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ON DISPERSIVE DERIVATION OF TRIANGLE ANOMALY

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The axial anomaly is one of the most intriguing phenomena in quantum field theory. The peculiar behaviour of the axial-vector current in quantum theory has been first observed in the papers [1] and subsequently explained in [2]. The problem has been reconsidered later and resolved independently by Adler, Bell, and Jackiw [3,4]. Since then, the anomalous divergence of the axial current or the closely related anomalous Ward identity for the VVA triangle diagram have been rederived in many different ways. Various approaches to the axial anomaly may be found, e.g., in refs. [5] through [10]; needless to say, our list of relevant literature is far from being complete.

In this note we will recover the anomalous axial Ward identity for the VVA triangle diagram, starting from its absorptive part and using dispersion relations. Some particular calculations along these lines have been performed in refs. [5,6,7]. Here we present a simple generalization of the earlier results [5,6]. A remarkable feature of such an approach is that everything can be straightforwardly expressed in terms of convergent integrals. In other treatments one deals with divergent quantities which must be given a precise meaning by an explicit regularization; cf., e.g., refs. [3,4,8-10]. (Of course, one may object that evaluating the triangle graph by means of dispersion relations can be viewed as a regularization procedure *sui generis*).

Let us begin with some definitions and basic formulae. The contribution of the familiar VVA triangle graph (with amputated external legs) is formally given by

$$T_{\alpha\mu\nu}(k,p) = \Gamma_{\alpha\mu\nu}(k,p) + \Gamma_{\alpha\mu\nu}(p,k); \tag{1}$$

$$\Gamma_{\alpha\mu\nu}(k,p) = \int \frac{d^4 r}{(2\pi)^4} \, \mathrm{Tr}\left(\frac{1}{\not{r} - \not{k} - m} \gamma_\mu \frac{1}{\not{r} - m} \gamma_\nu \frac{1}{\not{r} + \not{p} - m} \gamma_\alpha \gamma_5 \right), \tag{2}$$

where k, p are the external momenta outgoing from vector vertices and m is the fermion mass. In the sequel we shall also deal with the 2nd rank pseudotensor $T_{\mu\nu}(k,p)$ which is given by formulae analogous to (1), (2) with γ_5 replaced by the unit matrix. For the metric and γ-matrices we adopt the conventions of Bjorken and Drell [11]. The famous result [3,4] for the amplitude (1), (2) consists in the following: If one imposes the vector Ward identities

$$k^\mu T_{\alpha\mu\nu}(k,p) = p^\nu T_{\alpha\mu\nu}(k,p) = 0 \tag{3}$$

then the axial Ward identity picks up an anomalous term, i.e.,

$$q^{\alpha} T_{\alpha\mu\nu}(k,p) = 2m T_{\mu\nu}(k,p) + \frac{1}{2\pi^2} \epsilon_{\mu\nu\rho\sigma} k^{\rho} p^{\sigma}, \tag{4}$$

where $q = k + p$; the second term on the r.h.s. of eq.(4) is just the ABJ anomaly.

For simplicity we shall restrict ourselves to k, p such that $k^2 = p^2$. Invariant amplitudes (formfactors) corresponding to the 3rd rank Lorentz pseudotensor (1), (2) may be defined as follows (for a detailed discussion of this point, see ref. /12/):

$$T_{\alpha\mu\nu}(k,p) = F_1(q^2; k^2, m^2)\, \epsilon_{\alpha\mu\nu\rho}(k^{\rho} - p^{\rho}) + F_2(q^2; k^2, m^2)(\epsilon_{\alpha\mu\rho\sigma} p_{\nu} - \epsilon_{\alpha\mu\rho\sigma} k_{\mu}) k^{\rho} p^{\sigma} + F_3(q^2; k^2, m^2)\, \epsilon_{\mu\nu\rho\sigma} k^{\rho} p^{\sigma} q_{\alpha}. \tag{5}$$

Note that the notation employed in the definition (5) does not coincide with that of ref. /12/. For $\epsilon_{\mu\nu\rho\sigma}$ we adopt the convention $\epsilon_{0123} = +1$ (cf. /11/). The 2nd rank pseudotensor $T_{\mu\nu}(k,p)$ entering in the r.h.s. of eq.(4) is described by means of a single formfactor G, namely

$$T_{\mu\nu}(k,p) = G(q^2; k^2, m^2)\, \epsilon_{\mu\nu\rho\sigma} k^{\rho} p^{\sigma}. \tag{6}$$

Ward identities (3) and (4) may be then recast consecutively as

$$F_1 = k^2 F_2 \tag{7}$$

and

$$q^2 F_3 - 2F_1 = 2mG + \frac{1}{2\pi^2}. \tag{8}$$

When the formfactors F_j, $j = 1,2,3$ or G resp. are considered as functions of a complex variable q^2 (at a fixed value of k^2), these should possess a cut along the real axis, extending from $q^2 = 4m^2$ to infinity (see, e.g., /5,6,7,13/). The corresponding discontinuity of a formfactor F_j or G resp., divided by $2i$, will be called its absorptive (imaginary) part and denoted by A_j or B resp. In order to avoid the cuts with respect to the variable k^2, we shall consider the values $k^2 = p^2 \leq 0$ in what follows. The functions A_j, $j = 1,2,3$ and B can be calculated explicitly (see /5,6,12/; notice also that the formula (11.50) in ref. /7/ is incorrect). It turns out that the unsubtracted dispersion integrals

$$F_j^{(un)}(q^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{A_j(t)}{t - q^2}\, dt, \quad j = 1, 2, 3; \tag{9}$$

$$G^{(un)}(q^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{B(t)}{t - q^2}\, dt \tag{10}$$

are convergent. This is apparently related to the fact that despite a superficial linear divergence of the integral (2) this is finite when defined with the help of symmetric integration (cf. /6/). One may therefore try to define the formfactors appearing in eqs.(5), (6) by means of dispersion relations (9), (10). The only constraint is the vector Ward identity (7). From (7) and (9) it is easy to see that if we set

$$F_j(q^2; k^2, m^2) = F_j^{(un)}(q^2; k^2, m^2), \quad j = 1, 2, 3, \tag{11a}$$

$$G(q^2; k^2, m^2) = G^{(un)}(q^2; k^2, m^2) \tag{11b}$$

then (7) is satisfied automatically. Let us stress that this is due to our convenient choice of invariant amplitudes according to (5); there are other options, frequently encountered in current literature (cf., e.g., /3,6,12/), which would necessitate a modification of the definition (11a) through subtractions in dispersion relations (9) in order to satisfy the requirement (7). Of course, such subtractions have nothing to do with convergence properties of the integrals (9). Let us also remark that the formfactor G may be uniquely calculated directly from the Feynman graph for $T_{\mu\nu}(k,p)$, since the integral obtained from (2) by replacement $\gamma_5 \to 1$ is perfectly convergent after performing the trace. It may be verified that (10) coincides with the result of such a direct evaluation of G.

From the definitions (9) through (11b) one gets easily for the l.h.s. of the axial Ward identity (8) (taking into account that the absorptive parts obviously satisfy "normal" Ward identities):

$$q^2 F_3(q^2; k^2, m^2) - 2F_1(q^2; k^2, m^2) = -\frac{1}{\pi} \int_{4m^2}^{\infty} A_3(t; k^2, m^2)\, dt + 2mG(q^2; k^2, m^2). \tag{12}$$

We shall see below that the integral on the r.h.s. of eq.(12) is convergent. If we want to recover (8), we have to show that

$$\int_{4m^2}^{\infty} A_3(t; k^2, m^2)\, dt = -\frac{1}{2\pi}. \tag{13}$$

The "sum rule" (13) has been proved in ref. /5/ for $k^2 = 0$, $m \neq 0$ and in ref. /6/ for $k^2 < 0$, $m = 0$. In both cases it is obvious *a priori* (for dimensional reasons) that the integral in (13) is a constant. However, it is not clear, how such a statement could be proved on general grounds (using, e.g., some analyticity arguments) in the case when both k^2 and m are nonzero.

Below we shall demonstrate by means of an explicit calculation that eq. (13) is valid in the case $k^2 \leq 0$, $m > 0$. Our calculation thus encompasses the previous results [5,6,7].

The function $A_3(q^2; k^2, m^2)$ may be easily deduced from the results of ref. [12]:

$$A_3(q^2; k^2, m^2) = \frac{1}{2\pi}\,\frac{2k^2}{q^2}\Big[\frac{q^2+2k^2}{(q^2-4k^2)^2}\Big(1-\frac{4m^2}{q^2}\Big)^{1/2} + \frac{2k^2(q^2-2k^2)}{\sqrt{q^2}\,(q^2-4k^2)^{5/2}}\Big(\frac{q^2-k^2}{q^2-2k^2} + m^2\,\frac{q^2-4k^2}{2(k^2)^2}\Big)\ln S\Big], \tag{14}$$

where

$$S = \frac{q^2-2k^2-[(q^2-4m^2)(q^2-4k^2)]^{1/2}}{q^2-2k^2+[(q^2-4m^2)(q^2-4k^2)]^{1/2}}. \tag{15}$$

Introducing the dimensionless variables

$$\beta = \frac{-k^2}{m^2}, \quad x = \frac{q^2}{4m^2} \tag{16}$$

we obtain from (14), (15), (16)

$$\int_{4m^2}^{\infty} A_3(q^2; k^2, m^2)\,dq^2 = -\frac{1}{2\pi}\,[I_1(\beta) + I_2(\beta) + I_3(\beta)], \tag{17}$$

where

$$I_1(\beta) = \frac{1}{2}\beta\int_1^{\infty}\frac{x-\frac{1}{2}\beta}{x(x+\beta)^2}\Big(\frac{x-1}{x}\Big)^{1/2}dx,$$

$$I_2(\beta) = -\frac{1}{4}\beta^2\int_1^{\infty}\frac{x+\frac{1}{4}\beta}{x^{3/2}(x+\beta)^{5/2}}\ln\frac{x+\frac{1}{2}\beta-[(x-1)(x+\beta)]^{1/2}}{x+\frac{1}{2}\beta+[(x-1)(x+\beta)]^{1/2}}, \tag{18}$$

$$I_3(\beta) = -\frac{1}{2}\int_1^{\infty}\frac{x+\frac{1}{2}\beta}{x^{3/2}(x+\beta)^{3/2}}\ln\frac{x+\frac{1}{2}\beta-[(x-1)(x+\beta)]^{1/2}}{x+\frac{1}{2}\beta+[(x-1)(x+\beta)]^{1/2}}.$$

Calculation of the integrals (18) is elementary and the result is

$$I_1(\beta) = \frac{5}{4} - \frac{2\beta+5}{4\beta}\Big(\frac{\beta}{\beta+1}\Big)^{1/2}\ln[\beta^{1/2} + (\beta+1)^{1/2}], \tag{19}$$

$$I_2(\beta) = -\frac{1}{4} + \frac{2\beta^2-3\beta-8}{4\beta^2}\Big(\frac{\beta}{\beta+1}\Big)^{1/2}\ln[\beta^{1/2}+(\beta+1)^{1/2}] - \frac{2}{\beta}(\beta+1)^{1/2}I_0(\beta),$$

$$I_3(\beta) = \frac{2\beta+2}{\beta^2}\Big(\frac{\beta}{\beta+1}\Big)^{1/2}\ln[\beta^{1/2}+(\beta+1)^{1/2}] + \frac{2}{\beta}(\beta+1)^{1/2}I_0(\beta), \tag{19}$$

where

$$I_0(\beta) = \int_{-1}^{1}\frac{dt}{(1+\beta t^2)^{1/2}(\beta t-\beta-2)}.$$

The integral $I_0(\beta)$ can be also easily expressed in terms of elementary functions, but it is not necessary for our purposes. From (19) it is readily seen that

$$I_1(\beta) + I_2(\beta) + I_3(\beta) = 1. \tag{20}$$

Eqs. (20) and (17) then immediately imply (13) and this is the desired result.

Further, from (13) and (14) easily follows that

$$\lim_{\substack{m\to 0\\ k^2\to 0}} A_3(q^2; k^2, m^2) = -\frac{1}{2\pi}\,\delta(q^2). \tag{21}$$

Eq. (21) also represents a generalization of the earlier results [5,6,7] which in fact established (21) only for particular limiting procedure, namely $-k^2 \to 0$ followed by $m \to 0$ and vice versa.

To summarize, eqs. (13) and (21) (along with the intermediate formulae (19)) constitute the main results of the present paper, which demonstrate that the earlier calculations [5,6,7] may be extended in an elementary way.

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Горжейши И. E2-85-56

О дисперсионном выводе треугольной аномалии

Приводится простое обобщение результатов прежних работ других авторов, в которых восстанавливалась треугольная аномалия Адлера, Белла и Джекива при помощи дисперсионных соотношений. Рассматривается абсорбтивная часть треугольной диаграммы с двумя векторными и одной аксиальной вершиной, с внешними импульсами k, p в векторных вершинах такими, что $k^2 = p^2 \le 0$, и с массой фермиона $m \ge 0$. Явно вычисляется интеграл от мнимой части соответствующей инвариантной амплитуды и показано, что воспроизводится аномальный вклад в аксиальное тождество Уорда. Тем самым доказано дельтаобразное поведение такой мнимой части в пределе $k^2 \to 0$, $m \to 0$.

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On Dispersive Derivation of Triangle Anomaly

We present a straightforward generalization of the results of some previous treatments, in which the Adler-Bell-Jackiw triangle anomaly has been recovered with the help of dispersion relations. We consider the absorptive part of the VVA triangle diagram with the external momenta k, p at vector vertices such that $k^2 = p^2 \le 0$ and the fermion mass $m \ge 0$. An integral of the imaginary part of the relevant invariant amplitude is calculated explicitly and shown to produce the desired anomalous contribution to the axial Ward identity. This also enables one to demonstrate the δ-like behaviour of such an imaginary part in the limit $k^2 \to 0$, $m \to 0$.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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