properties of the first-class constraints (2.11).

In order to explore the equations of motion on the physical submanifold Γ^*, it is convenient to replace the set of the constraints (2.11), (2.12) and the gauge conditions (2.15) by the equivalent set of $2(m_1+m_2)$ equations in the canonical form. It is achieved by the special canonical transformation to the new canonical variables Q, P

$$Q_i = Q_i(q,p,t), \qquad P_i = P_i(q,p,t), \tag{2.17}$$

$$(Q_i, Q_j) = (P_i, P_j) = (Q_i, P_j) - \delta_{ij} = 0, \qquad i,j = 1,\dots,n \tag{2.18}$$

in terms of which the physical submanifold Γ^* is defined by the equations

$$Q_{\tau+\varkappa}(q,p,t) = 0, \qquad P_{\tau+\varkappa}(q,p,t) = 0, \qquad \tau = n-m_1-m_2, \qquad \varkappa = 1,\dots,m_1+m_2. \tag{2.19}$$

The canonical variables on Γ^* are Q_6, P_6, $1 \le 6 \le n-m_1-m_2$. We shall not prove here this statement; it can be found in the book /10/ theorem VII.24. The specific form of the functions $Q_i(q,p,t)$, $P_i(q,p,t)$, $i=1,\dots,n$ will not be used. We note only that they are explicitly time-dependent as the constraints (2.11), (2.12) and the gauge conditions (2.15) contain time explicitly.

For the abbreviation we shall mark sometimes the set of constraints in the canonical form (2.19) by one letter

$$\Omega_A(q,p,t) = 0, \qquad A = 1,\dots,2(m_1+m_2), \tag{2.20}$$

$$\Omega_{\varkappa}(q,p,t) = Q_{\tau+\varkappa}(q,p,t), \qquad \Omega_{m_1+m_2+\varkappa}(q,p,t) = P_{\tau+\varkappa}(q,p,t),$$
$$\tau = n-(m_1+m_2), \qquad \varkappa = 1,\dots,m_1+m_2.$$

The matrix constructed by the Poisson brackets of the constraints $\Omega_A(q,p,t)$ between themselves is equal to the unit symplectic matrix of the dimension $2(m_1+m_2)\times 2(m_1+m_2)$

$$\left\| (\Omega_A, \Omega_B) \right\| = \left\| \begin{matrix} 0 & I_{m_1+m_2} \\ -I_{m_1+m_2} & 0 \end{matrix} \right\| = J_{2(m_1+m_2)}. \tag{2.21}$$

Here $I_{m_1+m_2}$ is the unit $(m_1+m_2)\times(m_1+m_2)$ matrix.

3. Equations of motion in the phase space

We derive now the equation of motion in the phase space taking into account all the constraints and the gauge conditions written in the canonical form (2.19). The canonical Hamiltonian

$$H = p_i \dot{q}_i - L(q,\dot{q},t) \tag{3.1}$$

does not depend on the velocities $\dot{q}$ in the case of the degenerate Lagrangians as well. Indeed, differentiating (3.1) and using (2.4) we get

$$\begin{aligned} dH(q,\dot{q},p,t) &= dp_i\,\dot{q}_i + p_i\,d\dot{q}_i - \\ &- \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial t} dt = \\ &= dp_i\,\dot{q}_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt . \end{aligned} \tag{3.2}$$

Thus, dH does not contain the differentials of the velocities $d\dot{q}_i$. Therefore,

$$H = H(q,p,t), \tag{3.3}$$

$$dH=\frac{\partial H}{\partial q_i}dq_i+\frac{\partial H}{\partial p_i}dp_i+\frac{\partial H}{\partial t}dt. \tag{3.4}$$

Let us take into account eqs. (3.2) and (3.4) in the Euler equations

$$\frac{dp_i}{dt}=\frac{\partial L}{\partial q_i}, \quad i=1,\dots,n. \tag{3.5}$$

This gives the following equality:

$$\left(\frac{\partial H}{\partial q_i}+\dot{p}_i\right)dq_i+\left(\frac{\partial H}{\partial p_i}-\dot{q}_i\right)dp_i+\left(\frac{\partial H}{\partial t}+\frac{\partial L}{\partial t}\right)dt=0 \tag{3.6}$$

Besides the differentiation of the constraints (2.20) results in the equations

$$\frac{\partial \Omega_A}{\partial q_i}dq_i+\frac{\partial \Omega_A}{\partial p_i}dp_i+\frac{\partial \Omega_A}{\partial t}dt=0, \tag{3.7}$$

$$A=1,\dots,2(m_1+m_2).$$

The condition (2.21) enables one to use the Lagrange method of indefinite multipliers. Finally, the dynamics in the phase space[3)] is

$$\dot{q}_i=\frac{\partial H}{\partial p_i}+\lambda_A(t)\frac{\partial \Omega_A}{\partial p_i}, \quad \dot{p}_i=-\frac{\partial H}{\partial q_i}-\lambda_A(t)\frac{\partial \Omega_A}{\partial q_i}, \tag{3.8}$$

$$i=1,\dots,n,$$

$$\Omega_A(q,p,t)=0, \quad A=1,\dots,2(m_1+m_2). \tag{3.9}$$

The Lagrange multipliers $\lambda_A(t)$ in the equations of motion (3.8) are determined by the following conditions:

$$\frac{d\Omega_A}{dt}=\frac{\partial \Omega_A}{\partial t}+(\Omega_A,H)+\lambda_B(t)(\Omega_A,\Omega_B)=0, \tag{3.10}$$

$$A,B=1,\dots,2(m_1+m_2).$$

3) In addition to (3.8) equations (3.6) and (3.7) lead to the relation $\frac{\partial H}{\partial t}+\frac{\partial L}{\partial t}+\lambda_A(t)\frac{\partial \Omega_A}{\partial t}=0$ that is not the equation of motion.

As the constraints $\Omega_A(q,p,t)$ have the canonical form (2.20), (2.21) we obtain

$$\lambda_A(t)=-\bar{J}_{AB}\cdot\left[\frac{\partial \Omega_B}{\partial t}+(\Omega_B,H)\right], \tag{3.11}$$

$$\bar{J}=J^{-1}_{2(m_1+m_2)}, \quad A,B=1,\dots,2(m_1+m_2).$$

After substitution of (3.11) into (3.8) the equations of motion are written as follows:

$$\dot{q}_i=\frac{\partial H}{\partial p_i}-\frac{\partial \Omega_A}{\partial p_i}\bar{J}_{AB}\left[\frac{\partial \Omega_B}{\partial t}+(\Omega_B,H)\right], \tag{3.12}$$

$$\dot{p}_i=-\frac{\partial H}{\partial q_i}+\frac{\partial \Omega_A}{\partial q_i}\bar{J}_{AB}\left[\frac{\partial \Omega_B}{\partial t}+(\Omega_B,H)\right],$$

$$i=1,\dots,n, \quad A,B=1,\dots,2(m_1+m_2).$$

It is obvious that these equations are not the Hamiltonian ones. However, we are interested in the dynamics not in the whole phase space Γ but on its physical submanifold Γ^* defined by the constraint equations (3.9). On Γ^* eqs. (3.12) can be written in the explicitly Hamiltonian form

$$\dot{q}_i=\frac{\partial H_T}{\partial p_i}, \quad \dot{p}_i=-\frac{\partial H_T}{\partial q_i}, \quad i=1,\dots,n, \tag{3.13}$$

$$\Omega_A(q,p,t)=0, \quad A=1,\dots,2(m_1+m_2), \tag{3.14}$$

where

$$H_T(q,p,t)=H-\Omega_A\bar{J}_{AB}\left[\frac{\partial \Omega_B}{\partial t}+(\Omega_B,H)\right]. \tag{3.15}$$

The constraint equations (3.14) are noninvolutory invariant relations /13/ for the Hamiltonian equations of motion (3.13)

$$\frac{d\Omega_A}{dt}=\frac{\partial \Omega_A}{\partial t}+(\Omega_A,H_T)=0, \quad q,p\in\Gamma^*, \tag{3.16}$$

$$A=1,\dots,2(m_1+m_2).$$

Using them, one can reduce the number of the equations in the Hamiltonian system (3.13) by $2(m_1+m_2)$. For this purpose, the canonical transformation (2.17) should be used. In terms of the new variables Q_i, P_i the equations (3.13), (3.14) are written in the following form:

$$\dot{Q}_i = \frac{\partial \mathcal{H}}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial \mathcal{H}}{\partial Q_i}, \quad i = 1,\dots,n, \tag{3.17}$$

$$Q_{r+\varkappa} = 0, \quad P_{r+\varkappa} = 0, \quad r = n-(m_1+m_2), \tag{3.18}$$
$$1 \le \varkappa \le m_1+m_2 .$$

The new Hamiltonian $\mathcal{H}(Q,P,t)$ is

$$\mathcal{H}(Q,P,t) = H(q(Q,P,t), p(Q,P,t), t) + \tag{3.19}$$
$$+ R(Q,P,t).$$

Here the functions

$$q_i = q_i(Q,P,t), \qquad p_i = p_i(Q,P,t), \tag{3.20}$$
$$i = 1,\dots,n$$

determine the canonical transformation inverse to (2.17). The addition term $R(Q,P,t)$ in the Hamiltonian is due to the explicit time-dependence of the canonical transformation (2.17). This term is defined by the equations

$$\frac{\partial R(Q,P,t)}{\partial P_i} = \frac{\partial Q_i(q,p,t)}{\partial t}, \quad \frac{\partial R(Q,P,t)}{\partial Q_i} = -\frac{\partial P_i(q,p,t)}{\partial t}, \tag{3.21}$$
$$i = 1,\dots,n.$$

After differentiation with respect to t the right-hand sides of (3.21) have to be expressed by (3.20) as the functions of Q, P, t.

The equations of motion (3.17), (3.18) are in fact the Hamiltonian system with $2r = 2(n-m_1-m_2)$ equations only

$$\dot{Q}_\alpha = \frac{\partial K}{\partial P_\alpha}, \qquad \dot{P}_\alpha = -\frac{\partial K}{\partial Q_\alpha}, \tag{3.22}$$
$$\alpha = 1,\dots,r$$

Here

$$K(Q_\alpha, P_\alpha, t) = \mathcal{H}(Q,P,t)\Big|_{Q_{r+\varkappa}=P_{r+\varkappa}=0}, \tag{3.23}$$
$$\alpha = 1,\dots,r, \quad \varkappa = 1,\dots,m_1+m_2$$

is the corresponding Hamiltonian. On Γ^* the contribution to $K(Q_\alpha, P_\alpha, t)$ is given only by the canonical Hamiltonian and $R(Q,P,t)$

$$K(Q_\alpha,P_\alpha,t) = \big[H(q(Q,P,t), p(Q,P,t), t) + \tag{3.24}$$
$$+ R(Q,P,t)\big]\Big|_{Q_{r+\varkappa}=P_{r+\varkappa}=0}.$$

According to (3.16) the rest equations in the system (3.17) on Γ^* give

$$\dot{Q}_{r+\varkappa} = 0, \qquad \dot{P}_{r+\varkappa} = 0, \tag{3.25}$$
$$r = n - m_1 - m_2, \qquad \varkappa = 1,\dots, m_1+m_2 .$$

Thus, (3.18) are the invariant relations for (3.17).

In paper /9/ the wrong conclusion was made that R vanishes on the submanifold M and consequently equals zero on Γ^* as well. The clear example showing that it is not so is the case of degenerate Lagrangians homogeneous of the first-degree in the velocities /7/. Here the canonical Hamiltonian is identically zero and the dynamics on the physical submanifold Γ^* is generated by R only.

4. The construction of the path integral

At the outset we represent in a usual manner /14/ the matrix element of the evolution operator for the Hamiltonian system (3.22)

$$U(t'',t') = T\exp\Big[-i\int_{t'}^{t''} K(Q_\alpha, P_\alpha, t)\,dt\Big], \tag{4.1}$$
$$\alpha = 1,\dots,r$$

as the path integral on the physical submanifold Γ^* of the phase space

$$I = \langle Q_1'',\dots,Q_r'' | U(t'',t') | Q_1',\dots,Q_r' \rangle = \tag{4.2}$$
$$= \int \exp\Big\{ i\int_{t'}^{t''} [P_\alpha \dot{Q}_\alpha - K(Q_\beta, P_\beta, t)]\,dt \Big\} \prod_{t,\gamma} \frac{dQ_\gamma(t)\,dP_\gamma(t)}{(2\pi)^r},$$
$$\alpha, \beta, \gamma = 1,\dots, r = n - m_1 - m_2 .$$

Now, with the aid of the δ -functions, we extend the functional integration to the whole phase space Γ with the canonical coordinates $Q_i, P_i, \; i=1,\dots,n$

$$I=\int \exp\left\{i\int_{t'}^{t''}[P_i\dot{Q}_i - H(Q,P,t)-R(Q,P,t)]dt\right\}\cdot \tag{4.3}$$

$$\cdot\prod_{\varkappa=1}^{n-r}\delta(Q_{r+\varkappa})\cdot\delta(P_{r+\varkappa})\prod_{i=1}^{n}\frac{dQ_i(t)\,dP_i(t)}{(2\pi)^r}.$$

Taking into account the δ -functions in (4.3), we have spread the sum $\sum_{\alpha=1}^{r} P_\alpha \dot{Q}_\alpha$ in the exponent in (4.3) to all the variables $P_i, Q_i, \; i=1,\dots,n$ and the effective Hamiltonian K has been replaced according to (3.24) by the sum $H+R$.

Now we make the change of the functional integration variables with the aid of the canonical transformation (3.20) which is inverse to (2.17). We shall not discuss here the possibility of this procedure and refer only to the vast literature about this problem /15-17/. After this change, the additional term R in the canonical Hamiltonian (4.3) vanishes

$$I=\int \exp\left\{i\int_{t'}^{t''}[p_i\dot{q}_i - H(q,p,t)]dt\right\}\cdot \tag{4.4}$$

$$\cdot\prod_{\varkappa=1}^{n-r}\delta(Q_{r+\varkappa}(q,p,t))\,\delta(P_{r+\varkappa}(q,p,t))\prod_{i=1}^{n}\frac{dq_i(t)\,dp_i(t)}{(2\pi)^r}.$$

In order to use this formula we have to know the explicit form of the constraints and gauge conditions in the canonical form. Transition from the initial set of constraints (2.2) and gauge conditions (2.15) to the set of the constraints and the gauge conditions in the canonical form (2.18) is a rather difficult mathematical problem /10/. One must solve the partial differential equations. Therefore, it is very desirable to obtain a formula for the path integral in the phase space in terms of the initial set of constraints (2.2) and gauge conditions (2.19).

Let the initial set of constraints (2.2) and gauge conditions (2.19) can be written after the change q and p by (3.20) in the form

$$\eta_A(Q_{r+\varkappa}, P_{r+\varkappa}, t)=0, \tag{4.5}$$

$$A=1,\dots,2m_1+2m_2, \qquad \varkappa=1,\dots,n-r,$$

$$r=n-m_1-m_2.$$

Canonical variables Q_α and P_α, $\alpha=1,\dots,r$ on Γ^* do not enter obviously in the constraint equations (4.5). As the constraint equations (2.19) and (4.5) determine in the phase space the same submanifold Γ^*, then from (2.19) it follows (4.5) and vice versa. The variables $\eta_A(Q_{r+\varkappa}, P_{r+\varkappa}, t)$, $A=1,\dots,2(m_1+m_2)$ are by virtue of (2.3) and (2.16) functionally independent. Therefore, they can be considered as the noncanonical coordinates on the submanifold $\bar{\Gamma}$ which is the difference of Γ and Γ^*: $\bar{\Gamma}=\Gamma\setminus\Gamma^*$.

Now we use the following identity:

$$1=\int_{\bar{\Gamma}}\cdots\int\left(\det\|(\eta_A,\eta_{A'})\|\right)^{1/2}\cdot \tag{4.6}$$

$$\prod_{A=1}^{2(m_1+m_2)}\delta(\eta_A(Q_{r+\varkappa},P_{r+\varkappa},t))\prod_{\varkappa=1}^{n-r}dQ_{r+\varkappa}\,dP_{r+\varkappa}.$$

In order to prove (4.6) one must go in this formula to the integration over the noncanonical variables η_A and use the expression for the volume element of the phase space in terms of the noncanonical variables (see Appendix)

$$\prod_{\varkappa=1}^{n-r}dQ_{r+\varkappa}\,dP_{r+\varkappa}=\frac{\partial(Q_{r+\varkappa},P_{r+\varkappa})}{\partial(\eta_1,\dots,\eta_{2(m_1+m_2)})}\prod_{A=1}^{2(m_1+m_2)}d\eta_A= \tag{4.7}$$

$$=\left(\det\|(\eta_A,\eta_{A'})\|\right)^{-1/2}\prod_{A=1}^{2(m_1+m_2)}d\eta_A.$$

Substituting (4.6) into the integrand in (4.2), we obtain

$$I=\int\exp\left\{i\int_{t'}^{t''}[P_i\dot{Q}_i-H(Q,P,t)-R(Q,P,t)]dt\right\}\cdot \tag{4.8}$$

$$\cdot\prod_{A=1}^{2(m_1+m_2)}\delta(\eta_A(Q_{r+\varkappa},P_{r+\varkappa},t))\left(\det\|(\eta_A,\eta_{A'})\|\right)^{1/2}\prod_{i=1}^{n}\frac{dQ_i(t)\,dP_i(t)}{(2\pi)^r}.$$

As it has been noted above, from (4.5) it follows (2.19). Therefore, we were able to spread again the sum $\sum_{\alpha=1}^{r} P_\alpha \dot{Q}_\alpha$ in (4.7) to all the variables P_i, Q_i, $i=1,\dots,n$ and to substitute (3.24) instead of K. Using in (4.8) the change of the functional integration variables defined by the canonical transformation (3.20), we obtain the final formula for the path integral

$$I=\int \exp\left\{ i\int_{t'}^{t''}[p_i\dot{q}_i - H(q,p,t)dt]\right\} \prod_{A=1}^{2(m_1+m_2)} \delta(\eta_A(q,p,t))\cdot$$
$$\left(\det\|(\eta_A,\eta_{A'})_{q,p}\|\right)^{1/2} \prod_{i=1}^{n} \frac{dq_i(t)dp_i(t)}{(2\pi)^2}. \tag{4.9}$$

The additional term R in the exponent caused by the explicit time dependence of the constraints and the gauge conditions disappear after the change of the variables (3.20). In (4.8) only the initial complete set of constraints (2.2) and gauge conditions (2.19) enter

$$\eta_A(q,p,t) = \tag{4.10}$$
$$=\left\{\omega_s(q,p,t),\ s=1,\dots,m=m_1+2m_2;\ \chi_\alpha(q,p,t),\ \alpha=1,\dots,m_1\right\},$$
$$A=1,\dots,2(m_1+m_2).$$

Due to the δ-functions in (4.9) it is all the same what set of constraints (2.2) or (2.11), (2.12) is used in constructing the path integral.

Let us have another set of gauge conditions

$$\bar{\chi}_\alpha(q,p,t)=0,\quad \alpha=1,\dots,m_1, \tag{4.11}$$

$$\det\|(\bar{\chi}_\alpha,\varphi_\beta)\|_{|\bar{\chi}=\varphi=0}\neq 0,\quad \alpha,\beta=1,\dots,m_1. \tag{4.12}$$

Using (4.9) we can construct the Hamiltonian path integral in this gauge. The set of constraints (2.11), (2.12) and the new gauge conditions (4.11) can be replaced due to (4.12) by the equivalent set of equations in the canonical form. Making the inverse transition to

eq. (4.8) and integrating over $\bar{\Gamma}$ with the aid of the δ-functions, we will obtain formula (4.2) in which independent canonical variables Q_α, P_α, $\alpha=1,\dots,r$ will be replaced by some new canonical coordinates on Γ^* $\bar{Q}_\alpha, \bar{P}_\alpha$, $\alpha=1,\dots,r$. Thus, we have the same formula for I up to the change of the functional variables in the path integral defined by the canonical transformation from $\bar{Q}_\alpha, \bar{P}_\alpha$ to Q_α, P_α.

The integration measure in the final formula (4.9) is determined by

$$\Delta=\left(\det\|(\eta_A,\eta_{A'})\|\right)^{1/2}.$$

According to (4.7) Δ^{-1} is the volume element of the submanifold $\bar{\Gamma}=\Gamma\setminus\Gamma^*$ expressed in terms of the noncanonical coordinates which are defined by the constraints and gauge conditions (4.10).

Appendix

We derive here the expression for the volume element of the phase space in terms of arbitrary noncanonical coordinates. Let Γ be the $2n$-dimensional phase space with the canonical coordinates $q_1,\dots,q_n,p_1,\dots,p_n$ and $2n$ functions $\xi_\mu=\xi_\mu(q,p)$, $\mu=1,\dots,2n$ determine the transition to new variables in Γ which are in general case noncanonical. We suppose that the Jacobian of this transformation is different from zero

$$\det D\neq 0, \tag{A.1}$$

$$D=\left\|\begin{matrix}\dfrac{\partial q_i}{\partial \xi_k} & \dfrac{\partial q_i}{\partial \xi_s}\\[2mm] \dfrac{\partial p_i}{\partial \xi_k} & \dfrac{\partial p_i}{\partial \xi_s}\end{matrix}\right\|,\quad \begin{matrix} i,k=1,\dots,n,\\ s=n+1,\dots,2n.\end{matrix} \tag{A.2}$$

The volume element $d\Gamma$ is defined by

$$d\Gamma=\prod_{i=1}^{n} dq_i\,dp_i = \det D\cdot\prod_{\mu=1}^{2n} d\xi_\mu. \tag{A.3}$$

Let us show that det D can be represented in the form

$$\det D=\left(\det\|[\xi_\mu,\xi_\nu]\|\right)^{1/2}, \tag{A.4}$$

where $[\xi_\mu, \xi_\nu]$ are the Lagrange brackets

$$[\xi_\mu, \xi_\nu] = \sum_{i=1}^{n} \left(\frac{\partial q_i}{\partial \xi_\mu} \frac{\partial p_i}{\partial \xi_\nu} - \frac{\partial p_i}{\partial \xi_\mu} \frac{\partial q_i}{\partial \xi_\nu} \right). \tag{A.5}$$

For this purpose we consider the product of the matrices $D^T J D$, where D^T is the transposed matrix D and J is the symplectic unity matrix

$$J = \begin{Vmatrix} 0 & I_n \\ -I_n & 0 \end{Vmatrix}, \tag{A.6}$$

where I_n is the unity $n \times n$ matrix, $\det J = 1$. The direct calculation gives

$$D^T J D = \| [\xi_\mu, \xi_\nu] \|, \quad 1 \le \mu, \nu \le 2n. \tag{A.7}$$

If we evaluate the determinants in the left- and the right-hand sides of this equality, we get (A.4). The Lagrange brackets (A.5) and the Poisson brackets are connected by the formula

$$\sum_{\mu=1}^{2n} [\xi_\mu, \xi_\nu](\xi_\mu, \xi_\rho) = \delta_{\nu\rho}. \tag{A.8}$$

Hence eq.(A.4) can be represented in the form

$$\det D = \left(\det \| (\xi_\mu, \xi_\nu) \| \right)^{-1/2}. \tag{A.9}$$

The derivation of eq. (A.4) for the volume element of the symplectic manifold in terms of the noncanonical variables is analogous to great extent to the obtaining of the expression for the volume element of the Riemannian manifold. Let V_n be the Riemannian manifold with the coordinates η^μ, $1 \le \mu \le n$ and with the Riemannian structure (with the metric) $g_{\mu\nu}(\eta)$, $\det \| g_{\mu\nu} \| > 0$. We shall consider V_n as the submanifold of the flat space of the sufficiently large dimension R^{n+k}. Let $x_1^a(\eta), \dots, x_n^a(\eta)$, $a = 1, \dots, n+k$ be the coordinate tangent vectors to V_n at the point η and $N_1^a(\eta), \dots, N_n^a(\eta)$, $a = 1, \dots, n+k$ be the unit normals to V_n at this point. Then, the volume element of V_n can be defined as the volume of the parallelepiped constructed by the vectors $\vec{x}_1, \dots, \vec{x}_n, \vec{N}_1, \dots, \vec{N}_k$

$$dV_n = \det D \cdot d\eta_1 \dots d\eta_n, \tag{A.10}$$

where the $(n+k) \times (n+k)$ matrix $\mathcal{D}$ has the form

$$\mathcal{D} = \begin{Vmatrix} x_1^1 & x_1^2 & \dots & x_1^{n+k} \\ \dots & \dots & \dots & \dots \\ x_n^1 & x_{n_2}^2 & \dots & x_{n_{n+k}}^{n+k} \\ N_1^1 & N_1^2 & \dots & N_1^{n+k} \\ \dots & \dots & \dots & \dots \\ N_k^1 & N_k^2 & \dots & N_k^{n+k} \end{Vmatrix}. \tag{A.11}$$

By the simple transformations we obtain the well-known formula

$$\det \mathcal{D} = (\det \mathcal{D}^T \cdot \det \mathcal{D})^{1/2} = (\det(\mathcal{D}^T \cdot \mathcal{D}))^{1/2} = (\det \| (\vec{x}_\mu \cdot \vec{x}_\nu) \|)^{1/2} = (\det \| g_{\mu\nu} \|)^{1/2}. \tag{A.12}$$

References

1. Faddeev L.D. Teor.Mat.Fiz., 1969, 1, p. 3.
2. Faddeev L.D., Slavnov A.A. Gauge Fields, Introduction to Quantum Theory, Benjamin-Cummings: Reading, 1980.
3. Quantum Theory of Gauge Fields. Collection of papers. Moscow, "Mir", 1977.
4. Faddeev L.D., Popov V.N. Uspehi fiz. nauk (in Russian), 1973, v. 111, N 3, p. 427.
5. Dirac P.A.M. Lectures on Quantum Mechanics. Belfer Graduate School of Science. New York, Yeshiva University, 1964.
6. Mukunda N. Physica Scripta, 1980, v. 12, N 6, p. 801-804.
7. Barbashov B.M., Nesterenko V.V., Chervjakov A.M. Teor.Mat.Fiz., 1985, v. 63, N 1, p. 88-96.
8. Senjanovic P. - Ann.Phys., 1976, v. 100, N 1/2, 227-261.
9. Ditsas P. Canonical path integral quantization of gauge systems. Preprint TH.4042/84, Geneva: CERN, 1984.
10. Schouten J.A., v.d. Kulk W. Pfaff's problem and its generalizations. Oxford: Clarendon Press, 1949.

11. Maskawa T., Nakajima H. - Progr.Theor.Phys., 1976, v. 56, N 4, 1295 - 1309.
12. Shanmugadhasan S. - J.Math.Phys., 1973, v. 14, N 6, p.677-687.
13. Levi-Civita T. and Amaldi U. Lezioni di Meccanica Razionale, v. II, part 2. Bologna, Nicola Zanichelli, 1927.
14. Popov V.N. Continual Integrals in Quantum Theory of Fields and Statistical Physics (in Russian), M.: Atomizdat, 1976.
15. Prokhorov L.V. Physics of Element. Particles and Atomic Nuclei, 1982, v. 13, N 5, p.1094-1156.
16. Schulman L.S. Techniques an applications of path integration. New York: John Willey and Sons, 1981.
17. Gervais J., Jevicki A. - Nucl.Phys., 1976, v. B110, N 1, p. 93-112.

Received by Publishing Department
on August 8, 1985.

В Объединенном институте ядерных исследований начал выходить сборник *"Краткие сообщения ОИЯИ"*. В нем будут помещаться статьи, содержащие оригинальные научные, научно-технические, методические и прикладные результаты, требующие срочной публикации. Будучи частью "Сообщений ОИЯИ", статьи, вошедшие в сборник, имеют, как и другие издания ОИЯИ, статус официальных публикаций.

Сборник "Краткие сообщения ОИЯИ" будет выходить регулярно.

The Joint Institute for Nuclear Research begins publishing a collection of papers entitled *JINR Rapid Communications* which is a section of the JINR Communications and is intended for the accelerated publication of important results on the following subjects:

Physics of elementary particles and atomic nuclei.
Theoretical physics.
Experimental techniques and methods.
Accelerators.
Cryogenics.
Computing mathematics and methods.
Solid state physics. Liquids.
Theory of condensed matter.
Applied researches.

Being a part of the JINR Communications, the articles of new collection like all other publications of the Joint Institute for Nuclear Research have the status of official publications.

JINR Rapid Communications will be issued regularly.

SUBJECT CATEGORIES OF THE JINR PUBLICATIONS

Index	Subject
1.	High energy experimental physics
2.	High energy theoretical physics
3.	Low energy experimental physics
4.	Low energy theoretical physics
5.	Mathematics
6.	Nuclear spectroscopy and radiochemistry
7.	Heavy ion physics
8.	Cryogenics
9.	Accelerators
10.	Automatization of data processing
11.	Computing mathematics and technique
12.	Chemistry
13.	Experimental techniques and methods
14.	Solid state physics. Liquids
15.	Experimental physics of nuclear reactions at low energies
16.	Health physics. Shieldings
17.	Theory of condenced matter
18.	Applied researches
19.	Biophysics

Нестеренко В.В. E2-85-597

К выводу формулы для гамильтонова функционального интеграла в теориях со связями первого и второго рода

Дан простой последовательный вывод формулы для гамильтонова функционального интеграла в теориях со связями первого и второго рода /как стационарными, так и нестационарными/. Калибровочные условия могут быть неинволютивны между собой и содержать время явно. В отличие от других работ, большое внимание уделяется доказательству гамильтоновости теории на физическом подмногообразии Γ^* фазового пространства Γ. Показано, что Δ^{-1}, где Δ - детерминант Фаддеева-Попова, есть не что иное, как элемент объема подмногообразия $\bar{\Gamma} = \Gamma \setminus \Gamma^*$ в неканонических координатах. Доказана инвариантность формулы для гамильтонова функционального интеграла при конечных преобразованиях калибровочных условий.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1985

Nesterenko V.V. E2-85-597

On the Derivation of the Formula for the Hamiltonian Functional Integral in Theories with the First- and Second-Class Constraints

A simple and consistent derivation of the formula for the Hamiltonian functional integral in theories with the first- and second-class constraints is given. The gauge conditions may be noninvolutary, and the constraints and gauge conditions can be explicitly time-dependent. In contrast to other papers much attention will be paid to prove the Hamiltonian form of dynamics on the physical submanifold of the phase space. It will be shown that Δ^{-1}, where Δ is the Faddeev-Popov determinant, is just the volume element of the phase space in terms of the noncanonical coordinates. The invariance of the final formula for the functional integral under finite transformations of the gauge conditions will be proved.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1985