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MINIMAL QUANTIZATION AND BOUND-STATE EQUATIONS IN TWO-DIMENSIONAL MASSLESS QCD

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1. Introduction

The intermediate position QCD_{1+1} holds between an exactly solvable model - the Schwinger model, and the usual QCD_{3+1} motivates the interest in it. Thus, the non-Abelian structure of the gauge group is emphasized. Does such an intuitive conclusion take place in this theory? To anwer this question we study massless QCD_{1+1} in the frame of the minimal quantization scheme /1/ (in terms of gauge-invariant variables). An adequate and convenient description of two-dimensional field-theoretical models with fermions provides their bosonization /2,3/. In the model considered this procedure is troublesome because of the non-Abelian gauge group. A possible way out of this difficulty is some special choice of gauge conditions /4,5/. However, such an approach does not allow one to make general conclusions about gauge-dependent quantities, for example, about the coloured-objects Green functions. So, it is important to formulate the theory in terms of gauge-invariant variables and bosonize it in this general case. In the present paper we show that the solution of this problem consists in constructing the unitary irreducible representation of the Kac-Moody algebra (the current algebra in QCD_{1+1}) that satisfy dynamical equations of the theory.

The paper is organized as follows:

In section 2 by the Schwinger model example the minimal quantization scheme is illustrated. The topological confinement criterion based on the results about the quark Green function is formulated.

Section 3 is devoted to introduction of gauge-invariant variables in two-dimensional QCD.

The problems connected with the bosonization of the model are considered in section 4.

2. The minimal quantization scheme: Schwinger's model

Quantization of two-dimensional massless QCD - the Schwinger model, in terms of gauge-invariant variables (which we have called "the minimal quantization scheme") in the finite-volume space-time leads to an interesting result about Green's functions of charged objects [6]. Without entering into details, we shall give a brief review of the method used and discuss the main result.

The Schwinger model action

$$S = \int d^2x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\Psi}(x) \gamma^{\mu} (\partial_{\mu} - ie A_{\mu}) \Psi(x) \right] \tag{1}$$

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \qquad \mu, \nu = 0, 1$$

on the solutions of the constraint equation

$$\frac{\delta S}{\delta A_0} = 0 \qquad \Rightarrow \partial_1^2 A_0 = \partial_1 \partial_0 A_1 + e j_0 \tag{2}$$

takes the form

$$S = \int d^2x \left[\frac{1}{2} (\partial_0 A_1^I)^2 + i \bar{\Psi}^I \gamma^{\mu} \partial_{\mu} \Psi^I + \frac{e^2}{2} j_0^I \frac{1}{\partial_1^2} j_0^I \right]. \tag{3}$$

Note, that it is expressed in terms of the variables A_1^I, Ψ^I only, where

$$A_1^I = A_1 - \partial_1 \frac{1}{\partial_1^2} \partial_1 A_1 \equiv 0$$
$$\Psi^I = \exp(-ie \partial_1^{-1} A_1) \Psi. \tag{4}$$

The variables (4) are invariant under U(1)-gauge transformations. Their choice is rigidly fixed by the dynamics of the system and by the requirement of gauge invariance. However, in their construction procedure itself there is an ambiguity because the action of the inverse operator ∂_1^{-2} in (2) is determined up to a function satisfying homogeneous equations

$$\partial_1^2 \lambda(x) = 0 \quad , \quad \partial_1^2 \partial_0 \lambda(x) = 0. \tag{5}$$

We are interested in the solutions of equations (5) in the class of smooth function

$$g(x) = \exp\{i \lambda(x)\} \tag{6}$$

vanishing at the space-time boundaries, which corresponds to the absence of charges in it. At the ends of the time interval these functions determine a map of the line R(1) (with identified end-points- see [7]) onto the group U(1). The smoothness condition then transforms into the requirement of an integer value of the corresponding degree of mapping

$$\nu = n_+ - n_- = \frac{e}{4\pi} \int d^2x \, \varepsilon_{\mu\nu} F^{\mu\nu}$$

that may be written also as

$$\pi_1 (U(1)) = \mathbb{Z}, \tag{7}$$

π_1 being the first homotopy group.

In the finite-volume space-time

$$-\frac{T}{2} \le x_0 \le \frac{T}{2}$$
$$-\frac{R}{2} \le x_1 \le \frac{R}{2}$$

equations (5) have such nontrivial solutions

$$\lambda(x) = 2\pi N(x_0) \frac{x_1}{R} \tag{8}$$

$$N\left(\pm \frac{T}{2}\right) = n_{\pm} \qquad , \quad n_{\pm} = 0, \pm 1, \pm 2, \dots$$

which changes the form of variables (4)

$$(A_1^I)^{\lambda} = A_1^I + \frac{\partial_1 \lambda}{e}$$
$$(\Psi^I)^{\lambda} = e^{i\lambda} \Psi^I.$$

As a result, the sources of the charged fields in the Green-functions generating functional acquire phase factors (6), (8) which represents the topological vacuum degeneration in Schwinger's model [8]. After taking an average over this degeneration and removing the infrared regularization ($R, T \to \infty$), the quark Green function vanishes in the limit $p^2 \to 0$, whereas the neutral-current correlator preserves its one particle-pole (the quark bound state in the model) [6]. This may be considered as a manifestation of the confinement in Green's functions context [9,10].

The quark Green function vanishing in the Schwinger model is connected with topological degeneration of the gauge-field vacuum which follows from (7). So, its generalized form

$$\pi_{D-1}(G) = Z, \tag{9}$$

where D is the space-time dimension, and G - the symmetry group of the theory, may be considered as a criterion for the existence of topological confinement mechanism in it.

For example, in QED_{3+1} condition (9) is not valid,

$$\pi_3(U(1)) = 0,$$

which is in agreement with the observability of electrons. In QCD_{3+1} relation (9) takes place and if this criterion is really crucial it would allow us to make a conclusion about colour confinement there[11]. Thus, we have to compare the results of straightforward calculations and those following from the "topological" criterion in different field-theoretical models. The two-dimensional QCD is an interesting object from this point of view. Since

$$\pi_1(SU(N)) = 0, \tag{10}$$

the topological nature of confinement mechanism may be confirmed by the presence of free quarks in the model spectrum.

3. Gauge-invariant variables in QCD_{1+1}

The two-dimensional massless QCD with a gauge group SU(N) is described by the action

$$S = \int d^2x \left[-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + i\bar{\Psi}^A \gamma^\mu (D_\mu)_{AB} \Psi^B \right], \tag{11}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

$$(D_\mu)_{AB} = \partial_\mu \delta_{AB} + i g A^a_\mu \frac{\tau^a}{2}$$

$$[\tau^a, \tau^b] = 2 i f^{abc} \tau^c$$

$$a, b, \ldots = 1, 2, \ldots, N^2 - 1$$

$$A, B, \ldots = 1, 2, \ldots, N.$$

For subsequent considerations it is convenient to use the covariant derivative in the adjoint representation

$$(\nabla_\mu)^{ab} = \partial_\mu \delta^{ab} - g f^{abc} A^c_\mu$$

for rewriting the action (11) in the form

$$S = \int d^2x \left[\frac{1}{2} (\partial_0 A^a_1 - \nabla_1^{ab} A^b_0)^2 + i\bar{\Psi}\gamma^\mu \partial_\mu \Psi - g j^{\mu a} A^a_\mu \right], \tag{12}$$

where

$$j^a_\mu(x) = \bar{\Psi}(x) \gamma_\mu \frac{\tau^a}{2} \Psi(x).$$

Following the method from section 2, we eliminate components A^a_0 through the constraint equations

$$\frac{\delta S}{\delta A^a_0} = 0 \quad \Rightarrow \quad (\nabla_1^2 A_0)^a = (\nabla_1 \partial_0 A_1)^a - g j^a_0. \tag{13}$$

Here, we find a difference from QED_{1+1}. From (10) it comes out that there do not exist nontrivial solutions of the corresponding homogeneous equations. Thus, for the general solution of (13) we obtain

$$A_0^a = \left(\frac{1}{\nabla_1^2}\right)^{ab} (\nabla_1 \partial_0 A_1 - g j_0)^b.$$

The action (12) then becomes

$$S = \int d^2x \left\{ \frac{1}{2} g^2 \left(\frac{1}{\nabla_1^2} j_0\right)^a \left(\frac{1}{\nabla_1^2} j_0\right)^a + g^2 j^a_0 \left(\frac{1}{\nabla_1^2} j_0\right)^a + \right.$$
$$\left. + i\bar{\Psi}\gamma^\mu \partial_\mu \Psi - g j^a_0 \left(\frac{1}{\nabla_1} \partial_0 A_1\right)^a + g j^a_1 A^a_1 \right\}. \tag{14}$$

Let us introduce the variables

$$\hat{a}_1^I = h \left(\hat{A}_1 + \frac{1}{g} \nabla_1\right) h^{-1}$$
$$\Psi^I = h \Psi, \tag{15}$$

where the operator h is defined as

$$h = T \exp\left\{ g \int^{x_0} dx'_0 \left(\frac{1}{\nabla_1^2} \nabla_1 \partial_0 A_1\right)^a \tau^a \right\}.$$

Variables (15) are invariant under SU(N)-gauge transformations:

$$A_1^{a(\lambda)} = A_1^a + \frac{1}{g}(\nabla_1 \lambda)^a \qquad \qquad a_1^I(A_1^{(\lambda)}) = a_1^I(A_1)$$
$$\psi^{(\lambda)} = e^{\hat{\lambda}}\psi \qquad \Rightarrow \qquad \psi^I(A_1^{(\lambda)}, \psi^{(\lambda)}) = \psi^I(A_1, \psi).$$

The covariant derivative ∇_1 in these variables coincides with the usual one

$$\nabla_1(a_1^I) \equiv \partial_1 . \tag{16}$$

The action (14), when (15) and (16) are taken into account, takes the form

$$S = \int d^2x \left[\frac{1}{2} g^2 j_0^a \frac{1}{\partial_1^2} j_0^a + i\bar{\psi}\gamma^\mu \partial_\mu \psi \right] \tag{17}$$

that is the same as in the gauge $A_1^a = 0$. An analogous situation takes place in QED_{1+1} with the only difference that the axial gauge does not provide an explicit manifestation of the nontrivial dynamics of two-dimensional Abelian gauge field [12]. However, the gauge-invariant variables make possible another way for solving the confinement problem - study of analytical properties of the quark propagator and their interpretation in the spirit of quantum field theory and statistical physics. In particular, the existence of a single pole will be connected with the presence in the spectrum of a particle with the corresponding quantum numbers, the absence of such a pole will mean the absence of this particle in the asymptotic states [1].

4. Equations for the bound states in two-dimensional quantum chromodynamics

The action (17) is similar to the Schwinger model action (3), which may be bosonized, the theory being equivalent to the free massive scalar field one [3,13]. Bosonization is based on the appearance of an anomalous term in the axial current commutator

$$[j_{50}(x), j_{51}(y)] = \frac{1}{\pi i} \partial_y \delta(x-y) \tag{18}$$

due to the negative-energy states filling [12,14].

The substitution

$$j_{5\mu}(x) = \frac{1}{\sqrt{\pi}} \partial_\mu \phi(x)$$

transforms relation (18) into the scalar field $\phi(x)$ commutator. The axial current partial conservation law then takes the form of the Klein-Gordon equation

$$\partial^\mu j_{5\mu} \longrightarrow \Box\phi = m^2\phi \quad , \quad m^2 = \frac{e^2}{\pi} .$$

The non-Abelian gauge group in two-dimensional QCD makes the current commutation relations more complicated

$$\begin{aligned} [j_{50}^a(x), j_{51}^b(y)] &= i f^{abc} j_1^c(x)\delta(x-y) + \frac{\delta^{ab}}{2\pi i}\partial_y \delta(x-y) \\ [j_{50}^a(x), j_{50}^b(y)] &= i f^{abc} j_0^c(x)\delta(x-y) \\ [j_{51}^a(x), j_{51}^b(y)] &= i f^{abc} j_0^c(x)\delta(x-y) . \end{aligned} \tag{19}$$

The algebra (19) is known as the Kac-Moody algebra with a central extension [15]. This algebra also arises in the free fermion theory in two-dimensional space-time, in the nonlinear σ-model with the Wess-Zumino term added to the action [16], in two-dimensional supersymmetric fermionic models [17]. For the QCD_{1+1} bosonization we shall be interested in a subclass of irreducible unitary representations of the algebra (19), the separation being based on the dynamical equations of the theory - the vector-current conservation and the partial conservation of the axial one.

It is not difficult to find that

$$i[H_0, j_{50}^a(x)] = \partial_1 j_{51}^a(x), \tag{20}$$

where H_0 is the free Hamiltonian in the model,

$$H_0 = \int dx_1 \bar{\psi}(x)\gamma_1\partial_1\psi(x).$$

With the help of the Heisenberg equation of motion for the component $j_{50}^a(x)$ and eq.(20), we are led to the following expression for the axial current anomalous divergence:

$$\begin{aligned} \partial^\mu j_{5\mu}^a(x) &= \partial_0 j_{50}^a(x) - \partial_1 j_{51}^a(x) = \\ &= i[H, j_{50}^a(x)] - i[H_0, j_{50}^a(x)] = \\ &= i[H_{int}, j_{50}^a(x)] , \end{aligned} \tag{21}$$

where

$$H_{int} = \int dx_1 \frac{g^2}{2} j_0^a(x) \frac{1}{\partial_1^2} j_0^a(x) = \frac{g^2}{4} \int dx_1 dy_1 j_0^a(x) |x_1 - y_1| j_0^a(y).$$

With (19), (21) taken into account we find that

$$\partial^\mu j_{5\mu}^a(x) = \frac{g^2}{2\pi} \partial_1^{-1} j_{51}^a(x) - \frac{g^2}{2} f^{abc} \int dy_1 j_{50}^b(x) |x_1 - y_1| j_{51}^c(y). \tag{22}$$

Let us define the operator $\mathcal{D}_0(x)$ as

$$\mathcal{D}_0^{ab}(x) = \partial_0 \delta^{ab} - \frac{g^2}{2} f^{abc} \int dy_1 |x_1 - y_1| j_{51}^c(y).$$

Equation (22) and the vector-current conservation law are then written in a compact form

$$\mathcal{D}_0^{ab}(x) j_0^b(x) - \partial_1 j_1^a(x) = 0 \tag{23}$$

$$\mathcal{D}_0^{ab}(x) j_1^b(x) - \partial_1 j_0^a(x) = \frac{g^2}{2\pi} \partial_1^{-1} j_{51}^a(x).$$

Bosonization of two-dimensional QCD consists in constructing the unitary irreducible representations of algebra (19) as solutions of the system (23). So, this system describes the bosonic excitations in the model. The experience from QED_{1+1} and the proposed scheme itself are not compatible with the assumption that this mode is connected with the gauge field [18]. It should rather be interpreted as a bound state of fermions.

5. Conclusions

A relation of the confinement problem with the study of analytical properties of the coloured-objects Green functions seems natural from the point of view of quantum field theory and statistical physics. However, one of the difficulties along this way is the gauge dependence of these quantities. It may be eliminated after formulating the theory in terms of gauge-invariant variables. Their introduction and the results obtained give rise to an assumption of topological mechanism of confinement and to the criterion of its realization[1,6,11].

For checking the results of application of this criterion to the two-dimensional massless QCD, we have to bosnize the theory starting from its formulation in the frame of the minimal quantization scheme. We have shown that in this case the bosonization is reduced to the construction of the Kac-Moody algebra unitary representations as solutions of the dynamical equations for the currents. This approach has been successfully used in solving analogous problems in the Abelian case (the Schwinger model)[12] and also in the theory of free Majorana fermions with chiral O(N)xO(N) symmetry in two-dimensions [16]. This gives us a confidence that it will provide the correct interpretation of the bosonic mode in QCD_{1+1} and the verification of topological confinement mechanism and its criterion.

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Минимальная схема квантования и уравнения для связанных состояний в двумерной безмассовой КХД

Двумерная безмассовая квантовая хромодинамика рассматривается в качестве примера применения минимальной схемы квантования. В этой схеме теория формулируется в терминах калибровочно-инвариантных переменных. Процедура их введения и анализ результатов, полученных в КЭД$_{1+1}$ (модель Швингера) и в КХД, привели к предположению о топологическом механизме конфайнмента и позволили сформулировать критерий его реализуемости. Проверка результатов применения топологического критерия к безмассовой КХД$_{1+1}$ приводит к необходимости бозонизировать теорию,исходя из ее формулировки в рамках минимальной схемы квантования. Показано, что в этом случае бозонизация сводится к построению неприводимых унитарных представлений алгебры Каца - Муди как решения динамических уравнений для токов.

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Minimal Quantization and Bound-State Equations in Two-Dimensional Massless QCD

The massless two-dimensional quantum chromodynamics is considered as an example of application of the minimal quantization scheme. In its framework the theory is described in terms of gauge-invariant variables. Their construction and the results obtained in QCD$_{1+1}$ (the Schwinger model) and in QCD gave rise to an assumption of the topological mechanism of confinement and allowed us to formulate the criterion of its realization. For checking the results of application of this criterion to the massless QCD$_{1+1}$, a bosonization of the theory, starting from its formulation in the frame of the minimal quantization scheme is required. It is shown that in the case considered this procedure is reduced to the construction of the Kac-Moody algebra unitary representations as solutions of the dynamical equations for the currents.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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